Lecture 2

Elliptic curves over finite fields

The Group structure

Research School: Algebraic curves over finite fields
CIMPA-ICTP-UNESCO-MESR-MINECO-PHILIPPINES
University of the Phillipines Diliman, July 24, 2013

Francesco Pappalardi
Dipartimento di Matematica e Fisica
Università Roma Tre

Elliptic curves over $\mathbb{F}_q$

Definition (Elliptic curve)

An elliptic curve over a field K is the data of a non singular Weierstraß equation
$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, a_i \in K$

If $p = \text{char}\, K > 3$,

$$\Delta_E := \frac{1}{2^4}\big(-a_1^5a_3a_4 - 8a_1^3a_2a_3a_4 - 16a_1a_2^2a_3a_4 + 36a_1^2a_3^2a_4 - a_1^4a_4^2 - 8a_1^2a_2a_4^2 - 16a_2^2a_4^2 + 96a_1a_3a_4^2 + 64a_4^3 + a_1^6a_6 + 12a_1^4a_2a_6 + 48a_1^2a_2^2a_6 + 64a_2^3a_6 - 36a_1^3a_3a_6 - 144a_1a_2a_3a_6 - 72a_1^2a_4a_6 - 288a_2a_4a_6 + 432a_6^2\big)$$

Elliptic curves over K

After applying a suitable affine transformation we can always assume that E/K has a Weierstraß equation of the following form

Example (Classification ($p = \text{char } K$))

E	p	Δ_E
$y^2 = x^3 + Ax + B$	≥ 5	$4A^3 + 27B^2$
$y^2 + xy = x^3 + a_2x^2 + a_6$	2	a_6^2
$y^2 + a_3y = x^3 + a_4x + a_6$	2	a_3^4
$y^2 = x^3 + Ax^2 + Bx + C$	3	$4A^3C - A^2B^2 - 18ABC$ $+4B^3 + 27C^2$

Elliptic curves over K

After applying a suitable affine transformation we can always assume that E/K has a Weierstraß equation of the following form

Example (Classification ($p = \text{char } K$))

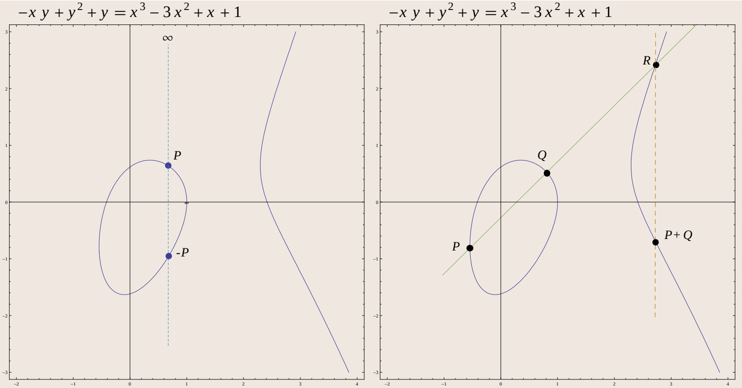
E	p	Δ_E
$y^2 = x^3 + Ax + B$	≥ 5	$4A^3 + 27B^2$
$y^2 + xy = x^3 + a_2x^2 + a_6$	2	a_6^2
$y^2 + a_3y = x^3 + a_4x + a_6$	2	a_3^4
$y^2 = x^3 + Ax^2 + Bx + C$	3	$4A^3C - A^2B^2 - 18ABC$ $+4B^3 + 27C^2$

Let $E/\mathbb{F}_q$ elliptic curve, $\infty := [0, 1, 0]$. Set

$$E(\mathbb{F}_q) = \{(x, y) \in \mathbb{F}_q^2 : \ y^2 = x^3 + Ax + B\} \cup \{\infty\}$$

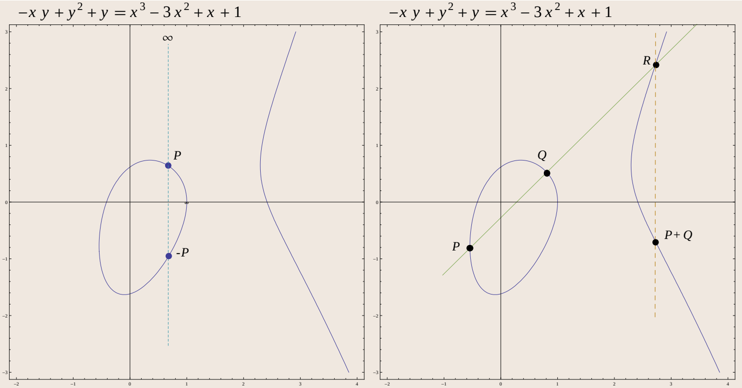
If $P, Q \in E(\mathbb{F}_q)$, $r_{P,Q}$: $\begin{cases} \text{line through } P \text{ and } Q & \text{if } P \neq Q \\ \text{tangent line to } E \text{ at } P & \text{if } P = Q, \end{cases}$

$r_{P,\infty}$: vertical line through P



If $P, Q \in E(\mathbb{F}_q)$, $r_{P,Q}$: $\begin{cases} \text{line through } P \text{ and } Q & \text{if } P \neq Q \\ \text{tangent line to } E \text{ at } P & \text{if } P = Q, \end{cases}$

$r_{P,\infty}$: vertical line through P



$r_{P,\infty} \cap E(\mathbb{F}_q) = \{P, \infty, P'\}$ $\rightsquigarrow$ $-P := P'$

$r_{P,Q} \cap E(\mathbb{F}_q) = \{P, Q, R\}$ $\rightsquigarrow$ $P +_E Q := -R$

Theorem

The addition law on E/K (K field) has the following properties:

(a) $P +_E Q \in E \qquad \forall P, Q \in E$

(b) $P +_E \infty = \infty +_E P = P \qquad \forall P \in E$

(c) $P +_E (-P) = \infty \qquad \forall P \in E$

(d) $P +_E (Q +_E R) = (P +_E Q) +_E R \qquad \forall P, Q, R \in E$

(e) $P +_E Q = Q +_E P \qquad \forall P, Q \in E$

So $(E(\bar{K}), +_E)$ is an abelian group.

Theorem

The addition law on E/K (K field) has the following properties:

(a) $P +_E Q \in E$ $\quad \forall P, Q \in E$

(b) $P +_E \infty = \infty +_E P = P$ $\quad \forall P \in E$

(c) $P +_E (-P) = \infty$ $\quad \forall P \in E$

(d) $P +_E (Q +_E R) = (P +_E Q) +_E R$ $\quad \forall P, Q, R \in E$

(e) $P +_E Q = Q +_E P$ $\quad \forall P, Q \in E$

So $(E(\bar{K}), +_E)$ is an abelian group.

Remark:

If $E/K \Rightarrow \forall L, K \subseteq L \subseteq \bar{K}, E(L)$ is an abelian group.

Theorem

The addition law on E/K (K field) has the following properties:

(a) $P +_E Q \in E$ $\quad \forall P, Q \in E$

(b) $P +_E \infty = \infty +_E P = P$ $\quad \forall P \in E$

(c) $P +_E (-P) = \infty$ $\quad \forall P \in E$

(d) $P +_E (Q +_E R) = (P +_E Q) +_E R$ $\quad \forall P, Q, R \in E$

(e) $P +_E Q = Q +_E P$ $\quad \forall P, Q \in E$

So $(E(\bar{K}), +_E)$ is an abelian group.

Remark:

If $E/K \Rightarrow \forall L, K \subseteq L \subseteq \bar{K}, E(L)$ is an abelian group.

$$-P = -(x_1, y_1) = (x_1, -a_1 x_1 - a_3 - y_1)$$

Formulas for Addition on E (Summary)

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(K) \setminus \{\infty\}$,

Addition Laws for the sum of affine points

- If $P_1 \neq P_2$
  - $x_1 = x_2$ $\Rightarrow$ $P_1 +_E P_2 = \infty$
  - $x_1 \neq x_2$

    $\lambda = \frac{y_2 - y_1}{x_2 - x_1} \qquad \nu = \frac{y_1x_2 - y_2x_1}{x_2 - x_1}$
- If $P_1 = P_2$
  - $2y_1 + a_1x + a_3 = 0$ $\Rightarrow$ $P_1 +_E P_2 = 2P_1 = \infty$
  - $2y_1 + a_1x + a_3 \neq 0$

    $\lambda = \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x + a_3}, \nu = -\frac{a_3y_1 + x_1^3 - a_4x_1 - 2a_6}{2y_1 + a_1x_1 + a_3}$

Formulas for Addition on E (Summary)

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(K) \setminus \{\infty\}$,

Addition Laws for the sum of affine points

- If $P_1 \neq P_2$
  - $x_1 = x_2$ $\Rightarrow$ $P_1 +_E P_2 = \infty$
  - $x_1 \neq x_2$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} \qquad \nu = \frac{y_1x_2 - y_2x_1}{x_2 - x_1}$$

- If $P_1 = P_2$
  - $2y_1 + a_1x + a_3 = 0$ $\Rightarrow$ $P_1 +_E P_2 = 2P_1 = \infty$
  - $2y_1 + a_1x + a_3 \neq 0$

$$\lambda = \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x + a_3}, \nu = -\frac{a_3y_1 + x_1^3 - a_4x_1 - 2a_6}{2y_1 + a_1x_1 + a_3}$$

Then

$$P_1 +_E P_2 = (\lambda^2 - a_1\lambda - a_2 - x_1 - x_2, -\lambda^3 - a_1^2\lambda + (\lambda + a_1)(a_2 + x_1 + x_2) - a_3 - \nu)$$

Formulas for Addition on E (Summary for special equation)

$$E : y^2 = x^3 + Ax + B$$

$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in E(K) \setminus \{\infty\}$,

Addition Laws for the sum of affine points

- If $P_1 \neq P_2$
  - $x_1 = x_2$ $\Rightarrow$ $P_1 +_E P_2 = \infty$
  - $x_1 \neq x_2$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} \qquad \nu = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$

- If $P_1 = P_2$
  - $y_1 = 0$ $\Rightarrow$ $P_1 +_E P_2 = 2P_1 = \infty$
  - $y_1 \neq 0$

$$\lambda = \frac{3x_1^2 + A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$$

Then

$$P_1 +_E P_2 = (\lambda^2 - x_1 - x_2, -\lambda^3 + \lambda(x_1 + x_2) - \nu)$$

Notations

Finite fields

1. $\mathbb{F}_p = \{0, 1, \dots, p-1\}$ is the prime field;

Algebraic Closure of $\mathbb{F}_q$

If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

Notations

Finite fields

1. $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ is the prime field;
2. $\mathbb{F}_q$ is a finite field with $q = p^n$ elements;

Algebraic Closure of $\mathbb{F}_q$

If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

F. Pappalardi

CIMPA

Notations

Finite fields

1. $\mathbb{F}_p = \{0, 1, \dots, p-1\}$ is the prime field;
2. $\mathbb{F}_q$ is a finite field with $q = p^n$ elements;
3. $\mathbb{F}_q = \mathbb{F}_p[\xi]$, $f(\xi) = 0$, $f \in \mathbb{F}_p[X]$ irreducible, $\partial f = n$;

Algebraic Closure of $\mathbb{F}_q$

If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

Notations

Finite fields

1. $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ is the prime field;
2. $\mathbb{F}_q$ is a finite field with $q = p^n$ elements;
3. $\mathbb{F}_q = \mathbb{F}_p[\xi]$, $f(\xi) = 0$, $f \in \mathbb{F}_p[X]$ irreducible, $\partial f = n$;
4. $\mathbb{F}_4 = \mathbb{F}_2[\xi]$, $\xi^2 = 1 + \xi$;

Algebraic Closure of $\mathbb{F}_q$

If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

Notations

Finite fields

1. $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ is the prime field;
2. $\mathbb{F}_q$ is a finite field with $q = p^n$ elements;
3. $\mathbb{F}_q = \mathbb{F}_p[\xi]$, $f(\xi) = 0$, $f \in \mathbb{F}_p[X]$ irreducible, $\partial f = n$;
4. $\mathbb{F}_4 = \mathbb{F}_2[\xi]$, $\xi^2 = 1 + \xi$;
5. $\mathbb{F}_8 = \mathbb{F}_2[\alpha]$, $\alpha^3 = \alpha + 1$ but also $\mathbb{F}_8 = \mathbb{F}_2[\beta]$, $\beta^3 = \beta^2 + 1$, $(\beta = \alpha^2 + 1)$;

Algebraic Closure of $\mathbb{F}_q$

If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

Notations

Finite fields

1. $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ is the prime field;
2. $\mathbb{F}_q$ is a finite field with $q = p^n$ elements;
3. $\mathbb{F}_q = \mathbb{F}_p[\xi]$, $f(\xi) = 0$, $f \in \mathbb{F}_p[X]$ irreducible, $\partial f = n$;
4. $\mathbb{F}_4 = \mathbb{F}_2[\xi]$, $\xi^2 = 1 + \xi$;
5. $\mathbb{F}_8 = \mathbb{F}_2[\alpha]$, $\alpha^3 = \alpha + 1$ but also $\mathbb{F}_8 = \mathbb{F}_2[\beta]$, $\beta^3 = \beta^2 + 1$, $(\beta = \alpha^2 + 1)$;
6. $\mathbb{F}_{101^{101}} = \mathbb{F}_{101}[\omega], \omega^{101} = \omega + 1$

Algebraic Closure of $\mathbb{F}_q$

If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

Notations

Finite fields

1. $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ is the prime field;
2. $\mathbb{F}_q$ is a finite field with $q = p^n$ elements;
3. $\mathbb{F}_q = \mathbb{F}_p[\xi]$, $f(\xi) = 0$, $f \in \mathbb{F}_p[X]$ irreducible, $\partial f = n$;
4. $\mathbb{F}_4 = \mathbb{F}_2[\xi]$, $\xi^2 = 1 + \xi$;
5. $\mathbb{F}_8 = \mathbb{F}_2[\alpha]$, $\alpha^3 = \alpha + 1$ but also $\mathbb{F}_8 = \mathbb{F}_2[\beta]$, $\beta^3 = \beta^2 + 1$, $(\beta = \alpha^2 + 1)$;
6. $\mathbb{F}_{101^{101}} = \mathbb{F}_{101}[\omega], \omega^{101} = \omega + 1$

Algebraic Closure of $\mathbb{F}_q$

1. $\forall n \in \mathbb{N}$, we fix an $\mathbb{F}_{q^n}$

If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

Notations

Finite fields

1. $\mathbb{F}_p = \{0, 1, \dots, p-1\}$ is the prime field;
2. $\mathbb{F}_q$ is a finite field with $q = p^n$ elements;
3. $\mathbb{F}_q = \mathbb{F}_p[\xi]$, $f(\xi) = 0$, $f \in \mathbb{F}_p[X]$ irreducible, $\partial f = n$;
4. $\mathbb{F}_4 = \mathbb{F}_2[\xi]$, $\xi^2 = 1 + \xi$;
5. $\mathbb{F}_8 = \mathbb{F}_2[\alpha]$, $\alpha^3 = \alpha + 1$ but also $\mathbb{F}_8 = \mathbb{F}_2[\beta]$, $\beta^3 = \beta^2 + 1$, $(\beta = \alpha^2 + 1)$;
6. $\mathbb{F}_{101^{101}} = \mathbb{F}_{101}[\omega], \omega^{101} = \omega + 1$

Algebraic Closure of $\mathbb{F}_q$

1. $\forall n \in \mathbb{N}$, we fix an $\mathbb{F}_{q^n}$
2. We also require that $\mathbb{F}_{q^n} \subseteq \mathbb{F}_{q^m}$ if $n \mid m$

If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

Notations

Finite fields

1. $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ is the prime field;
2. $\mathbb{F}_q$ is a finite field with $q = p^n$ elements;
3. $\mathbb{F}_q = \mathbb{F}_p[\xi]$, $f(\xi) = 0$, $f \in \mathbb{F}_p[X]$ irreducible, $\partial f = n$;
4. $\mathbb{F}_4 = \mathbb{F}_2[\xi]$, $\xi^2 = 1 + \xi$;
5. $\mathbb{F}_8 = \mathbb{F}_2[\alpha]$, $\alpha^3 = \alpha + 1$ but also $\mathbb{F}_8 = \mathbb{F}_2[\beta]$, $\beta^3 = \beta^2 + 1$, $(\beta = \alpha^2 + 1)$;
6. $\mathbb{F}_{101^{101}} = \mathbb{F}_{101}[\omega], \omega^{101} = \omega + 1$

Algebraic Closure of $\mathbb{F}_q$

1. $\forall n \in \mathbb{N}$, we fix an $\mathbb{F}_{q^n}$
2. We also require that $\mathbb{F}_{q^n} \subseteq \mathbb{F}_{q^m}$ if $n \mid m$
3. We let $\overline{\mathbb{F}}_q = \bigcup_{n \in \mathbb{N}} \mathbb{F}_{q^n}$

If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

Notations

Finite fields

1. $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$ is the prime field;
2. $\mathbb{F}_q$ is a finite field with $q = p^n$ elements;
3. $\mathbb{F}_q = \mathbb{F}_p[\xi]$, $f(\xi) = 0$, $f \in \mathbb{F}_p[X]$ irreducible, $\partial f = n$;
4. $\mathbb{F}_4 = \mathbb{F}_2[\xi]$, $\xi^2 = 1 + \xi$;
5. $\mathbb{F}_8 = \mathbb{F}_2[\alpha]$, $\alpha^3 = \alpha + 1$ but also $\mathbb{F}_8 = \mathbb{F}_2[\beta]$, $\beta^3 = \beta^2 + 1$, $(\beta = \alpha^2 + 1)$;
6. $\mathbb{F}_{101^{101}} = \mathbb{F}_{101}[\omega], \omega^{101} = \omega + 1$

Algebraic Closure of $\mathbb{F}_q$

1. $\forall n \in \mathbb{N}$, we fix an $\mathbb{F}_{q^n}$
2. We also require that $\mathbb{F}_{q^n} \subseteq \mathbb{F}_{q^m}$ if $n \mid m$
3. We let $\overline{\mathbb{F}}_q = \bigcup_{n\in\mathbb{N}} \mathbb{F}_{q^n}$
4. $\overline{\mathbb{F}}_q$ is *algebraically closed*

If $F(x, y) \in \mathbb{F}_q[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \overline{\mathbb{F}}_q^2$ s.t. $F(x_0, y_0) = 0$.

The j-invariant

Let $E/K : y^2 = x^3 + Ax + B$, $p \geq 5$ and $\Delta_E := 4A^3 + 27B^2$.

The j-invariant

Let $E/K : y^2 = x^3 + Ax + B$, $p \geq 5$ and $\Delta_E := 4A^3 + 27B^2$.

$$\begin{cases} x \longleftarrow u^{-2}x \\ y \longleftarrow u^{-3}y \end{cases} \quad u \in K^* \;\Rightarrow\; E \longrightarrow E_u : y^2 = x^3 + u^4 Ax + u^6 B$$

The j-invariant

Let $E/K : y^2 = x^3 + Ax + B$, $p \geq 5$ and $\Delta_E := 4A^3 + 27B^2$.

$$\begin{cases} x \longleftarrow u^{-2}x \\ y \longleftarrow u^{-3}y \end{cases} \quad u \in K^* \Rightarrow E \longrightarrow E_u : y^2 = x^3 + u^4 Ax + u^6 B$$

Definition

The j–invariant of E is $j = j(E) = 1728 \frac{4A^3}{4A^3 + 27B^2}$

Properties of j–invariants

1. $j(E) = j(E_u), \forall u \in K^*$

The j-invariant

Let $E/K : y^2 = x^3 + Ax + B$, $p \geq 5$ and $\Delta_E := 4A^3 + 27B^2$.

$$\begin{cases} x \longleftarrow u^{-2}x \\ y \longleftarrow u^{-3}y \end{cases} \quad u \in K^* \;\Rightarrow\; E \longrightarrow E_u : y^2 = x^3 + u^4Ax + u^6B$$

Definition

The j–invariant of E is $j = j(E) = 1728\frac{4A^3}{4A^3+27B^2}$

Properties of j–invariants

1. $j(E) = j(E_u), \forall u \in K^*$
2. $j(E'/K) = j(E''/K) \;\Rightarrow\; \exists u \in \bar{K}^*$ s.t. $E'' = E'_u$

The j-invariant

Let $E/K : y^2 = x^3 + Ax + B$, $p \geq 5$ and $\Delta_E := 4A^3 + 27B^2$.

$$\begin{cases} x \longleftarrow u^{-2}x \\ y \longleftarrow u^{-3}y \end{cases} \quad u \in K^* \;\Rightarrow\; E \longrightarrow E_u : y^2 = x^3 + u^4Ax + u^6B$$

Definition

The j–invariant of E is $j = j(E) = 1728\frac{4A^3}{4A^3+27B^2}$

Properties of j–invariants

1. $j(E) = j(E_u), \forall u \in K^*$
2. $j(E'/K) = j(E''/K) \;\Rightarrow\; \exists u \in \bar{K}^*$ s.t. $E'' = E'_u$

if $K = \mathbb{F}_q$ can take $u \in \mathbb{F}_{q^{12}}$

The j-invariant

Let $E/K : y^2 = x^3 + Ax + B$, $p \geq 5$ and $\Delta_E := 4A^3 + 27B^2$.

$$\begin{cases} x \longleftarrow u^{-2}x \\ y \longleftarrow u^{-3}y \end{cases} \quad u \in K^* \Rightarrow E \longrightarrow E_u : y^2 = x^3 + u^4Ax + u^6B$$

Definition

The j–invariant of E is $j = j(E) = 1728\frac{4A^3}{4A^3+27B^2}$

Properties of j–invariants

1. $j(E) = j(E_u), \forall u \in K^*$
2. $j(E'/K) = j(E''/K) \Rightarrow \exists u \in \bar{K}^*$ s.t. $E'' = E'_u$

   if $K = \mathbb{F}_q$ can take $u \in \mathbb{F}_{q^{12}}$
3. $j \neq 0, 1728 \Rightarrow E : y^2 = x^3 + \frac{3j}{1728-j}x + \frac{2j}{1728-j}, j(E) = j$

The j-invariant

Let $E/K : y^2 = x^3 + Ax + B$, $p \geq 5$ and $\Delta_E := 4A^3 + 27B^2$.

$$\begin{cases} x \longleftarrow u^{-2}x \\ y \longleftarrow u^{-3}y \end{cases} \quad u \in K^* \;\Rightarrow\; E \longrightarrow E_u : y^2 = x^3 + u^4Ax + u^6B$$

Definition

The j–invariant of E is $j = j(E) = 1728\frac{4A^3}{4A^3+27B^2}$

Properties of j–invariants

1. $j(E) = j(E_u), \forall u \in K^*$
2. $j(E'/K) = j(E''/K) \;\Rightarrow\; \exists u \in \bar{K}^*$ s.t. $E'' = E'_u$

   if $K = \mathbb{F}_q$ can take $u \in \mathbb{F}_{q^{12}}$
3. $j \neq 0, 1728 \Rightarrow E : y^2 = x^3 + \frac{3j}{1728-j}x + \frac{2j}{1728-j}, j(E) = j$
4. $j = 0 \;\Rightarrow\; E : y^2 = x^3 + B, \quad j = 1728 \;\Rightarrow\; E : y^2 = x^3 + Ax$

The j-invariant

Let $E/K : y^2 = x^3 + Ax + B$, $p \geq 5$ and $\Delta_E := 4A^3 + 27B^2$.

$$\begin{cases} x \longleftarrow u^{-2}x \\ y \longleftarrow u^{-3}y \end{cases} \quad u \in K^* \;\Rightarrow\; E \longrightarrow E_u : y^2 = x^3 + u^4Ax + u^6B$$

Definition

The j–invariant of E is $j = j(E) = 1728\frac{4A^3}{4A^3+27B^2}$

Properties of j–invariants

1. $j(E) = j(E_u), \forall u \in K^*$
2. $j(E'/K) = j(E''/K) \;\Rightarrow\; \exists u \in \bar{K}^*$ s.t. $E'' = E'_u$

   if $K = \mathbb{F}_q$ can take $u \in \mathbb{F}_{q^{12}}$
3. $j \neq 0, 1728 \Rightarrow E : y^2 = x^3 + \frac{3j}{1728-j}x + \frac{2j}{1728-j}, j(E) = j$
4. $j = 0 \;\Rightarrow\; E : y^2 = x^3 + B, \quad j = 1728 \;\Rightarrow\; E : y^2 = x^3 + Ax$
5. $j : K \longleftrightarrow \{\bar{K}$–affinely equivalent classes of $E/K\}$.

The j-invariant

Let $E/K : y^2 = x^3 + Ax + B$, $p \geq 5$ and $\Delta_E := 4A^3 + 27B^2$.

$$\begin{cases} x \longleftarrow u^{-2}x \\ y \longleftarrow u^{-3}y \end{cases} \quad u \in K^* \Rightarrow E \longrightarrow E_u : y^2 = x^3 + u^4Ax + u^6B$$

Definition

The j–invariant of E is $j = j(E) = 1728\frac{4A^3}{4A^3+27B^2}$

Properties of j–invariants

1. $j(E) = j(E_u), \forall u \in K^*$
2. $j(E'/K) = j(E''/K) \Rightarrow \exists u \in \bar{K}^*$ s.t. $E'' = E'_u$

   if $K = \mathbb{F}_q$ can take $u \in \mathbb{F}_{q^{12}}$
3. $j \neq 0, 1728 \Rightarrow E : y^2 = x^3 + \frac{3j}{1728-j}x + \frac{2j}{1728-j}, j(E) = j$
4. $j = 0 \Rightarrow E : y^2 = x^3 + B, \quad j = 1728 \Rightarrow E : y^2 = x^3 + Ax$
5. $j : K \longleftrightarrow \{\bar{K}$–affinely equivalent classes of $E/K\}$.
6. $p = 2, 3$ different definition

Examples of j invariants

From monday $E_1 : y^2 = x^3 + 1$ and $E_2 : y^2 = x^3 + 2$

Examples of j invariants

From monday $E_1 : y^2 = x^3 + 1$ and $E_2 : y^2 = x^3 + 2$

$$\#E_1(\mathbb{F}_5) = \#E_2(\mathbb{F}_5) = 6 \qquad \text{and} \qquad j(E_1) = j(E_2) = 0$$

$$\begin{cases} x \longleftarrow 2x \\ y \longleftarrow \sqrt{3}y \end{cases}$$

E_1 and E_2 affinely equivalent over $\mathbb{F}_5[\sqrt{3}] = \mathbb{F}_{25}$ (*twists*)

Examples of j invariants

From monday $E_1 : y^2 = x^3 + 1$ and $E_2 : y^2 = x^3 + 2$

$$\#E_1(\mathbb{F}_5) = \#E_2(\mathbb{F}_5) = 6 \qquad \text{and} \qquad j(E_1) = j(E_2) = 0$$

$$\begin{cases} x \longleftarrow 2x \\ y \longleftarrow \sqrt{3}y \end{cases}$$

E_1 and E_2 affinely equivalent over $\mathbb{F}_5[\sqrt{3}] = \mathbb{F}_{25}$ (*twists*)

Definition (twisted curve)

Let $E/\mathbb{F}_q : y^2 = x^3 + Ax + B, \mu \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$.

$$E_\mu : y^2 = x^3 + \mu^2 Ax + \mu^3 B$$

is called twisted curve.

Examples of j invariants

From monday $E_1 : y^2 = x^3 + 1$ and $E_2 : y^2 = x^3 + 2$

$$\#E_1(\mathbb{F}_5) = \#E_2(\mathbb{F}_5) = 6 \qquad \text{and} \qquad j(E_1) = j(E_2) = 0$$

$$\begin{cases} x \longleftarrow 2x \\ y \longleftarrow \sqrt{3}y \end{cases}$$

E_1 and E_2 affinely equivalent over $\mathbb{F}_5[\sqrt{3}] = \mathbb{F}_{25}$ (*twists*)

Definition (twisted curve)

Let $E/\mathbb{F}_q : y^2 = x^3 + Ax + B, \mu \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$.

$$E_\mu : y^2 = x^3 + \mu^2 Ax + \mu^3 B$$

is called twisted curve.

Exercise: prove that

- $j(E) = j(E_\mu)$

Examples of j invariants

From monday $E_1 : y^2 = x^3 + 1$ and $E_2 : y^2 = x^3 + 2$

$$\#E_1(\mathbb{F}_5) = \#E_2(\mathbb{F}_5) = 6 \quad \text{and} \quad j(E_1) = j(E_2) = 0$$

$$\begin{cases} x \longleftarrow 2x \\ y \longleftarrow \sqrt{3}y \end{cases}$$

E_1 and E_2 affinely equivalent over $\mathbb{F}_5[\sqrt{3}] = \mathbb{F}_{25}$ (*twists*)

Definition (twisted curve)

Let $E/\mathbb{F}_q : y^2 = x^3 + Ax + B, \mu \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$.

$$E_\mu : y^2 = x^3 + \mu^2 Ax + \mu^3 B$$

is called twisted curve.

Exercise: prove that

- $j(E) = j(E_\mu)$
- E and E_μ are $\mathbb{F}_q[\sqrt{\mu}]$–affinely equivalent

Examples of j invariants

From monday $E_1 : y^2 = x^3 + 1$ and $E_2 : y^2 = x^3 + 2$

$\#E_1(\mathbb{F}_5) = \#E_2(\mathbb{F}_5) = 6$ and $j(E_1) = j(E_2) = 0$

$$\begin{cases} x \longleftarrow 2x \\ y \longleftarrow \sqrt{3}y \end{cases}$$

E_1 and E_2 affinely equivalent over $\mathbb{F}_5[\sqrt{3}] = \mathbb{F}_{25}$ (*twists*)

Definition (twisted curve)

Let $E/\mathbb{F}_q : y^2 = x^3 + Ax + B, \mu \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$.

$$E_\mu : y^2 = x^3 + \mu^2 Ax + \mu^3 B$$

is called twisted curve.

Exercise: prove that

- $j(E) = j(E_\mu)$
- E and E_μ are $\mathbb{F}_q[\sqrt{\mu}]$–affinely equivalent
- $\#E(\mathbb{F}_{q^2}) = \#E_\mu(\mathbb{F}_{q^2})$

Examples of j invariants

From monday $E_1 : y^2 = x^3 + 1$ and $E_2 : y^2 = x^3 + 2$

$\#E_1(\mathbb{F}_5) = \#E_2(\mathbb{F}_5) = 6$ and $j(E_1) = j(E_2) = 0$

$$\begin{cases} x \longleftarrow 2x \\ y \longleftarrow \sqrt{3}y \end{cases}$$

E_1 and E_2 affinely equivalent over $\mathbb{F}_5[\sqrt{3}] = \mathbb{F}_{25}$ (*twists*)

Definition (twisted curve)

Let $E/\mathbb{F}_q : y^2 = x^3 + Ax + B, \mu \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$.

$$E_\mu : y^2 = x^3 + \mu^2 Ax + \mu^3 B$$

is called twisted curve.

Exercise: prove that

- $j(E) = j(E_\mu)$
- E and E_μ are $\mathbb{F}_q[\sqrt{\mu}]$–affinely equivalent
- $\#E(\mathbb{F}_{q^2}) = \#E_\mu(\mathbb{F}_{q^2})$
- usually $\#E(\mathbb{F}_q) \neq \#E_\mu(\mathbb{F}_q)$

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

$$P \text{ has order } 2 \iff 2P = \infty \iff P = -P$$

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

$$P \text{ has order } 2 \iff 2P = \infty \iff P = -P$$

So

$$-P = (x_1, -a_1x_1 - a_3 - y_1) = (x_1, y_1) = P$$

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

$$P \text{ has order } 2 \iff 2P = \infty \iff P = -P$$

So

$$-P = (x_1, -a_1x_1 - a_3 - y_1) = (x_1, y_1) = P \implies 2y_1 = -a_1x_1 - a_3$$

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

$$P \text{ has order } 2 \iff 2P = \infty \iff P = -P$$

So

$$-P = (x_1, -a_1x_1 - a_3 - y_1) = (x_1, y_1) = P \implies 2y_1 = -a_1x_1 - a_3$$

If $p \neq 2$, can assume $E : y^2 = x^3 + Ax^2 + Bx + C$

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

$$P \text{ has order } 2 \iff 2P = \infty \iff P = -P$$

So

$$-P = (x_1, -a_1x_1 - a_3 - y_1) = (x_1, y_1) = P \implies 2y_1 = -a_1x_1 - a_3$$

If $p \neq 2$, can assume $E : y^2 = x^3 + Ax^2 + Bx + C$

$$-P = (x_1, -y_1) = (x_1, y_1) = P$$

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

$$P \text{ has order } 2 \iff 2P = \infty \iff P = -P$$

So

$$-P = (x_1, -a_1 x_1 - a_3 - y_1) = (x_1, y_1) = P \implies 2y_1 = -a_1 x_1 - a_3$$

If $p \neq 2$, can assume $E : y^2 = x^3 + Ax^2 + Bx + C$

$$-P = (x_1, -y_1) = (x_1, y_1) = P \implies y_1 = 0, x_1^3 + Ax_1^2 + Bx_1 + C = 0$$

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

$$P \text{ has order } 2 \iff 2P = \infty \iff P = -P$$

So

$$-P = (x_1, -a_1x_1 - a_3 - y_1) = (x_1, y_1) = P \implies 2y_1 = -a_1x_1 - a_3$$

If $p \neq 2$, can assume $E : y^2 = x^3 + Ax^2 + Bx + C$

$$-P = (x_1, -y_1) = (x_1, y_1) = P \implies y_1 = 0, x_1^3 + Ax_1^2 + Bx_1 + C = 0$$

Note

- the number of points of order 2 in $E(\mathbb{F}_q)$ equals the number of roots of $X^3 + Ax^2 + Bx + C$ in $\mathbb{F}_q$

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

$$P \text{ has order } 2 \iff 2P = \infty \iff P = -P$$

So

$$-P = (x_1, -a_1x_1 - a_3 - y_1) = (x_1, y_1) = P \implies 2y_1 = -a_1x_1 - a_3$$

If $p \neq 2$, can assume $E : y^2 = x^3 + Ax^2 + Bx + C$

$$-P = (x_1, -y_1) = (x_1, y_1) = P \implies y_1 = 0, x_1^3 + Ax_1^2 + Bx_1 + C = 0$$

Note

- the number of points of order 2 in $E(\mathbb{F}_q)$ equals the number of roots of $X^3 + Ax^2 + Bx + C$ in $\mathbb{F}_q$
- roots are distinct since discriminant $\Delta_E \neq 0$

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

$$P \text{ has order } 2 \iff 2P = \infty \iff P = -P$$

So

$$-P = (x_1, -a_1x_1 - a_3 - y_1) = (x_1, y_1) = P \implies 2y_1 = -a_1x_1 - a_3$$

If $p \neq 2$, can assume $E : y^2 = x^3 + Ax^2 + Bx + C$

$$-P = (x_1, -y_1) = (x_1, y_1) = P \implies y_1 = 0, x_1^3 + Ax_1^2 + Bx_1 + C = 0$$

Note

- the number of points of order 2 in $E(\mathbb{F}_q)$ equals the number of roots of $X^3 + Ax^2 + Bx + C$ in $\mathbb{F}_q$
- roots are distinct since discriminant $\Delta_E \neq 0$
- $E(\mathbb{F}_{q^6})$ has always 3 points of order 2 if $E/\mathbb{F}_q$

Determining points of order 2

Let $P = (x_1, y_1) \in E(\mathbb{F}_q) \setminus \{\infty\}$,

$$P \text{ has order } 2 \iff 2P = \infty \iff P = -P$$

So

$$-P = (x_1, -a_1x_1 - a_3 - y_1) = (x_1, y_1) = P \implies 2y_1 = -a_1x_1 - a_3$$

If $p \neq 2$, can assume $E : y^2 = x^3 + Ax^2 + Bx + C$

$$-P = (x_1, -y_1) = (x_1, y_1) = P \implies y_1 = 0, x_1^3 + Ax_1^2 + Bx_1 + C = 0$$

Note

- the number of points of order 2 in $E(\mathbb{F}_q)$ equals the number of roots of $X^3 + Ax^2 + Bx + C$ in $\mathbb{F}_q$
- roots are distinct since discriminant $\Delta_E \neq 0$
- $E(\mathbb{F}_{q^6})$ has always 3 points of order 2 if $E/\mathbb{F}_q$
- $E[2] := \{P \in E(\bar{\mathbb{F}}_q) : 2P = \infty\} \cong C_2 \oplus C_2$

Determining points of order 2 (continues)

- If $p = 2$ and $E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6$

Determining points of order 2 (continues)

- If $p = 2$ and $E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6$

Determining points of order 2 (continues)

- If $p = 2$ and $E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6$

$$-P = (x_1, a_3 + y_1) = (x_1, y_1) = P$$

Determining points of order 2 (continues)

- If $p = 2$ and $E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6$

$$-P = (x_1, a_3 + y_1) = (x_1, y_1) = P \implies a_3 = 0$$

Determining points of order 2 (continues)

- If $p = 2$ and $E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6$

$$-P = (x_1, a_3 + y_1) = (x_1, y_1) = P \implies a_3 = 0$$

Absurd ($a_3 = 0$) and there are no points of order 2.

- If $p = 2$ and $E : y^2 + xy = x^3 + a_4 x + a_6$

Determining points of order 2 (continues)

- If $p = 2$ and $E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6$

$$-P = (x_1, a_3 + y_1) = (x_1, y_1) = P \implies a_3 = 0$$

Absurd ($a_3 = 0$) and there are no points of order 2.

- If $p = 2$ and $E : y^2 + xy = x^3 + a_4 x + a_6$

Determining points of order 2 (continues)

- If $p = 2$ and $E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6$

$$-P = (x_1, a_3 + y_1) = (x_1, y_1) = P \implies a_3 = 0$$

  Absurd ($a_3 = 0$) and there are no points of order 2.

- If $p = 2$ and $E : y^2 + xy = x^3 + a_4 x + a_6$

$$-P = (x_1, x_1 + y_1) = (x_1, y_1) = P$$

Determining points of order 2 (continues)

- If $p = 2$ and $E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6$

$$-P = (x_1, a_3 + y_1) = (x_1, y_1) = P \implies a_3 = 0$$

Absurd ($a_3 = 0$) and there are no points of order 2.

- If $p = 2$ and $E : y^2 + xy = x^3 + a_4 x + a_6$

$$-P = (x_1, x_1 + y_1) = (x_1, y_1) = P \implies x_1 = 0, y_1^2 = a_6$$

Determining points of order 2 (continues)

- If $p = 2$ and $E : y^2 + a_3y = x^3 + a_2x^2 + a_6$

$$-P = (x_1, a_3 + y_1) = (x_1, y_1) = P \implies a_3 = 0$$

Absurd ($a_3 = 0$) and there are no points of order 2.

- If $p = 2$ and $E : y^2 + xy = x^3 + a_4x + a_6$

$$-P = (x_1, x_1 + y_1) = (x_1, y_1) = P \implies x_1 = 0, y_1^2 = a_6$$

So there is exactly one point of order 2 namely $(0, \sqrt{a_6})$

Determining points of order 2 (continues)

- If $p = 2$ and $E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6$

$$-P = (x_1, a_3 + y_1) = (x_1, y_1) = P \implies a_3 = 0$$

Absurd ($a_3 = 0$) and there are no points of order 2.

- If $p = 2$ and $E : y^2 + xy = x^3 + a_4 x + a_6$

$$-P = (x_1, x_1 + y_1) = (x_1, y_1) = P \implies x_1 = 0, y_1^2 = a_6$$

So there is exactly one point of order 2 namely $(0, \sqrt{a_6})$

Definition

2–torsion points

$$E[2] = \{P \in E : 2P = \infty\}.$$

In conclusion

$$E[2] \cong \begin{cases} C_2 \oplus C_2 & \text{if } p > 2 \\ C_2 & \text{if } p = 2, E : y^2 + xy = x^3 + a_4 x + a_6 \\ \{\infty\} & \text{if } p = 2, E : y^2 + a_3 y = x^3 + a_2 x^2 + a_6 \end{cases}$$

Elliptic curves over $\mathbb{F}_2, \mathbb{F}_3$ and $\mathbb{F}_5$

Each curve $/\mathbb{F}_2$ has cyclic $E(\mathbb{F}_2)$.

E	$E(\mathbb{F}_2)$	$\lvert E(\mathbb{F}_2)\rvert$
$y^2 + xy = x^3 + x^2 + 1$	$\{\infty, (0,1)\}$	2
$y^2 + xy = x^3 + 1$	$\{\infty, (0,1), (1,0), (1,1)\}$	4
$y^2 + y = x^3 + x$	$\{\infty, (0,0), (0,1), (1,0), (1,1)\}$	5
$y^2 + y = x^3 + x + 1$	$\{\infty\}$	1
$y^2 + y = x^3$	$\{\infty, (0,0), (0,1)\}$	3

Elliptic curves over $\mathbb{F}_2, \mathbb{F}_3$ and $\mathbb{F}_5$

Each curve $/\mathbb{F}_2$ has cyclic $E(\mathbb{F}_2)$.

E	$E(\mathbb{F}_2)$	$\|E(\mathbb{F}_2)\|$
$y^2+xy=x^3+x^2+1$	$\{\infty,(0,1)\}$	2
$y^2+xy=x^3+1$	$\{\infty,(0,1),(1,0),(1,1)\}$	4
$y^2+y=x^3+x$	$\{\infty,(0,0),(0,1),(1,0),(1,1)\}$	5
$y^2+y=x^3+x+1$	$\{\infty\}$	1
$y^2+y=x^3$	$\{\infty,(0,0),(0,1)\}$	3

- $E_1: y^2=x^3+x \qquad E_2: y^2=x^3-x$

  $E_1(\mathbb{F}_3)\cong C_4$ and $E_2(\mathbb{F}_3)\cong C_2\oplus C_2$

- $E_3: y^2=x^3+x \qquad E_4: y^2=x^3+x+2$

  $E_3(\mathbb{F}_5)\cong C_2\oplus C_2$ and $E_4(\mathbb{F}_5)\cong C_4$

- $E_5: y^2=x^3+4x \qquad E_6: y^2=x^3+4x+1$

  $E_5(\mathbb{F}_5)\cong C_2\oplus C_4$ and $E_6(\mathbb{F}_5)\cong C_8$

Determining points of order 3

Let $P = (x_1, y_1) \in E(\mathbb{F}_q)$

$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

Determining points of order 3

Let $P = (x_1, y_1) \in E(\mathbb{F}_q)$

$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

So, if $p > 3$ and $E : y^2 = x^2 + Ax + B$

$$2P = (x_{2P}, y_{2P}) = 2(x_1, y_1) = (\lambda^2 - 2x_1, -\lambda^3 + 2\lambda x_1 - \nu)$$

Determining points of order 3

Let $P = (x_1, y_1) \in E(\mathbb{F}_q)$

$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

So, if $p > 3$ and $E : y^2 = x^2 + Ax + B$

$$2P = (x_{2P}, y_{2P}) = 2(x_1, y_1) = (\lambda^2 - 2x_1, -\lambda^3 + 2\lambda x_1 - \nu)$$

where $\lambda = \frac{3x_1^2+A}{2y_1}, \nu = -\frac{x_1^3-Ax_1-2B}{2y_1}$.

Determining points of order 3

Let $P = (x_1, y_1) \in E(\mathbb{F}_q)$

$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

So, if $p > 3$ and $E : y^2 = x^2 + Ax + B$

$$2P = (x_{2P}, y_{2P}) = 2(x_1, y_1) = (\lambda^2 - 2x_1, -\lambda^3 + 2\lambda x_1 - \nu)$$

where $\lambda = \frac{3x_1^2+A}{2y_1}, \nu = -\frac{x_1^3-Ax_1-2B}{2y_1}$.

$$P \text{ has order } 3 \iff x_{2P} = x_1$$

Determining points of order 3

Let $P = (x_1, y_1) \in E(\mathbb{F}_q)$

$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

So, if $p > 3$ and $E : y^2 = x^2 + Ax + B$

$$2P = (x_{2P}, y_{2P}) = 2(x_1, y_1) = (\lambda^2 - 2x_1, -\lambda^3 + 2\lambda x_1 - \nu)$$

where $\lambda = \frac{3x_1^2+A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$.

$$P \text{ has order } 3 \iff x_{2P} = x_1$$

Substituting λ,

Determining points of order 3

Let $P = (x_1, y_1) \in E(\mathbb{F}_q)$

$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

So, if $p > 3$ and $E : y^2 = x^2 + Ax + B$

$$2P = (x_{2P}, y_{2P}) = 2(x_1, y_1) = (\lambda^2 - 2x_1, -\lambda^3 + 2\lambda x_1 - \nu)$$

where $\lambda = \frac{3x_1^2 + A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$.

$$P \text{ has order } 3 \iff x_{2P} = x_1$$

$$\text{Substituting } \lambda, \quad x_{2P} - x_1 = \frac{-3x_1^4 - 6Ax_1^2 - 12Bx_1 + A^2}{4(x_1^3 + Ax_1 + 4B)} = 0$$

Determining points of order 3

Let $P = (x_1, y_1) \in E(\mathbb{F}_q)$

$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

So, if $p > 3$ and $E : y^2 = x^2 + Ax + B$

$$2P = (x_{2P}, y_{2P}) = 2(x_1, y_1) = (\lambda^2 - 2x_1, -\lambda^3 + 2\lambda x_1 - \nu)$$

where $\lambda = \frac{3x_1^2+A}{2y_1}, \nu = -\frac{x_1^3-Ax_1-2B}{2y_1}$.

$$P \text{ has order } 3 \iff x_{2P} = x_1$$

$$\text{Substituting } \lambda, \quad x_{2P} - x_1 = \frac{-3x_1^4-6Ax_1^2-12Bx_1+A^2}{4(x_1^3+Ax_1+4B)} = 0$$

Note

- $\psi_3(x) := 3x^4 + 6Ax^2 + 12Bx - A^2$ the 3rd *division* polynomial

Determining points of order 3

Let $P = (x_1, y_1) \in E(\mathbb{F}_q)$

$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

So, if $p > 3$ and $E : y^2 = x^2 + Ax + B$

$$2P = (x_{2P}, y_{2P}) = 2(x_1, y_1) = (\lambda^2 - 2x_1, -\lambda^3 + 2\lambda x_1 - \nu)$$

where $\lambda = \frac{3x_1^2 + A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$.

$$P \text{ has order } 3 \iff x_{2P} = x_1$$

$$\text{Substituting } \lambda, \quad x_{2P} - x_1 = \frac{-3x_1^4 - 6Ax_1^2 - 12Bx_1 + A^2}{4(x_1^3 + Ax_1 + 4B)} = 0$$

Note

- $\psi_3(x) := 3x^4 + 6Ax^2 + 12Bx - A^2$ the 3rd *division* polynomial
- $(x_1, y_1) \in E(\mathbb{F}_q)$ has order 3 $\Rightarrow \psi_3(x_1) = 0$

Determining points of order 3

Let $P = (x_1, y_1) \in E(\mathbb{F}_q)$

$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

So, if $p > 3$ and $E : y^2 = x^2 + Ax + B$

$$2P = (x_{2P}, y_{2P}) = 2(x_1, y_1) = (\lambda^2 - 2x_1, -\lambda^3 + 2\lambda x_1 - \nu)$$

where $\lambda = \frac{3x_1^2 + A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$.

$$P \text{ has order } 3 \iff x_{2P} = x_1$$

$$\text{Substituting } \lambda, \quad x_{2P} - x_1 = \frac{-3x_1^4 - 6Ax_1^2 - 12Bx_1 + A^2}{4(x_1^3 + Ax_1 + 4B)} = 0$$

Note

- $\psi_3(x) := 3x^4 + 6Ax^2 + 12Bx - A^2$ the 3rd *division* polynomial
- $(x_1, y_1) \in E(\mathbb{F}_q)$ has order 3 $\quad \Rightarrow \psi_3(x_1) = 0$
- $E(\mathbb{F}_q)$ has at most 8 points of order 3

Determining points of order 3

Let $P = (x_1, y_1) \in E(\mathbb{F}_q)$

$$P \text{ has order } 3 \iff 3P = \infty \iff 2P = -P$$

So, if $p > 3$ and $E : y^2 = x^2 + Ax + B$

$$2P = (x_{2P}, y_{2P}) = 2(x_1, y_1) = (\lambda^2 - 2x_1, -\lambda^3 + 2\lambda x_1 - \nu)$$

where $\lambda = \frac{3x_1^2+A}{2y_1}, \nu = -\frac{x_1^3-Ax_1-2B}{2y_1}$.

$$P \text{ has order } 3 \iff x_{2P} = x_1$$

$$\text{Substituting } \lambda, \quad x_{2P} - x_1 = \frac{-3x_1^4-6Ax_1^2-12Bx_1+A^2}{4(x_1^3+Ax_1+4B)} = 0$$

Note

- $\psi_3(x) := 3x^4 + 6Ax^2 + 12Bx - A^2$ the 3rd *division* polynomial
- $(x_1, y_1) \in E(\mathbb{F}_q)$ has order 3 $\quad \Rightarrow \psi_3(x_1) = 0$
- $E(\mathbb{F}_q)$ has at most 8 points of order 3
- If $p \neq 3$, $E[3] := \{P \in E : 3P = \infty\} \cong C_3 \oplus C_3$

Determining points of order 3 (continues)

Exercise

Let $E: y^2 = x^3 + Ax^2 + Bx + C, A, B, C \in \mathbb{F}_{3^n}$. Prove that if $P = (x_1, y_1) \in E(\mathbb{F}_{3^n})$ has order 3, then

1. $Ax_1^3 + AC - B^2 = 0$

Determining points of order 3 (continues)

Exercise

Let $E : y^2 = x^3 + Ax^2 + Bx + C, A, B, C \in \mathbb{F}_{3^n}$. Prove that if $P = (x_1, y_1) \in E(\mathbb{F}_{3^n})$ has order 3, then

1. $Ax_1^3 + AC - B^2 = 0$
2. $E[3] \cong C_3$ if $A \neq 0$ and $E[3] = \{\infty\}$ otherwise

Determining points of order 3 (continues)

Exercise

Let $E : y^2 = x^3 + Ax^2 + Bx + C, A, B, C \in \mathbb{F}_{3^n}$. Prove that if $P = (x_1, y_1) \in E(\mathbb{F}_{3^n})$ has order 3, then

1. $Ax_1^3 + AC - B^2 = 0$
2. $E[3] \cong C_3$ if $A \neq 0$ and $E[3] = \{\infty\}$ otherwise

Determining points of order 3 (continues)

Exercise

Let $E : y^2 = x^3 + Ax^2 + Bx + C, A, B, C \in \mathbb{F}_{3^n}$. Prove that if $P = (x_1, y_1) \in E(\mathbb{F}_{3^n})$ has order 3, then

1. $Ax_1^3 + AC - B^2 = 0$
2. $E[3] \cong C_3$ if $A \neq 0$ and $E[3] = \{\infty\}$ otherwise

Example (from Monday)

If $E : y^2 = x^3 + x + 1$, then $\#E(\mathbb{F}_5) = 9$.

Determining points of order 3 (continues)

Exercise

Let $E : y^2 = x^3 + Ax^2 + Bx + C, A, B, C \in \mathbb{F}_{3^n}$. Prove that if $P = (x_1, y_1) \in E(\mathbb{F}_{3^n})$ has order 3, then

1. $Ax_1^3 + AC - B^2 = 0$
2. $E[3] \cong C_3$ if $A \neq 0$ and $E[3] = \{\infty\}$ otherwise

Example (from Monday)

If $E : y^2 = x^3 + x + 1$, then $\#E(\mathbb{F}_5) = 9$.

$$\psi_3(x) = (x+3)(x+4)(x^2+3x+4)$$

Hence

$$E[3] = \left\{\infty, (2, \pm 1), (1, \pm\sqrt{3}), (1 \pm 2\sqrt{3}, \pm(1 \pm \sqrt{3}))\right\}$$

Determining points of order 3 (continues)

Exercise

Let $E : y^2 = x^3 + Ax^2 + Bx + C, A, B, C \in \mathbb{F}_{3^n}$. Prove that if $P = (x_1, y_1) \in E(\mathbb{F}_{3^n})$ has order 3, then

1. $Ax_1^3 + AC - B^2 = 0$
2. $E[3] \cong C_3$ if $A \neq 0$ and $E[3] = \{\infty\}$ otherwise

Example (from Monday)

If $E : y^2 = x^3 + x + 1$, then $\#E(\mathbb{F}_5) = 9$.

$$\psi_3(x) = (x+3)(x+4)(x^2+3x+4)$$

Hence

$$E[3] = \left\{\infty, (2, \pm 1), (1, \pm\sqrt{3}), (1 \pm 2\sqrt{3}, \pm(1 \pm \sqrt{3}))\right\}$$

1. $E(\mathbb{F}_5) = \{\infty, (2, \pm 1), (0, \pm 1), (3, \pm 1), (4, \pm 2)\} \cong C_9$

Determining points of order 3 (continues)

Exercise

Let $E : y^2 = x^3 + Ax^2 + Bx + C$, $A, B, C \in \mathbb{F}_{3^n}$. Prove that if $P = (x_1, y_1) \in E(\mathbb{F}_{3^n})$ has order 3, then

1. $Ax_1^3 + AC - B^2 = 0$
2. $E[3] \cong C_3$ if $A \neq 0$ and $E[3] = \{\infty\}$ otherwise

Example (from Monday)

If $E : y^2 = x^3 + x + 1$, then $\#E(\mathbb{F}_5) = 9$.

$$\psi_3(x) = (x+3)(x+4)(x^2+3x+4)$$

Hence

$$E[3] = \left\{\infty, (2, \pm 1), (1, \pm\sqrt{3}), (1 \pm 2\sqrt{3}, \pm(1 \pm \sqrt{3}))\right\}$$

1. $E(\mathbb{F}_5) = \{\infty, (2, \pm 1), (0, \pm 1), (3, \pm 1), (4, \pm 2)\} \cong C_9$
2. Since $\mathbb{F}_{25} = \mathbb{F}_5[\sqrt{3}] \quad \Rightarrow \quad E[3] \subset E(\mathbb{F}_{25})$

Determining points of order 3 (continues)

Exercise

Let $E: y^2 = x^3 + Ax^2 + Bx + C, A, B, C \in \mathbb{F}_{3^n}$. Prove that if $P = (x_1, y_1) \in E(\mathbb{F}_{3^n})$ has order 3, then

1. $Ax_1^3 + AC - B^2 = 0$
2. $E[3] \cong C_3$ if $A \neq 0$ and $E[3] = \{\infty\}$ otherwise

Example (from Monday)

If $E: y^2 = x^3 + x + 1$, then $\#E(\mathbb{F}_5) = 9$.

$$\psi_3(x) = (x+3)(x+4)(x^2+3x+4)$$

Hence

$$E[3] = \left\{\infty, (2, \pm 1), (1, \pm\sqrt{3}), (1 \pm 2\sqrt{3}, \pm(1 \pm \sqrt{3}))\right\}$$

1. $E(\mathbb{F}_5) = \{\infty, (2, \pm 1), (0, \pm 1), (3, \pm 1), (4, \pm 2)\} \cong C_9$
2. Since $\mathbb{F}_{25} = \mathbb{F}_5[\sqrt{3}] \quad \Rightarrow \quad E[3] \subset E(\mathbb{F}_{25})$
3. $\#E(\mathbb{F}_{25}) = 27 \quad \Rightarrow \quad E(\mathbb{F}_{25}) \cong C_3 \oplus C_9$

Determining points of order 3 (continues)

Inequivalent curves $/\mathbb{F}_7$ with $\#E(\mathbb{F}_7) = 9$.

E	$\psi_3(x)$	$E[3] \cap E(\mathbb{F}_7)$	$E(\mathbb{F}_7) \cong$
$y^2 = x^3 + 2$	$x(x+1)(x+2)(x+4)$	$\left\{ \begin{array}{l} \infty, (0, \pm 3), (-1, \pm 1), \\ (5, \pm 1), (3, \pm 1) \end{array} \right\}$	$C_3 \oplus C_3$
$y^2 = x^3 + 3x + 2$	$(x+2)(x^3 + 5x^2 + 3x + 2)$	$\{\infty, (5, \pm 3)\}$	C_9
$y^2 = x^3 + 5x + 2$	$(x+4)(x^3 + 3x^2 + 5x + 2)$	$\{\infty, (3, \pm 3)\}$	C_9
$y^2 = x^3 + 6x + 2$	$(x+1)(x^3 + 6x^2 + 6x + 2)$	$\{\infty, (6, \pm 3)\}$	C_9

Determining points of order 3 (continues)

Inequivalent curves $/\mathbb{F}_7$ with $\#E(\mathbb{F}_7) = 9$.

E	$\psi_3(x)$	$E[3] \cap E(\mathbb{F}_7)$	$E(\mathbb{F}_7) \cong$
$y^2 = x^3 + 2$	$x(x+1)(x+2)(x+4)$	$\left\{ \begin{array}{l} \infty, (0, \pm 3), (-1, \pm 1), \\ (5, \pm 1), (3, \pm 1) \end{array} \right\}$	$C_3 \oplus C_3$
$y^2 = x^3 + 3x + 2$	$(x+2)(x^3 + 5x^2 + 3x + 2)$	$\{\infty, (5, \pm 3)\}$	C_9
$y^2 = x^3 + 5x + 2$	$(x+4)(x^3 + 3x^2 + 5x + 2)$	$\{\infty, (3, \pm 3)\}$	C_9
$y^2 = x^3 + 6x + 2$	$(x+1)(x^3 + 6x^2 + 6x + 2)$	$\{\infty, (6, \pm 3)\}$	C_9

Can one count the number of inequivalent $E/\mathbb{F}_q$ with $\#E(\mathbb{F}_q) = r$?

Example (A curve over $\mathbb{F}_4 = \mathbb{F}_2(\xi), \xi^2 = \xi + 1$; $E : y^2 + y = x^3$)

Determining points of order 3 (continues)

Inequivalent curves $/\mathbb{F}_7$ with $\#E(\mathbb{F}_7) = 9$.

E	$\psi_3(x)$	$E[3] \cap E(\mathbb{F}_7)$	$E(\mathbb{F}_7) \cong$
$y^2 = x^3 + 2$	$x(x+1)(x+2)(x+4)$	$\left\{ \begin{array}{l} \infty, (0, \pm 3), (-1, \pm 1), \\ (5, \pm 1), (3, \pm 1) \end{array} \right\}$	$C_3 \oplus C_3$
$y^2 = x^3 + 3x + 2$	$(x+2)(x^3 + 5x^2 + 3x + 2)$	$\{\infty, (5, \pm 3)\}$	C_9
$y^2 = x^3 + 5x + 2$	$(x+4)(x^3 + 3x^2 + 5x + 2)$	$\{\infty, (3, \pm 3)\}$	C_9
$y^2 = x^3 + 6x + 2$	$(x+1)(x^3 + 6x^2 + 6x + 2)$	$\{\infty, (6, \pm 3)\}$	C_9

Can one count the number of inequivalent $E/\mathbb{F}_q$ with $\#E(\mathbb{F}_q) = r$?

Example (A curve over $\mathbb{F}_4 = \mathbb{F}_2(\xi), \xi^2 = \xi + 1; \qquad E : y^2 + y = x^3$)

We know $E(\mathbb{F}_2) = \{\infty, (0, 0), (0, 1)\} \subset E(\mathbb{F}_4)$.

Determining points of order 3 (continues)

Inequivalent curves $/\mathbb{F}_7$ with $\#E(\mathbb{F}_7) = 9$.

E	$\psi_3(x)$	$E[3] \cap E(\mathbb{F}_7)$	$E(\mathbb{F}_7) \cong$
$y^2 = x^3 + 2$	$x(x+1)(x+2)(x+4)$	$\left\{ \begin{array}{l} \infty, (0, \pm 3), (-1, \pm 1), \\ (5, \pm 1), (3, \pm 1) \end{array} \right\}$	$C_3 \oplus C_3$
$y^2 = x^3 + 3x + 2$	$(x+2)(x^3 + 5x^2 + 3x + 2)$	$\{\infty, (5, \pm 3)\}$	C_9
$y^2 = x^3 + 5x + 2$	$(x+4)(x^3 + 3x^2 + 5x + 2)$	$\{\infty, (3, \pm 3)\}$	C_9
$y^2 = x^3 + 6x + 2$	$(x+1)(x^3 + 6x^2 + 6x + 2)$	$\{\infty, (6, \pm 3)\}$	C_9

Can one count the number of inequivalent $E/\mathbb{F}_q$ with $\#E(\mathbb{F}_q) = r$?

Example (A curve over $\mathbb{F}_4 = \mathbb{F}_2(\xi), \xi^2 = \xi + 1; \qquad E : y^2 + y = x^3$)

We know $E(\mathbb{F}_2) = \{\infty, (0,0), (0,1)\} \subset E(\mathbb{F}_4)$.

$E(\mathbb{F}_4) = \{\infty, (0,0), (0,1), (1,\xi), (1,\xi+1), (\xi,\xi), (\xi,\xi+1), (\xi+1,\xi), (\xi+1,\xi+1)\}$

CIMPA

Determining points of order 3 (continues)

Inequivalent curves $/\mathbb{F}_7$ with $\#E(\mathbb{F}_7) = 9$.

E	$\psi_3(x)$	$E[3] \cap E(\mathbb{F}_7)$	$E(\mathbb{F}_7) \cong$
$y^2 = x^3 + 2$	$x(x+1)(x+2)(x+4)$	$\left\{ \begin{array}{l} \infty, (0, \pm 3), (-1, \pm 1), \\ (5, \pm 1), (3, \pm 1) \end{array} \right\}$	$C_3 \oplus C_3$
$y^2 = x^3 + 3x + 2$	$(x+2)(x^3 + 5x^2 + 3x + 2)$	$\{\infty, (5, \pm 3)\}$	C_9
$y^2 = x^3 + 5x + 2$	$(x+4)(x^3 + 3x^2 + 5x + 2)$	$\{\infty, (3, \pm 3)\}$	C_9
$y^2 = x^3 + 6x + 2$	$(x+1)(x^3 + 6x^2 + 6x + 2)$	$\{\infty, (6, \pm 3)\}$	C_9

Can one count the number of inequivalent $E/\mathbb{F}_q$ with $\#E(\mathbb{F}_q) = r$?

Example (A curve over $\mathbb{F}_4 = \mathbb{F}_2(\xi), \xi^2 = \xi + 1; \quad E : y^2 + y = x^3$)

We know $E(\mathbb{F}_2) = \{\infty, (0,0), (0,1)\} \subset E(\mathbb{F}_4)$.

$E(\mathbb{F}_4) = \{\infty, (0,0), (0,1), (1,\xi), (1,\xi+1), (\xi,\xi), (\xi,\xi+1), (\xi+1,\xi), (\xi+1,\xi+1)\}$

$$\psi_3(x) = x^4 + x = x(x+1)(x+\xi)(x+\xi+1) \Rightarrow E(\mathbb{F}_4) \cong C_3 \oplus C_3$$

Exercise (Suppose $(x_0, y_0) \in E/\mathbb{F}_{2^n}$ has order 3. Show that)

1. $E : y^2 + a_3 y = x^3 + a_4 x + a_6 \;\Rightarrow\; x_0^4 + a_3^2 x_0 + (a_4 a_3)^2 = 0$

Determining points of order 3 (continues)

Inequivalent curves $/\mathbb{F}_7$ with $\#E(\mathbb{F}_7) = 9$.

E	$\psi_3(x)$	$E[3] \cap E(\mathbb{F}_7)$	$E(\mathbb{F}_7) \cong$
$y^2 = x^3 + 2$	$x(x+1)(x+2)(x+4)$	$\left\{\begin{matrix}\infty, (0, \pm 3), (-1, \pm 1), \\ (5, \pm 1), (3, \pm 1)\end{matrix}\right\}$	$C_3 \oplus C_3$
$y^2 = x^3 + 3x + 2$	$(x+2)(x^3 + 5x^2 + 3x + 2)$	$\{\infty, (5, \pm 3)\}$	C_9
$y^2 = x^3 + 5x + 2$	$(x+4)(x^3 + 3x^2 + 5x + 2)$	$\{\infty, (3, \pm 3)\}$	C_9
$y^2 = x^3 + 6x + 2$	$(x+1)(x^3 + 6x^2 + 6x + 2)$	$\{\infty, (6, \pm 3)\}$	C_9

Can one count the number of inequivalent $E/\mathbb{F}_q$ with $\#E(\mathbb{F}_q) = r$?

Example (A curve over $\mathbb{F}_4 = \mathbb{F}_2(\xi), \xi^2 = \xi + 1; \quad E: y^2 + y = x^3$)

We know $E(\mathbb{F}_2) = \{\infty, (0,0), (0,1)\} \subset E(\mathbb{F}_4)$.

$E(\mathbb{F}_4) = \{\infty, (0,0), (0,1), (1, \xi), (1, \xi+1), (\xi, \xi), (\xi, \xi+1), (\xi+1, \xi), (\xi+1, \xi+1)\}$

$$\psi_3(x) = x^4 + x = x(x+1)(x+\xi)(x+\xi+1) \Rightarrow E(\mathbb{F}_4) \cong C_3 \oplus C_3$$

Exercise (Suppose $(x_0, y_0) \in E/\mathbb{F}_{2^n}$ has order 3. Show that)

1. $E: y^2 + a_3 y = x^3 + a_4 x + a_6 \;\Rightarrow\; x_0^4 + a_3^2 x_0 + (a_4 a_3)^2 = 0$
2. $E: y^2 + xy = x^3 + a_2 x^2 + a_6 \;\Rightarrow\; x_0^4 + x_0^3 + a_6 = 0$

Determining points of order (dividing) m

Determining points of order (dividing) m

Definition (m–torsion point)

Let E/K and let $\bar{K}$ an *algebraic closure of* K.

$$E[m] = \{P \in E(\bar{K}) : \ mP = \infty\}$$

Determining points of order (dividing) m

Definition (m–torsion point)

Let E/K and let $\bar{K}$ an *algebraic closure of* K.

$$E[m] = \{P \in E(\bar{K}) : \; mP = \infty\}$$

Theorem (Structure of Torsion Points)

Let E/K *and* $m \in \mathbb{N}$. *If* $p = \mathrm{char}(K) \nmid m$,

Determining points of order (dividing) m

Definition (m–torsion point)

Let E/K and let $\bar{K}$ an *algebraic closure of* K.

$$E[m] = \{P \in E(\bar{K}) : \ mP = \infty\}$$

Theorem (Structure of Torsion Points)

Let E/K and $m \in \mathbb{N}$. If $p = \mathrm{char}(K) \nmid m$,

$$E[m] \cong C_m \oplus C_m$$

If $m = p^r m', p \nmid m'$,

$$E[m] \cong C_m \oplus C_{m'} \qquad \text{or} \qquad E[m] \cong C_{m'} \oplus C_{m'}$$

Determining points of order (dividing) m

Definition (m–torsion point)

Let E/K and let $\bar{K}$ an *algebraic closure of* K.

$$E[m] = \{P \in E(\bar{K}) : \ mP = \infty\}$$

Theorem (Structure of Torsion Points)

Let E/K and $m \in \mathbb{N}$. If $p = \mathrm{char}(K) \nmid m$,

$$E[m] \cong C_m \oplus C_m$$

If $m = p^r m', p \nmid m'$,

$$E[m] \cong C_m \oplus C_{m'} \qquad or \qquad E[m] \cong C_{m'} \oplus C_{m'}$$

$$E/\mathbb{F}_p \text{ is called } \begin{cases} \textit{ordinary} & \text{if } E[p] \cong C_p \\ \textit{supersingular} & \text{if } E[p] = \{\infty\} \end{cases}$$

Group Structure of $E(\mathbb{F}_q)$

Corollary

Let $E/\mathbb{F}_q$. $\exists n, k \in \mathbb{N}$ *are such that*

$$E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$$

Group Structure of $E(\mathbb{F}_q)$

Corollary

Let $E/\mathbb{F}_q$. $\exists n, k \in \mathbb{N}$ *are such that*

$$E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$$

Proof.

From classification Theorem of finite abelian group

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$$

with $n_i | n_{i+1}$ for $i \geq 1$.

Group Structure of $E(\mathbb{F}_q)$

Corollary

Let $E/\mathbb{F}_q$*.* $\exists n, k \in \mathbb{N}$ *are such that*

$$E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$$

Proof.

From classification Theorem of finite abelian group

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$$

with $n_i | n_{i+1}$ for $i \geq 1$.

Hence $E(\mathbb{F}_q)$ contains n_1^r points of order dividing n_1. From *Structure of Torsion Theorem*, $\#E[n_1] \leq n_1^2$. So $r \leq 2$ □

Group Structure of $E(\mathbb{F}_q)$

Corollary

Let $E/\mathbb{F}_q$. $\exists n, k \in \mathbb{N}$ are such that

$$E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$$

Proof.

From classification Theorem of finite abelian group

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$$

with $n_i | n_{i+1}$ for $i \geq 1$.

Hence $E(\mathbb{F}_q)$ contains n_1^r points of order dividing n_1. From *Structure of Torsion Theorem*, $\#E[n_1] \leq n_1^2$. So $r \leq 2$ □

Theorem (Corollary of Weil Pairing)

Let $E/\mathbb{F}_q$ and $n, k \in \mathbb{N}$ s.t. $E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$. Then $n \mid q - 1$.

Group Structure of $E(\mathbb{F}_q)$

Corollary

Let $E/\mathbb{F}_q$. $\exists n, k \in \mathbb{N}$ are such that

$$E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$$

Proof.

From classification Theorem of finite abelian group

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2} \oplus \cdots \oplus C_{n_r}$$

with $n_i | n_{i+1}$ for $i \geq 1$.

Hence $E(\mathbb{F}_q)$ contains n_1^r points of order dividing n_1. From *Structure of Torsion Theorem*, $\#E[n_1] \leq n_1^2$. So $r \leq 2$ □

Theorem (Corollary of Weil Pairing)

Let $E/\mathbb{F}_q$ and $n, k \in \mathbb{N}$ s.t. $E(\mathbb{F}_q) \cong C_n \oplus C_{nk}$. Then $n \mid q - 1$.

We shall discuss the proof of the latter tomorrow

Sketch of the proof of Structure Theorem of Torsion Points
The division polynomials

Sketch of the proof of Structure Theorem of Torsion Points

The division polynomials

The proof generalizes previous ideas and determine the points $P \in E(\mathbb{F}_q)$ such that $mP = \infty$ or equivalently $(m-1)P = -P$.

Definition (Division Polynomials of $E : y^2 = x^3 + Ax + B$ ($p > 3$))

$$\begin{aligned}
\psi_0 &= 0 \\
\psi_1 &= 1 \\
\psi_2 &= 2y \\
\psi_3 &= 3x^4 + 6Ax^2 + 12Bx - A^2 \\
\psi_4 &= 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3) \\
&\ \vdots \\
\psi_{2m+1} &= \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3 \qquad \text{for } m \geq 2 \\
\psi_{2m} &= \left(\frac{\psi_m}{2y}\right) \cdot (\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2) \qquad \text{for } m \geq 3
\end{aligned}$$

The polynomial $\psi_m \in \mathbb{Z}[x, y]$ is called the m^{th} *division polynomial*

The division polynomials

Lemma

Let $E : y^2 = x^3 + Ax + B$, $(p > 3)$ and let $\psi_m \in \mathbb{Z}[x, y]$ the m^{th} division polynomial. Then

$$\psi_{2m+1} \in \mathbb{Z}[x] \qquad \text{and} \qquad \psi_{2m} \in 2y\mathbb{Z}[x]$$

The division polynomials

Lemma

Let $E : y^2 = x^3 + Ax + B$, $(p > 3)$ and let $\psi_m \in \mathbb{Z}[x, y]$ the m^{th} division polynomial. Then

$$\psi_{2m+1} \in \mathbb{Z}[x] \qquad \textit{and} \qquad \psi_{2m} \in 2y\mathbb{Z}[x]$$

Proof is an exercise.

True $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$ and for the rest apply induction, the identity $y^2 = x^3 + Ax + B \cdots$ and consider the cases m odd and m even. □

The division polynomials

Lemma

Let $E : y^2 = x^3 + Ax + B$, *(*$p > 3$*) and let* $\psi_m \in \mathbb{Z}[x, y]$ *the* m^{th} *division polynomial. Then*

$$\psi_{2m+1} \in \mathbb{Z}[x] \qquad \textit{and} \qquad \psi_{2m} \in 2y\mathbb{Z}[x]$$

Proof is an exercise.

True $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$ and for the rest apply induction, the identity $y^2 = x^3 + Ax + B \cdots$ and consider the cases m odd and m even. □

Lemma

$$\psi_m = \begin{cases} y(mx^{(m^2-4)/2} + \cdots) & \text{if } m \text{ is even} \\ mx^{(m^2-1)/2} + \cdots & \text{if } m \text{ is odd.} \end{cases}$$

Hence $\psi_m^2 = m^2 x^{m^2-1} + \cdots$

The division polynomials

Lemma

Let $E : y^2 = x^3 + Ax + B$, $(p > 3)$ *and let* $\psi_m \in \mathbb{Z}[x, y]$ *the* m^{th} *division polynomial. Then*

$$\psi_{2m+1} \in \mathbb{Z}[x] \qquad \text{and} \qquad \psi_{2m} \in 2y\mathbb{Z}[x]$$

Proof is an exercise.

True $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$ and for the rest apply induction, the identity $y^2 = x^3 + Ax + B \cdots$ and consider the cases m odd and m even. □

Lemma

$$\psi_m = \begin{cases} y(mx^{(m^2-4)/2} + \cdots) & \text{if } m \text{ is even} \\ mx^{(m^2-1)/2} + \cdots & \text{if } m \text{ is odd.} \end{cases}$$

Hence $\psi_m^2 = m^2 x^{m^2-1} + \cdots$

Proof is another exercise on induction:

□

Theorem ($E : Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)

Theorem ($E: Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)

$$m(x,y) = \left(x - \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2(x)}, \frac{\psi_{2m}(x,y)}{2\psi_m^4(x)}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m^3(x,y)}\right)$$

Theorem ($E : Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)

$$m(x,y) = \left(x - \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2(x)}, \frac{\psi_{2m}(x,y)}{2\psi_m^4(x)}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m^3(x,y)}\right)$$

where

$$\phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1}, \omega_m = \frac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{4y}$$

Theorem ($E: Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)

$$m(x,y) = \left(x - \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2(x)}, \frac{\psi_{2m}(x,y)}{2\psi_m^4(x)}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m^3(x,y)}\right)$$

where

$$\phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1}, \omega_m = \frac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{4y}$$

We will omit the proof of the above (see [8, Section 9.5])

Theorem ($E : Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)

$$m(x,y) = \left(x - \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2(x)}, \frac{\psi_{2m}(x,y)}{2\psi_m^4(x)}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m^3(x,y)}\right)$$

where

$$\phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1}, \omega_m = \frac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{4y}$$

We will omit the proof of the above (see [8, Section 9.5])

Exercise (Prove that after substituting $y^2 = x^3 + Ax + B$)

1. $\phi_m(x) \in \mathbb{Z}[x]$

Theorem ($E : Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)

$$m(x,y) = \left(x - \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2(x)}, \frac{\psi_{2m}(x,y)}{2\psi_m^4(x)}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m^3(x,y)}\right)$$

where

$$\phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1}, \omega_m = \frac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{4y}$$

We will omit the proof of the above (see [8, Section 9.5])

Exercise (Prove that after substituting $y^2 = x^3 + Ax + B$)

1. $\phi_m(x) \in \mathbb{Z}[x]$
2. $\phi_m(x) = x^{m^2} + \cdots \qquad \psi_m(x)^2 = m^2 x^{m^2-1} + \cdots$

Theorem ($E : Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)

$$m(x,y) = \left(x - \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2(x)}, \frac{\psi_{2m}(x,y)}{2\psi_m^4(x)}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m^3(x,y)}\right)$$

where

$$\phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1}, \omega_m = \frac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{4y}$$

We will omit the proof of the above (see [8, Section 9.5])

Exercise (Prove that after substituting $y^2 = x^3 + Ax + B$)

1. $\phi_m(x) \in \mathbb{Z}[x]$
2. $\phi_m(x) = x^{m^2} + \cdots \qquad \psi_m(x)^2 = m^2 x^{m^2-1} + \cdots$
3. $\omega_{2m+1} \in y\mathbb{Z}[x]$, $\omega_{2m} \in \mathbb{Z}[x]$

Theorem ($E: Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)

$$m(x,y) = \left(x - \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2(x)}, \frac{\psi_{2m}(x,y)}{2\psi_m^4(x)}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m^3(x,y)}\right)$$

where

$$\phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1}, \omega_m = \frac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{4y}$$

We will omit the proof of the above (see [8, Section 9.5])

Exercise (Prove that after substituting $y^2 = x^3 + Ax + B$)

1. $\phi_m(x) \in \mathbb{Z}[x]$
2. $\phi_m(x) = x^{m^2} + \cdots \qquad \psi_m(x)^2 = m^2 x^{m^2-1} + \cdots$
3. $\omega_{2m+1} \in y\mathbb{Z}[x]$, $\omega_{2m} \in \mathbb{Z}[x]$
4. $\frac{\omega_m(x,y)}{\psi_m^3(x,y)} \in y\mathbb{Z}(x)$

Theorem ($E : Y^2 = X^3 + AX + B$ elliptic curve, $P = (x, y) \in E$)

$$m(x,y) = \left(x - \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2(x)}, \frac{\psi_{2m}(x,y)}{2\psi_m^4(x)}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m^3(x,y)}\right)$$

where

$$\phi_m = x\psi_m^2 - \psi_{m+1}\psi_{m-1}, \omega_m = \frac{\psi_{m+2}\psi_{m-1}^2 - \psi_{m-2}\psi_{m+1}^2}{4y}$$

We will omit the proof of the above (see [8, Section 9.5])

Exercise (Prove that after substituting $y^2 = x^3 + Ax + B$)

1. $\phi_m(x) \in \mathbb{Z}[x]$
2. $\phi_m(x) = x^{m^2} + \cdots \qquad \psi_m(x)^2 = m^2 x^{m^2-1} + \cdots$
3. $\omega_{2m+1} \in y\mathbb{Z}[x]$, $\omega_{2m} \in \mathbb{Z}[x]$
4. $\frac{\omega_m(x,y)}{\psi_m^3(x,y)} \in y\mathbb{Z}(x)$
5. $\gcd(\psi_m^2(x), \phi_m(x)) = 1$
   this is not really an exercise!! - see [8, Corollary 3.7]

Lemma

$$\#E[m] = \#\{P \in E(\bar{K}) : mP = \infty\} \begin{cases} = m^2 & \text{if } p \nmid m \\ < m^2 & \text{if } p \mid m \end{cases}$$

Proof.

Consider the homomorphism:

$$[m] : E(\bar{K}) \to E(\bar{K}), P \mapsto mP$$

Lemma

$$\#E[m] = \#\{P \in E(\bar{K}) : mP = \infty\} \begin{cases} = m^2 & \text{if } p \nmid m \\ < m^2 & \text{if } p \mid m \end{cases}$$

Proof.

Consider the homomorphism:

$$[m] : E(\bar{K}) \to E(\bar{K}), P \mapsto mP$$

If $p \nmid m$, need to show that

$$\#\,\mathrm{Ker}[m] = \#E[m] = m^2$$

Lemma

$$\#E[m] = \#\{P \in E(\bar{K}) : mP = \infty\} \begin{cases} = m^2 & \text{if } p \nmid m \\ < m^2 & \text{if } p \mid m \end{cases}$$

Proof.

Consider the homomorphism:

$$[m] : E(\bar{K}) \to E(\bar{K}), P \mapsto mP$$

If $p \nmid m$, need to show that

$$\#\operatorname{Ker}[m] = \#E[m] = m^2$$

We shall prove that $\exists P_0 = (a, b) \in [m](E(\bar{K})) \setminus \{\infty\}$ s.t.

$$\#\{P \in E(\bar{K}) : \ mP = P_0\} = m^2$$

Lemma

$$\#E[m] = \#\{P \in E(\bar{K}) : mP = \infty\} \begin{cases} = m^2 & \text{if } p \nmid m \\ < m^2 & \text{if } p \mid m \end{cases}$$

Proof.

Consider the homomorphism:

$$[m] : E(\bar{K}) \to E(\bar{K}), P \mapsto mP$$

If $p \nmid m$, need to show that

$$\# \operatorname{Ker}[m] = \#E[m] = m^2$$

We shall prove that $\exists P_0 = (a, b) \in [m](E(\bar{K})) \setminus \{\infty\}$ s.t.

$$\#\{P \in E(\bar{K}) : \; mP = P_0\} = m^2$$

Since $E(\bar{K})$ infinite, we can choose $(a, b) \in [m](E(\bar{K}))$ s.t.

1. $ab \neq 0$

□

Lemma

$$\#E[m] = \#\{P \in E(\bar{K}) : mP = \infty\} \begin{cases} = m^2 & \text{if } p \nmid m \\ < m^2 & \text{if } p \mid m \end{cases}$$

Proof.

Consider the homomorphism:

$$[m] : E(\bar{K}) \to E(\bar{K}), P \mapsto mP$$

If $p \nmid m$, need to show that

$$\#\,\mathrm{Ker}[m] = \#E[m] = m^2$$

We shall prove that $\exists P_0 = (a, b) \in [m](E(\bar{K})) \setminus \{\infty\}$ s.t.

$$\#\{P \in E(\bar{K}) : \ mP = P_0\} = m^2$$

Since $E(\bar{K})$ infinite, we can choose $(a, b) \in [m](E(\bar{K}))$ s.t.

1. $ab \neq 0$
2. $\forall x_0 \in \bar{K} : (\phi'_m\psi_m - 2\phi_m\psi'_m)(x_0)\psi_m(x_0) = 0 \Rightarrow a \neq \frac{\phi_m(x_0)}{\psi_m^2(x_0)}$

if $p \nmid m$, conditions imply that $\phi_m(x) - a\psi_m^2(x)$
has $m^2 = \partial(\phi_m(x) - a\psi_m^2(x))$ distinct roots
in fact $\partial\phi_m(x) = m^2$ and $\partial\psi_m^2(x) = m^2 - 1$ □

Proof continues.

Write

$$mP = m(x, y) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m(x)^3}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, yr(x)\right)$$

Proof continues.

Write

$$mP = m(x,y) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m(x)^3}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, yr(x)\right)$$

The map

$$\{\alpha \in \bar{K} : \phi_m(\alpha) - a\psi_m(\alpha)^2 = 0\} \leftrightarrow \{P \in E(\bar{K}) : mP = (a,b)\}$$
$$\alpha_0 \mapsto (\alpha_0, br(\alpha_0)^{-1})$$

is a well defined bijection.

Proof continues.

Write

$$mP = m(x,y) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m(x)^3}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, yr(x)\right)$$

The map

$$\{\alpha \in \bar{K} : \phi_m(\alpha) - a\psi_m(\alpha)^2 = 0\} \leftrightarrow \{P \in E(\bar{K}) : mP = (a,b)\}$$
$$\alpha_0 \mapsto (\alpha_0, br(\alpha_0)^{-1})$$

is a well defined bijection.

Hence there are m^2 points $P \in E(\bar{K})$ with $mP = (a,b)$

Proof continues.

Write

$$mP = m(x,y) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m(x)^3}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, yr(x)\right)$$

The map

$$\{\alpha \in \bar{K} : \phi_m(\alpha) - a\psi_m(\alpha)^2 = 0\} \leftrightarrow \{P \in E(\bar{K}) : mP = (a,b)\}$$
$$\alpha_0 \mapsto (\alpha_0, br(\alpha_0)^{-1})$$

is a well defined bijection.

Hence there are m^2 points $P \in E(\bar{K})$ with $mP = (a,b)$

So there are m^2 elements in $\mathrm{Ker}[m]$.

Proof continues.

Write

$$mP = m(x,y) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, \frac{\omega_m(x,y)}{\psi_m(x)^3}\right) = \left(\frac{\phi_m(x)}{\psi_m^2(x)}, yr(x)\right)$$

The map

$$\{\alpha \in \bar{K} : \phi_m(\alpha) - a\psi_m(\alpha)^2 = 0\} \leftrightarrow \{P \in E(\bar{K}) : mP = (a,b)\}$$
$$\alpha_0 \mapsto (\alpha_0, br(\alpha_0)^{-1})$$

is a well defined bijection.

Hence there are m^2 points $P \in E(\bar{K})$ with $mP = (a,b)$

So there are m^2 elements in $\mathrm{Ker}[m]$.

If $p \mid m$, the proof is the same except that $\phi_m(x) - a\psi_m(x)^2$ has multiple roots!!
In fact $\phi'_m(x) - a\psi'_m(x)^2 = 0$ □

From Lemma, Theorem follows:

If $p \nmid m$, apply classification Theorem of finite Groups:

$$E[m] \cong C_{n_1} \oplus C_{n_2} \oplus \cdots C_{n_k},$$

$n_i \mid n_{i+1}$. Let $\ell \mid n_1$, then $E[\ell] \subset E[m]$. Hence $\ell^k = \ell^2 \Rightarrow k = 2$. So

$$E[m] \cong C_{n_1} \oplus C_{n_2}$$

From Lemma, Theorem follows:

If $p \nmid m$, apply classification Theorem of finite Groups:

$$E[m] \cong C_{n_1} \oplus C_{n_2} \oplus \cdots C_{n_k},$$

$n_i \mid n_{i+1}$. Let $\ell \mid n_1$, then $E[\ell] \subset E[m]$. Hence $\ell^k = \ell^2 \;\Rightarrow\; k = 2$. So

$$E[m] \cong C_{n_1} \oplus C_{n_2}$$

Finally $n_2 \mid m$ and $n_1 n_2 = m^2$ so $m = n_1 = n_2$.

From Lemma, Theorem follows:

If $p \nmid m$, apply classification Theorem of finite Groups:

$$E[m] \cong C_{n_1} \oplus C_{n_2} \oplus \cdots C_{n_k},$$

$n_i \mid n_{i+1}$. Let $\ell \mid n_1$, then $E[\ell] \subset E[m]$. Hence $\ell^k = \ell^2 \Rightarrow k = 2$. So

$$E[m] \cong C_{n_1} \oplus C_{n_2}$$

Finally $n_2 \mid m$ and $n_1 n_2 = m^2$ so $m = n_1 = n_2$.

If $p \mid m$, write $m = p^j m'$, $p \nmid m'$ and

$$E[m] \cong E[m'] \oplus E[p^j] \cong C_{m'} \oplus C_{m'} \oplus E[p^j]$$

The statement follows from:

$$E[p^j] \cong \begin{cases} \{\infty\} \\ C_{p^j} \end{cases} \qquad \text{and} \qquad C_{m'} \oplus C_{p^j} \cong C_{m'p^j}$$

From Lemma, Theorem follows:

If $p \nmid m$, apply classification Theorem of finite Groups:

$$E[m] \cong C_{n_1} \oplus C_{n_2} \oplus \cdots C_{n_k},$$

$n_i \mid n_{i+1}$. Let $\ell \mid n_1$, then $E[\ell] \subset E[m]$. Hence $\ell^k = \ell^2 \Rightarrow k = 2$. So

$$E[m] \cong C_{n_1} \oplus C_{n_2}$$

Finally $n_2 \mid m$ and $n_1 n_2 = m^2$ so $m = n_1 = n_2$.

If $p \mid m$, write $m = p^j m'$, $p \nmid m'$ and

$$E[m] \cong E[m'] \oplus E[p^j] \cong C_{m'} \oplus C_{m'} \oplus E[p^j]$$

The statement follows from:

$$E[p^j] \cong \begin{cases} \{\infty\} \\ C_{p^j} \end{cases} \quad \text{and} \quad C_{m'} \oplus C_{p^j} \cong C_{m' p^j}$$

which is done by induction.

From Lemma, Theorem follows (continues)

Induction base:

$$E[p] \cong \begin{cases} \{\infty\} \\ C_p \end{cases}$$

From Lemma, Theorem follows (continues)

Induction base:

$$E[p] \cong \begin{cases} \{\infty\} \\ C_p \end{cases} \qquad \text{if follows from } \#E[p] < p^2$$

From Lemma, Theorem follows (continues)

Induction base:

$$E[p] \cong \begin{cases} \{\infty\} \\ C_p \end{cases} \qquad \text{if follows from } \#E[p] < p^2$$

- If $E[p] = \{\infty\} \Rightarrow E[p^j] = \{\infty\}\ \forall j \geq 2$:
  In fact if $E[p^j] \neq \{\infty\}$ then it would contain some element of order p

From Lemma, Theorem follows (continues)

Induction base:

$$E[p] \cong \begin{cases} \{\infty\} \\ C_p \end{cases} \qquad \text{if follows from } \#E[p] < p^2$$

- If $E[p] = \{\infty\} \Rightarrow E[p^j] = \{\infty\}\ \forall j \geq 2$:
  In fact if $E[p^j] \neq \{\infty\}$ then it would contain some element of order p

From Lemma, Theorem follows (continues)

Induction base:

$$E[p] \cong \begin{cases} \{\infty\} \\ C_p \end{cases} \qquad \text{if follows from } \#E[p] < p^2$$

- If $E[p] = \{\infty\} \Rightarrow E[p^j] = \{\infty\}\ \forall j \geq 2$:
  In fact if $E[p^j] \neq \{\infty\}$ then it would contain some element of order p(contradiction).
- If $E[p] \cong C_p$, then $E[p^j] \cong C_{p^j}\ \forall j \geq 2$:
  In fact $E[p^j]$ is cyclic (otherwise $E[p]$ would not be cyclic!)

From Lemma, Theorem follows (continues)

Induction base:

$$E[p] \cong \begin{cases} \{\infty\} \\ C_p \end{cases} \qquad \text{if follows from } \#E[p] < p^2$$

- If $E[p] = \{\infty\} \Rightarrow E[p^j] = \{\infty\}\ \forall j \geq 2$:
  In fact if $E[p^j] \neq \{\infty\}$ then it would contain some element of order p(contradiction).
- If $E[p] \cong C_p$, then $E[p^j] \cong C_{p^j}\ \forall j \geq 2$:
  In fact $E[p^j]$ is cyclic (otherwise $E[p]$ would not be cyclic!)

From Lemma, Theorem follows (continues)

Induction base:

$$E[p] \cong \begin{cases} \{\infty\} \\ C_p \end{cases} \qquad \text{if follows from } \#E[p] < p^2$$

- If $E[p] = \{\infty\} \Rightarrow E[p^j] = \{\infty\}\ \forall j \geq 2$:
  In fact if $E[p^j] \neq \{\infty\}$ then it would contain some element of order p(contradiction).
- If $E[p] \cong C_p$, then $E[p^j] \cong C_{p^j}\ \forall j \geq 2$:
  In fact $E[p^j]$ is cyclic (otherwise $E[p]$ would not be cyclic!)

Fact: $[p] : E(\bar{K}) \rightarrow E(\bar{K})$ is surjective (to be proven tomorrow)

From Lemma, Theorem follows (continues)

Induction base:

$$E[p] \cong \begin{cases} \{\infty\} \\ C_p \end{cases} \qquad \text{if follows from } \#E[p] < p^2$$

- If $E[p] = \{\infty\} \Rightarrow E[p^j] = \{\infty\}\ \forall j \geq 2$:
  In fact if $E[p^j] \neq \{\infty\}$ then it would contain some element of order p(contradiction).
- If $E[p] \cong C_p$, then $E[p^j] \cong C_{p^j}\ \forall j \geq 2$:
  In fact $E[p^j]$ is cyclic (otherwise $E[p]$ would not be cyclic!)

Fact: $[p] : E(\bar{K}) \to E(\bar{K})$ is surjective (to be proven tomorrow)

If $P \in E$ and $\text{ord}\, P = p^{j-1} \Rightarrow \exists Q \in E$ s.t. $pQ = P$ and $Q = p^j$.
Hence $E[p^j] \cong C_{p^j}$ since it contains an element of order p^j.

Remark:

- $E[2m+1] \setminus \{\infty\} = \{(x, y) \in E(\bar{K}) : \psi_{2m+1}(x) = 0\}$
- $E[2m] \setminus E[2] = \{(x, y) \in E(\bar{K}) : y^{-1}\psi_{2m}(x) = 0\}$

Theorem (Hasse)

Let E be an elliptic curve over the finite field $\mathbb{F}_q$. Then the order of $E(\mathbb{F}_q)$ satisfies

$$|q + 1 - \#E(\mathbb{F}_q)| \leq 2\sqrt{q}.$$

Theorem (Hasse)

Let E be an elliptic curve over the finite field $\mathbb{F}_q$. Then the order of $E(\mathbb{F}_q)$ satisfies

$$|q+1-\#E(\mathbb{F}_q)| \le 2\sqrt{q}.$$

So $\#E(\mathbb{F}_q) \in [(\sqrt{q}-1)^2, (\sqrt{q}+1)^2]$ the *Hasse interval* $\mathcal{I}_q$

Example (Hasse Intervals)

q	$\mathcal{I}_q$
2	$\{1, 2, 3, 4, 5\}$
3	$\{1, 2, 3, 4, 5, 6, 7\}$
4	$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
5	$\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$
7	$\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$
8	$\{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$
9	$\{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$
11	$\{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$
13	$\{7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$
16	$\{9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25\}$
17	$\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26\}$
19	$\{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28\}$
23	$\{15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33\}$
25	$\{16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36\}$
27	$\{18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38\}$
29	$\{20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40\}$
31	$\{21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43\}$
32	$\{22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44\}$

Theorem (Waterhouse)

Theorem (Waterhouse)

Let $q = p^n$ and let $N = q + 1 - a$.

$$\exists E/\mathbb{F}_q \text{ s.t.} \#E(\mathbb{F}_q) = N \Leftrightarrow |a| \leq 2\sqrt{q} \text{ and}$$

one of the following is satisfied:

Theorem (Waterhouse)

Let $q = p^n$ and let $N = q + 1 - a$.

$$\exists E/\mathbb{F}_q \text{ s.t.} \#E(\mathbb{F}_q) = N \Leftrightarrow |a| \leq 2\sqrt{q} \text{ and}$$

one of the following is satisfied:

(i) $\gcd(a, p) = 1$;

Example (q prime $\forall N \in I_q, \exists E/\mathbb{F}_q, \#E(\mathbb{F}_q) = N$. q not prime:)

q	$a \in$
$4 = 2^2$	$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$8 = 2^3$	$\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
$9 = 3^2$	$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
$16 = 2^4$	$\{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$25 = 5^2$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$27 = 3^3$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$32 = 2^5$	$\{-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Theorem (Waterhouse)

Let $q = p^n$ and let $N = q + 1 - a$.

$$\exists E/\mathbb{F}_q \text{ s.t.} \#E(\mathbb{F}_q) = N \Leftrightarrow |a| \leq 2\sqrt{q} \text{ and}$$

one of the following is satisfied:

(i) $\gcd(a, p) = 1$;

(ii) *n even and one of the following is satisfied:*

Example (q prime $\forall N \in I_q, \exists E/\mathbb{F}_q, \#E(\mathbb{F}_q) = N$. q not prime:)

q	$a \in$
$4 = 2^2$	$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$8 = 2^3$	$\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
$9 = 3^2$	$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
$16 = 2^4$	$\{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$25 = 5^2$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$27 = 3^3$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$32 = 2^5$	$\{-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Theorem (Waterhouse)

Let $q = p^n$ *and let* $N = q + 1 - a$.

$$\exists E/\mathbb{F}_q \text{ s.t.} \#E(\mathbb{F}_q) = N \Leftrightarrow |a| \leq 2\sqrt{q} \text{ and}$$

one of the following is satisfied:

(i) $\gcd(a, p) = 1$;

(ii) *n even and one of the following is satisfied:*

1. $a = \pm 2\sqrt{q}$;

Example (q prime $\forall N \in I_q, \exists E/\mathbb{F}_q, \#E(\mathbb{F}_q) = N$. q not prime:)

q	$a \in$
$4 = 2^2$	$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$8 = 2^3$	$\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
$9 = 3^2$	$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
$16 = 2^4$	$\{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$25 = 5^2$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$27 = 3^3$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$32 = 2^5$	$\{-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Theorem (Waterhouse)

Let $q = p^n$ and let $N = q + 1 - a$.

$$\exists E/\mathbb{F}_q \text{ s.t.} \#E(\mathbb{F}_q) = N \Leftrightarrow |a| \leq 2\sqrt{q} \text{ and}$$

one of the following is satisfied:

(i) $\gcd(a, p) = 1$;

(ii) *n even and one of the following is satisfied:*

1. $a = \pm 2\sqrt{q}$;
2. $p \not\equiv 1 \pmod 3$, *and* $a = \pm\sqrt{q}$;

Example (q prime $\forall N \in I_q, \exists E/\mathbb{F}_q, \#E(\mathbb{F}_q) = N$. q not prime:)

q	$a \in$
$4 = 2^2$	$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$8 = 2^3$	$\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
$9 = 3^2$	$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
$16 = 2^4$	$\{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$25 = 5^2$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$27 = 3^3$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$32 = 2^5$	$\{-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Theorem (Waterhouse)

Let $q = p^n$ and let $N = q + 1 - a$.

$$\exists E/\mathbb{F}_q \text{ s.t.} \#E(\mathbb{F}_q) = N \Leftrightarrow |a| \leq 2\sqrt{q} \text{ and}$$

one of the following is satisfied:

(i) $\gcd(a, p) = 1$;

(ii) *n even and one of the following is satisfied:*

1. $a = \pm 2\sqrt{q}$;
2. $p \not\equiv 1 \pmod 3$, *and* $a = \pm\sqrt{q}$;
3. $p \not\equiv 1 \pmod 4$, *and* $a = 0$;

Example (q prime $\forall N \in I_q, \exists E/\mathbb{F}_q, \#E(\mathbb{F}_q) = N$. q not prime:)

q	$a \in$
$4 = 2^2$	$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$8 = 2^3$	$\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
$9 = 3^2$	$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
$16 = 2^4$	$\{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$25 = 5^2$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$27 = 3^3$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$32 = 2^5$	$\{-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Theorem (Waterhouse)

Let $q = p^n$ and let $N = q + 1 - a$.

$$\exists E/\mathbb{F}_q \text{ s.t.} \#E(\mathbb{F}_q) = N \Leftrightarrow |a| \leq 2\sqrt{q} \text{ and}$$

one of the following is satisfied:

(i) $\gcd(a, p) = 1$;

(ii) *n even and one of the following is satisfied:*

1. $a = \pm 2\sqrt{q}$;
2. $p \not\equiv 1 \pmod 3$, *and* $a = \pm\sqrt{q}$;
3. $p \not\equiv 1 \pmod 4$, *and* $a = 0$;

(iii) *n is odd, and one of the following is satisfied:*

Example (q prime $\forall N \in I_q, \exists E/\mathbb{F}_q, \#E(\mathbb{F}_q) = N$. q not prime:)

q	$a \in$
$4 = 2^2$	$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$8 = 2^3$	$\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
$9 = 3^2$	$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
$16 = 2^4$	$\{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$25 = 5^2$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$27 = 3^3$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$32 = 2^5$	$\{-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Theorem (Waterhouse)

Let $q = p^n$ and let $N = q + 1 - a$.

$$\exists E/\mathbb{F}_q \text{ s.t.} \#E(\mathbb{F}_q) = N \Leftrightarrow |a| \le 2\sqrt{q} \text{ and}$$

one of the following is satisfied:

(i) $\gcd(a, p) = 1$;

(ii) *n even and one of the following is satisfied:*

1. $a = \pm 2\sqrt{q}$;
2. $p \not\equiv 1 \pmod 3$, *and* $a = \pm\sqrt{q}$;
3. $p \not\equiv 1 \pmod 4$, *and* $a = 0$;

(iii) *n is odd, and one of the following is satisfied:*

1. $p = 2$ *or* 3, *and* $a = \pm p^{(n+1)/2}$;

Example (q prime $\forall N \in I_q, \exists E/\mathbb{F}_q, \#E(\mathbb{F}_q) = N$. q not prime:)

q	$a \in$
$4 = 2^2$	$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$8 = 2^3$	$\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
$9 = 3^2$	$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
$16 = 2^4$	$\{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$25 = 5^2$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$27 = 3^3$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$32 = 2^5$	$\{-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Theorem (Waterhouse)

Let $q = p^n$ and let $N = q + 1 - a$.

$$\exists E/\mathbb{F}_q \text{ s.t.} \#E(\mathbb{F}_q) = N \Leftrightarrow |a| \leq 2\sqrt{q} \text{ and}$$

one of the following is satisfied:

(i) $\gcd(a, p) = 1$;

(ii) *n even and one of the following is satisfied:*

1. $a = \pm 2\sqrt{q}$;
2. $p \not\equiv 1 \pmod 3$, *and* $a = \pm\sqrt{q}$;
3. $p \not\equiv 1 \pmod 4$, *and* $a = 0$;

(iii) *n is odd, and one of the following is satisfied:*

1. $p = 2$ *or* 3, *and* $a = \pm p^{(n+1)/2}$;
2. $a = 0$.

Example (q prime $\forall N \in I_q, \exists E/\mathbb{F}_q, \#E(\mathbb{F}_q) = N$. q not prime:)

q	$a \in$
$4 = 2^2$	$\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$8 = 2^3$	$\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
$9 = 3^2$	$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$
$16 = 2^4$	$\{-8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$
$25 = 5^2$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$27 = 3^3$	$\{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$32 = 2^5$	$\{-11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Theorem (Rück)

Suppose N is a possible order of an elliptic curve $/\mathbb{F}_q$, $q = p^n$. *Write*

$$N = p^e n_1 n_2, \quad p \nmid n_1 n_2 \quad \text{and} \quad n_1 \mid n_2 \text{ (possibly } n_1 = 1).$$

There exists $E/\mathbb{F}_q$ *s.t.*

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2 p^e}$$

if and only if

1. $n_1 = n_2$ *in the case (ii).1 of Waterhouse's Theorem;*

Theorem (Rück)

Suppose N is a possible order of an elliptic curve $/\mathbb{F}_q$, $q = p^n$. *Write*

$$N = p^e n_1 n_2, \quad p \nmid n_1 n_2 \quad \text{and} \quad n_1 \mid n_2 \text{ (possibly } n_1 = 1).$$

There exists $E/\mathbb{F}_q$ *s.t.*

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2 p^e}$$

if and only if

1. $n_1 = n_2$ *in the case (ii).1 of Waterhouse's Theorem;*
2. $n_1 | q - 1$ *in all other cases of Waterhouse's Theorem.*

Theorem (Rück)

Suppose N is a possible order of an elliptic curve $/\mathbb{F}_q$, $q = p^n$. Write

$$N = p^e n_1 n_2, \quad p \nmid n_1 n_2 \quad \text{and} \quad n_1 \mid n_2 \text{ (possibly } n_1 = 1).$$

There exists $E/\mathbb{F}_q$ s.t.

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2 p^e}$$

if and only if

1. *$n_1 = n_2$ in the case (ii).1 of Waterhouse's Theorem;*
2. *$n_1 | q - 1$ in all other cases of Waterhouse's Theorem.*

Theorem (Rück)

Suppose N is a possible order of an elliptic curve $/\mathbb{F}_q$, $q = p^n$. *Write*

$$N = p^e n_1 n_2, \quad p \nmid n_1 n_2 \quad \text{and} \quad n_1 \mid n_2 \text{ (possibly } n_1 = 1).$$

There exists $E/\mathbb{F}_q$ *s.t.*

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2 p^e}$$

if and only if

1. $n_1 = n_2$ *in the case (ii).1 of Waterhouse's Theorem;*
2. $n_1 | q - 1$ *in all other cases of Waterhouse's Theorem.*

Example

- If $q = p^{2n}$ and $\#E(\mathbb{F}_q) = q + 1 \pm 2\sqrt{q} = (p^n \pm 1)^2$, then $E(\mathbb{F}_q) \cong C_{p^n \pm 1} \oplus C_{p^n \pm 1}$.

Theorem (Rück)

Suppose N is a possible order of an elliptic curve $/\mathbb{F}_q$, $q = p^n$. Write

$$N = p^e n_1 n_2, \quad p \nmid n_1 n_2 \quad \textit{and} \quad n_1 \mid n_2 \ (\textit{possibly } n_1 = 1).$$

There exists $E/\mathbb{F}_q$ s.t.

$$E(\mathbb{F}_q) \cong C_{n_1} \oplus C_{n_2 p^e}$$

if and only if

1. *$n_1 = n_2$ in the case (ii).1 of Waterhouse's Theorem;*
2. *$n_1 | q - 1$ in all other cases of Waterhouse's Theorem.*

Example

- If $q = p^{2n}$ and $\#E(\mathbb{F}_q) = q + 1 \pm 2\sqrt{q} = (p^n \pm 1)^2$, then
  $$E(\mathbb{F}_q) \cong C_{p^n \pm 1} \oplus C_{p^n \pm 1}.$$
- Let $N = 100$ and $q = 101 \Rightarrow \exists E_1, E_2, E_3, E_4/\mathbb{F}_{101}$ s.t.
  $$E_1(\mathbb{F}_{101}) \cong C_{10} \oplus C_{10} \qquad E_2(\mathbb{F}_{101}) \cong C_2 \oplus C_{50}$$
  $$E_3(\mathbb{F}_{101}) \cong C_5 \oplus C_{20} \qquad E_4(\mathbb{F}_{101}) \cong C_{100}$$

Further Reading...

- Ian F. Blake, Gadiel Seroussi, and Nigel P. Smart, Advances in elliptic curve cryptography, London Mathematical Society Lecture Note Series, vol. 317, Cambridge University Press, Cambridge, 2005.
- J. W. S. Cassels, Lectures on elliptic curves, London Mathematical Society Student Texts, vol. 24, Cambridge University Press, Cambridge, 1991.
- John E. Cremona, Algorithms for modular elliptic curves, 2nd ed., Cambridge University Press, Cambridge, 1997.
- Anthony W. Knapp, Elliptic curves, Mathematical Notes, vol. 40, Princeton University Press, Princeton, NJ, 1992.
- Neal Koblitz, Introduction to elliptic curves and modular forms, Graduate Texts in Mathematics, vol. 97, Springer-Verlag, New York, 1984.
- Joseph H. Silverman, The arithmetic of elliptic curves, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 1986.
- Joseph H. Silverman and John Tate, Rational points on elliptic curves, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992.
- Lawrence C. Washington, Elliptic curves: Number theory and cryptography, 2nd ED. Discrete Mathematics and Its Applications, Chapman & Hall/CRC, 2008.
- Horst G. Zimmer, Computational aspects of the theory of elliptic curves, Number theory and applications (Banff, AB, 1988) NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 265, Kluwer Acad. Publ., Dordrecht, 1989, pp. 279–324.