An amalgamated duplication of a ring along an ideal

presented by

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1. INTRODUCTION. By extending a notion introduced in the integral domain case by W. Heinzer, J. Huckaba and I. Papick [HHP-1998], we can say that a regular ideal I of a ring R is *a multiplicative-canonical ideal* (or simply, *an m–canonical ideal*) *of R* if each regular fractional ideal J of R is I–reflexive, i.e.

$$J = (I : (I : J)) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(J, I), I)\,,$$

(where $(I : J) := \{x \in T(R) \mid xJ \subseteq I\}$, and $T(R)$ denotes the total ring of fractions of R).

Recall that, given a 1-dimensional Cohen-Macaulay ring R, R is *a Gorenstein ring* if and only if R has an m–canonical ideal isomorphic to R, i.e.

$$J = (R : (R : J)) = J_{\boldsymbol{v}}\,,$$

for each regular fractional ideal J of R (cf. J. Herzog and E. Kunz [HK-1971, Korollar 3.4] and E. Matlis [M-1973, Chapter XIII]).

In higher dimension the problem of the relations between Cohen-Macaulay and Gorenstein rings is more delicate.

Given a Cohen-Macaulay local ring (R, M, k) of dimension d, *a canonical module* E is an R–module such that

$$\dim_k(\mathrm{Ext}^i_R(k, E)) = \delta_{i,d}\,.$$

It is wellknown that if a Cohen-Macaulay local ring has a canonical module this is uniquelly determined, up to isomorphisms. In general, *given a Cohen-Macaulay local ring R,*

R is a Gorenstein ring $\Leftrightarrow$ R has a canonical module isomorphic to R

(cf. W. Bruns and J. Herzog [BH-1993, Section 3.3]).

Let (R, M) be a Cohen-Macaulay local ring admitting a canonical module E, and let $R \ltimes E$ be the idealization of E in R (M. Nagata [N-1962, page 2]) then I. Reiten [R-1972] proved that *$R \ltimes E$ is a Gorenstein ring.*

Later, in 1975, R. Fossum, P. Griffith and I. Reiten in [FGR-1975] proved a more precise statement:

If (R, M) is a Cohen-Macaulay local ring and E a R–module, then $R \ltimes E$ is a Gorenstein ring if and only if the R–module E is a canonical module of R.

But, it is easy to see that $R \ltimes E$ is *not* a reduced ring, even if R is an integral domain.

In this talk, I will introduce a new general construction, called the amalgamated duplication of a ring R along an R–module E, which is an ideal in some overring of R (and so E is submodule of the total ring of fractions $T(R)$ of R), and denoted by $R \bowtie E$.

(When $E^2 = 0$, the new construction $R \bowtie E$ coincides with the idealization $R \ltimes E$.)

M. D'Anna [D'A-2005] has applied this construction to give an explicit method for constructing a *reduced* Gorenstein local ring associated in a natural way to a Cohen-Macaulay local domain.

2. THE GENERAL CONSTRUCTION

Let R be a commutative ring with unit, $T(R)$ its total ring of fractions, let E be an R-submodule of $T(R)$ such that $E \cdot E \subseteq E$ (note that the last condition is equivalent to require that E is an ideal in some overring S of R).

In the R-module direct sum $R \oplus E$, we can introduce a multiplicative structure by setting:

$$(r,e)(s,f) := (rs, rf + se + ef)\,, \text{ where } r,s \in R \text{ and } e,f \in E\ .$$

We denote by $R \dot{\oplus} E$ the direct sum $R \oplus E$ endowed also with the multiplication defined above.

The following properties are easy to check:

Lemma 1 (a) *$R\dot{\bigoplus}E$ is a ring.*

(b) *The map $i : R \to R\dot{\bigoplus}E$, defined by $r \mapsto (r, 0)$, is an injective ring homomorphism (and so $R\dot{\bigoplus}E$ is an R–algebra).*

(c) *The map $j : R\dot{\bigoplus}E \to T(R) \times T(R)$, defined by $(r, e) \mapsto (r, r + e)$, is an injective ring homomorphism.* □

Set

$$R^{\triangle} := \{(r, r) \mid r \in R\}$$
$$R \bowtie E := j(R\dot{\bigoplus}E) = \{(r, r + e) \mid r \in R,\ e \in E\}.$$

Clearly, we have the following inclusions of subrings of $T(R) \times T(R)$:

$$R^{\triangle} \subseteq R \bowtie E \subseteq R \times (R+E) \subseteq T(R) \times T(R).$$

Remark 2 For an *arbitrary* R-module E, M. Nagata introduced in 1955 [N-1955] *the idealization of E in R*, denoted here by $R \ltimes E$, which is the R–module $R \oplus E$ endowed with a multiplicative structure defined by:

$$(r,e)(s,f) := (rs, rf + se)\,, \quad \text{where } r,s \in R \text{ and } e,f \in E \ .$$

The idealization $R \ltimes E$, is called by Fossum [F-1973] *the trivial extension of R by E*, since it is a ring such that the following sequence of canonical homomorphisms:

$$0 \to E \xrightarrow{\iota_E} R \ltimes E \xrightarrow{\pi_R} R \to 0\,, \quad (\iota_E : e \mapsto (0,e)\,; \quad \pi_R : (r,e) \mapsto r\,),$$

is an exact sequence.

Note that $\iota_E(E) =: E^{\ltimes}$ is an ideal in $R \ltimes E$ (isomorphic as an R-module to E), which is nilpotent of index 2 (i.e. $E^{\ltimes} \cdot E^{\ltimes} = 0$).

Therefore, even if R is reduced, the idealization $R \ltimes E$ is *not* a reduced ring (except in the trivial case for $E = (0)$, since $R \ltimes (0) = R$).

Note that the idealization $R \ltimes E$ coincides with the ring $R \dot{\oplus} E$ (Lemma 1) if and only if E is an R-submodule of $T(R)$ that is nilpotent of index 2 (i.e. $E \cdot E = (0)$).

Proposition 3 *Let R be a ring and E a R–submodule of $T(R)$ such that $E \cdot E \subseteq E$. Then:*

(a) $R \bowtie E$ *is a subdirect product of the ring $R \times (R+E)$,*
i.e. if π_i $(i = 1, 2)$ are the projections of $R \times (R+E)$ onto R and $(R+E)$, respectively, and if $\mathfrak{O}_i := \mathrm{Ker}(\pi_i|_{R \bowtie E})$, then $(R \bowtie E)/\mathfrak{O}_1 \cong R$, $(R \bowtie E)/\mathfrak{O}_2 \cong R+E$ and $\mathfrak{O}_1 \cap \mathfrak{O}_2 = 0$.

(b) *The following properties are equivalent:*

(i) *R is a domain (or, equivalently, $R+E$ is a domain);*

(ii) *$\mathfrak{O}_1$ is a prime ideal of $R \bowtie E$;*

(iii) *$\mathfrak{O}_2$ is a prime ideal of $R \bowtie E$;*

(iv) *$R \bowtie E$ is a reduced ring and $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are prime ideals of $R \bowtie E$.* □

Note that it can be shown that R is a domain if and only if $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are the only minimal prime ideals $R \bowtie E$.

Theorem 4 *In the situation of previous Proposition 3, let* $v : R \times (R+E) \twoheadrightarrow R \times ((R+E)/E)$ *and* $u : R \hookrightarrow R \times ((R+E)/E)$ *be the natural ring homomorphisms (defined respectively by* $v((x, r+e)) := (x, r+E)$ *and* $u(r) := (r, r+E)$*, for all* $x, r \in R$ *and* $e \in E$*), then* $v^{-1}(u(R)) = R \bowtie E$.
Therefore, if $v' \ (:= \pi_1|_{R\bowtie E}) : R \bowtie E \twoheadrightarrow R$ *is the canonical map defined by* $(r, r+e) \mapsto r$ *and* $u' : R \bowtie E \hookrightarrow R \times (R+E)$ *is the natural embedding, then the following diagram:*

$$\begin{array}{ccc} R\bowtie E & \xrightarrow{v'} & R \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ R \times (R+E) & \xrightarrow{v} & R \times ((R+E)/E) \end{array}$$

is a pullback. □

Example 5 *Let k be a field and X an indeterminate over k. Set:*

$$R := k[X^4, X^6, X^7, X^9]\,, \quad S := k[X^2, X^3]\,, \quad E := X^2 S = X^2 k + X^4 k[X]\,.$$

Then, it is easy to see that:

$$R+E = k[X^2, X^5]$$

and

$$R \bowtie E = \{(f, g) \in R \times (R+E) \mid f(0) = g(0)\}\,.$$

3. THE CONSTRUCTION $R\bowtie E$ WHEN E IS AN IDEAL IN R

Proposition 6 *Let I be an ideal of a ring R. Using the notation of Proposition 3 and Theorem 4, we have that $R+I = R$ and the following commutative diagram of canonical ring homomorphisms*

$$\begin{array}{ccc} R\bowtie I & \xrightarrow{v'} & R \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ R\times R & \xrightarrow{v} & R\times(R/I) \end{array}$$

is a pullback. The ideal $\mathfrak{O}_1 = (0)\times I = \mathrm{Ker}(v) = \mathrm{Ker}(v')$ is a common ideal of $R\bowtie I$ and $R\times R$,

the ideal $\mathfrak{O}_2 := \mathrm{Ker}(R\bowtie I \xrightarrow{u'} R\times R \xrightarrow{\pi_2} R)$ coincides with $I\times(0) = (I\times(0))\cap(R\bowtie I)$ and $(R\bowtie I)/\mathfrak{O}_i \cong R$, for $i = 1, 2$.

If R is a domain then $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are the only minimal primes of $R\bowtie I$.$\square$

Corollary 7 *In the situation of Proposition 6, let* R' *(respectively,* R^**) be the integral closure (respectively, the complete integral closure) of* R *in* $T(R)$*, we have:*

(a) $\dim(R \bowtie I) = \dim(R)$.

(b) R *is Noetherian if and only if* $R \bowtie I$ *is Noetherian.*

(c) *The integral closure of* $R^{\triangle}$ *and of* $R \bowtie I$ *in* $T(R) \times T(R)$ *coincide with* $R' \times R'$.

(d) *If* I *contains a nonzero regular element, then* $T(R \bowtie I) = T(R) \times T(R)$ *and the complete integral closure of* $R \bowtie I$ *in* $T(R) \times T(R)$ *coincide with* $R^* \times R^*$*, which is the complete integral closure of* $R \times R$ *in* $T(R) \times T(R)$.
$\square$

We can now use the pullback presentation of $R \bowtie I$ to describe Spec$(R\bowtie I)$.
Note that, if $\mathcal{Q} \in$ Spec$(R\bowtie I)$, then either $\mathcal{Q} \not\supseteq \mathfrak{O}_{\mathbf{1}}$ or $\mathcal{Q} \supseteq \mathfrak{O}_{\mathbf{1}}$.

- **Case 1**. $\mathcal{Q} \not\supseteq \mathfrak{O}_{\mathbf{1}}$ $(= (0) \times I)$.

In this case, there exists a unique prime ideal Q of $R \times R$ such that $\mathcal{Q} = Q \cap (R\bowtie I)$ and $Q \not\supseteq (0) \times I$. Hence, it is not difficult to see that $Q = R \times P$ for some prime P of R such that $P \not\supseteq I$.
(More precisely P is the trace of Q and of $\mathcal{Q}$ in R, under the diagonal embedding.)
Moreover,

$$\mathcal{Q} = \{(p+i, p) \mid p \in P,\ i \in I\} = (R \times P) \cap (R\bowtie I)\ .$$

$$(R\bowtie I)_{\mathcal{Q}} \cong (R \times R)_Q = (R \times R)_{R\times P} \cong R_P\ .$$

▶ **Case 2**. $\mathcal{Q} \supseteq \mathfrak{O}_1$ ($= (0) \times I$).
In this case, here exists a unique prime ideal P of R such that $\mathcal{Q} = v'^{-1}(P)$ (or, equivalently, $P = v'(\mathcal{Q})$; where $v' : R \bowtie I \to R$ is the canonical projection). Hence:

$$\mathcal{Q} = \{(p, p+i) \mid p \in P,\ i \in I\} = (P \times R) \cap (R \bowtie I) \qquad \text{and}$$

$$(R \bowtie I)/\mathcal{Q} \cong R/P\,.$$

Furthermore, it is easy to see that:

- if $P \supseteq I$,
$$\mathcal{Q} = (P \times R) \cap (R \bowtie I)) = (R \times P) \cap (R \bowtie I)\,.$$

- if $P \not\supseteq I$,
$$\mathcal{Q} = (P \times R) \cap (R \bowtie I) \neq (R \times P) \cap (R \bowtie I)\,.$$

After studying the relation between Spec($R \times R$) and Spec($R \bowtie I$), under the continuous map $(u')^a$, associated the canonical embedding $u' : R \bowtie I \hookrightarrow R \times R$, next goal is to investigate directly the relation between Spec($R \bowtie I$) and Spec(R), under the canonical map associated to the diagonal embedding $\delta : R \hookrightarrow R \bowtie I$, $(r \ \mapsto (r, r))$.

For the sake of simplicity, we will identify R with its isomorphic image $R^{\triangle}$ in $R \bowtie I$ and we will denote the contraction to R of an ideal $\boldsymbol{\mathcal{H}}$ of $R \bowtie I$ by $\boldsymbol{\mathcal{H}} \cap R$ (instead of $\delta^{-1}(\boldsymbol{\mathcal{H}})$).

Notation. In the following, the residue field at the prime ideal Q of a ring A (i.e. the field A_Q/QA_Q) will be denoted by $\boldsymbol{k}_A(Q)$.

Theorem 8 *Let I be an ideal of a ring R and let $R \bowtie I$ be as in Proposition 6. Let P be a prime ideal of R and consider the following ideals:*

- $\mathcal{P}_1 := v'^{-1}(P) = u'^{-1}(P \times R) = u'^{-1}(P \times (P + I)) =$
  $$= \{(p, p + i) \mid p \in P,\ i \in I\} =: P \bowtie I.$$
- $\mathcal{P}_2 := u'^{-1}(R \times P) = \{(p + i, p) \mid p \in P,\ i \in I\}.$
- $\mathcal{P} := \mathcal{P}_1 \cap \mathcal{P}_2 = u'^{-1}(P \times P) = \{(p, p + i') \mid p \in P,\ i' \in I \cap P\} = \{(p_1, p_2) \mid p_1, p_2 \in P,\ p_1 - p_2 \in I\}.$
- $P^e := P(R \bowtie I) = \{(p, p + i'') \mid p \in P,\ i'' \in PI\}.$

Obviously $P^e \subseteq \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}$.

Then, we have:

(a) $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are the only prime ideals of* $R\bowtie I$ *lying over* P *.*

(b) *If* $P \supseteq I$, *then* $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P} = \sqrt{P^e} = P \bowtie I$. *Moreover,*

$$\boldsymbol{k}_R(P) \cong \boldsymbol{k}_{R\bowtie I}(\mathcal{P}) \, .$$

(c) *If* $P \not\supseteq I$ *then* $\mathcal{P}_1 \neq \mathcal{P}_2$. *Moreover* $\mathcal{P} = \sqrt{P^e}$ *and*

$$\boldsymbol{k}_R(P) \cong \boldsymbol{k}_{R\bowtie I}(\mathcal{P}_1) \cong \boldsymbol{k}_{R\bowtie I}(\mathcal{P}_2) \, .$$

(d) *If* P *is a maximal ideal of* R *then* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are maximal ideals of* $R\bowtie I$ *.*

(e) *If* R *is a local ring with maximal ideal* M *then* $R \bowtie I$ *is a local ring with maximal ideal* $\mathcal{M} := \sqrt{M^e} = M \bowtie I$. □

As a consequence of Corollary 7 (c) and (d), and Proposition 3 (b), we obtain the following.

Corollary 9 *Let R be an integral domain with quotient field K and let I be a nonzero ideal of R. We denote by R' (respectively, R^*) the integral closure (respectively, the complete integral closure) of R in K. Then:*

(a) *$R \bowtie I$ is a reduced ring (with two distinct minimal primes $\mathfrak{O}_1$ and $\mathfrak{O}_2$ such that $R \bowtie I / \mathfrak{O}_i \cong R$, $i = 1, 2$).*

(b) *$T(R \bowtie I) = K \times K$ and the integral closure (respectively, the complete integral closure) of $R \bowtie I$ in $K \times K$ is $R' \times R'$ (respectively, $R^* \times R^*$).* □

Next goal is to give a complete description of the affine scheme Spec$(R \bowtie I)$, determining the localizations of $R \bowtie I$ in each of its prime ideals.

Theorem 10 *In the situation of Proposition 6, let* $X := \mathrm{Spec}(R\bowtie I)$, $Y := \mathrm{Spec}(R \times R) \cong \mathrm{Spec}(R) \amalg \mathrm{Spec}(R)$ *and* $X_0 := \mathrm{Spec}(R)$ *and let* $\alpha : Y \twoheadrightarrow X$ *and* $\beta : X \twoheadrightarrow X_0$ *be the canonical surjective maps associated to the integral embeddings* $R\bowtie I \hookrightarrow R \times R$ *and* $R \cong R^{\Delta} \hookrightarrow R\bowtie I$.

(a) *Since* $\mathfrak{O}_1 = \{0\} \times I$ *is a common ideal of* $R \times R$ *and* $R\bowtie I$, *then*

$$\alpha\Big|_{Y\setminus V_Y(\mathfrak{O}_1)} : Y \setminus V_Y(\mathfrak{O}_1) \longrightarrow X \setminus V_X(\mathfrak{O}_1)$$

is a scheme isomorphism, where

$$Y \setminus V_Y(\mathfrak{O}_1) \cong \Big((X_0 \amalg X_0) \setminus (X_0 \amalg V_{X_0}(I))\Big) = X_0 \setminus V_{X_0}(I)\,.$$

In particular, for each prime ideal P of R, such that $P \not\supseteq I$,

if we set $\overline{P_1} := P \times R$ and $\overline{P_2} := R \times P$, and

if $\boldsymbol{\mathcal{P}}_i := \overline{P}_i \cap (R \bowtie I)$, for $1 \leq i \leq 2$, then

$\boldsymbol{\mathcal{P}}_1$ and $\boldsymbol{\mathcal{P}}_2$ are distict prime ideal of $R \bowtie I$ and they are the only prime ideals of $R \bowtie I$ contracting onto P.

Moreover, the following canonical ring homomorphisms are isomorphisms:

$$R_P \to (R \bowtie I)_{\boldsymbol{\mathcal{P}}_i} \longrightarrow (R \times R)_{\overline{P}_i}, \quad \text{for } 1 \leq i \leq 2.$$

(b) *Let $P \in \mathrm{Spec}(R)$ be such that $P \supseteq I$.*

Then, in $R \bowtie I$, there exists a unique prime ideal $\boldsymbol{\mathcal{P}}$ $(= \boldsymbol{\mathcal{P}_1} = \boldsymbol{\mathcal{P}_2}) = \sqrt{P^e} = P \bowtie I$ such that $\boldsymbol{\mathcal{P}} \cap R = P$.

In this case, we have that the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} (R \bowtie I)_{\boldsymbol{\mathcal{P}}} & \longrightarrow & R_P \\ \downarrow & & \downarrow{\scriptstyle u_P} \\ R_P \times R_P & \xrightarrow{v_P} & R_P \times (R_P/I_P) \end{array}$$

is a pullback (where $I_P = IR_P$, $u_P(x) := (x, x + I_P)$ and $v_P((x,y)) := (x, y + I_P)$, for $x, y \in R_P$), i.e. (by Proposition 6)

$$(R \bowtie I)_{\boldsymbol{\mathcal{P}}} \cong R_P \bowtie I_P\,.$$

□

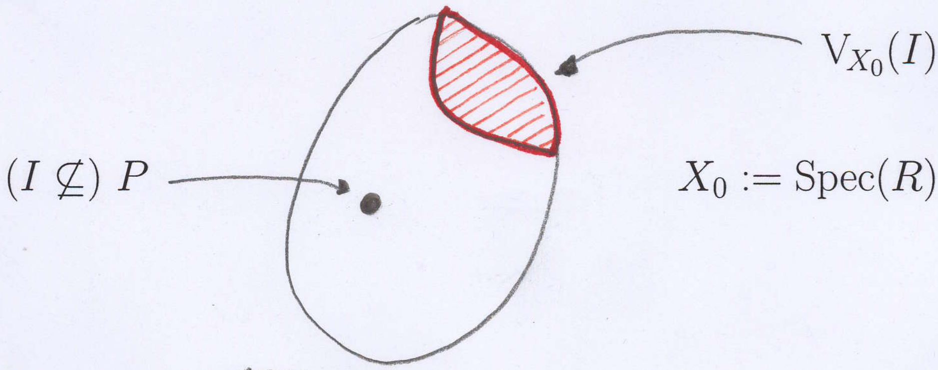
Example 11 If R is a local ring, with maximal ideal M and residue field k, then $R \bowtie M$ is local and it can be obtained as a pullback of the following diagram of canonical homomorphisms:

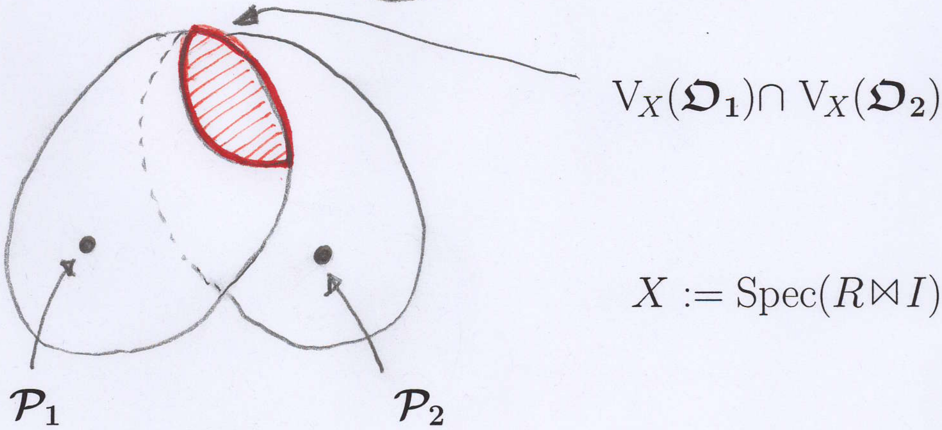
$$\begin{array}{ccc} R \bowtie M & \longrightarrow & k \\ \big\downarrow & & \big\downarrow \alpha \\ R \times R & \xrightarrow{\beta} & k \times k \end{array}$$

(where α is the diagonal embedding, β is the canonical surjection $(x, y) \mapsto (x + M, y + M)$).

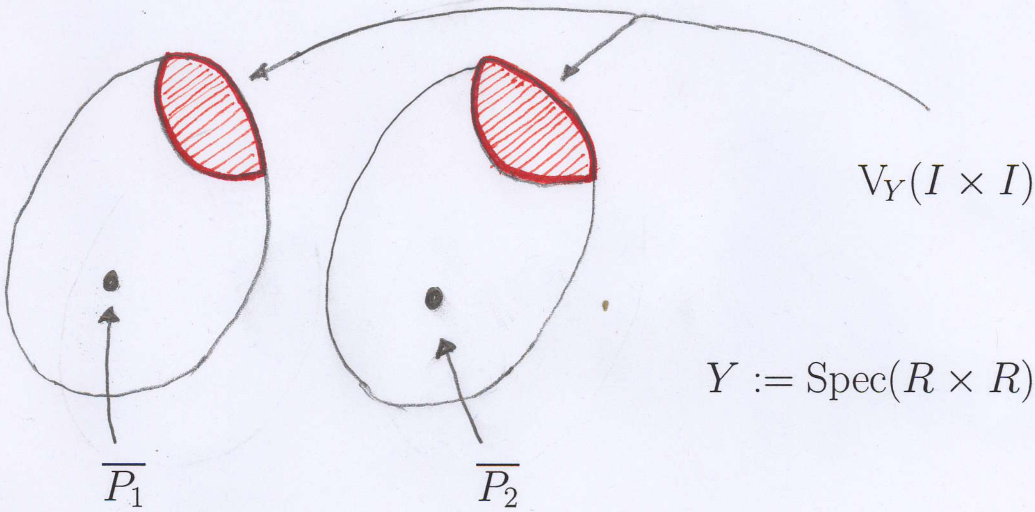
Moreover, if we assume that R is integrally closed in $T(R)$, then $R \bowtie M$ is seminormal in its integral closure inside $T(R) \times T(R)$ (which, in this situation, coincides with $R \times R$).

CASE " $I \not\subseteq$ "



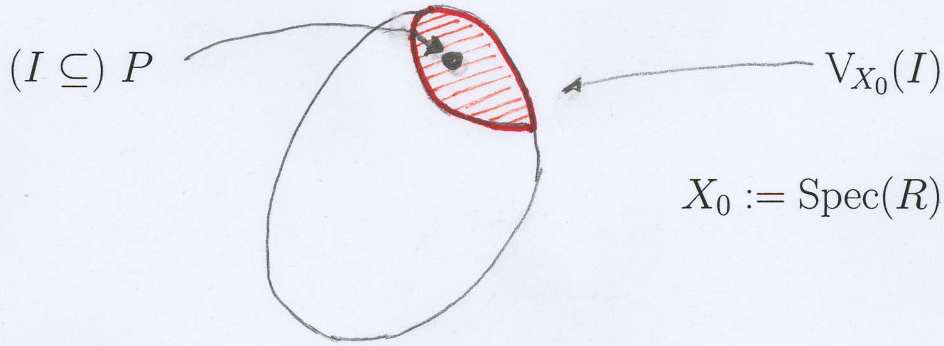


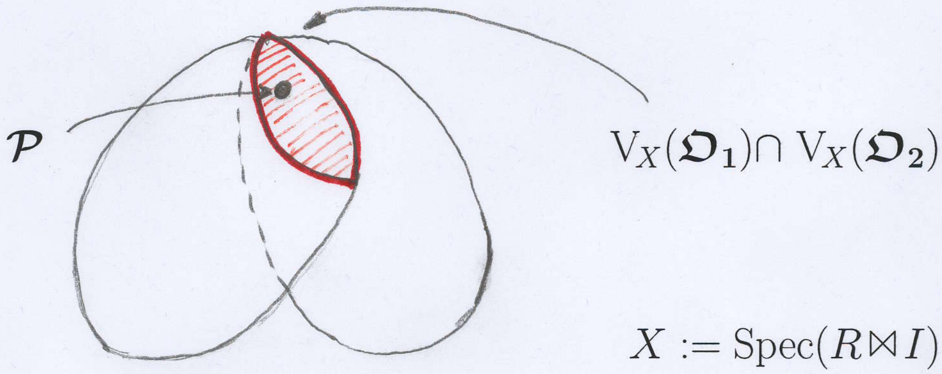
$\mathcal{P}_1 = \{(p, p+i) \mid p \in P,\ i \in I\}$ $\quad$ $\mathcal{P}_2 = \{(p+i, p) \mid p \in P,\ i \in I\}$



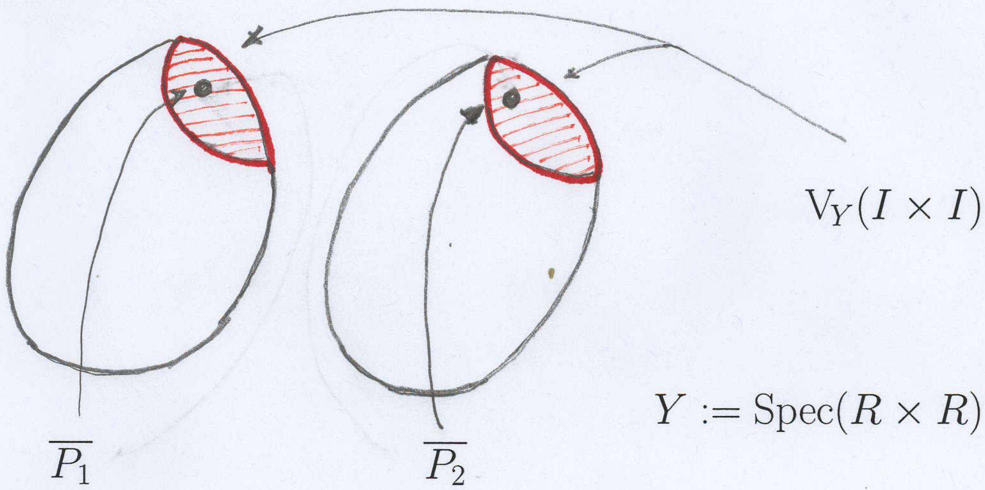
$\overline{P_1} = P \times R$ $\quad$ $\overline{P_2} = R \times P$ $\quad$ $I \times I = \mathfrak{O}_1 + \mathfrak{O}_2$

CASE " $I \subseteq$ "





$$\boldsymbol{\mathcal{P}} = P \bowtie I = \{(p, p+i) \mid p \in P,\ i \in I\} = \sqrt{P^e}$$



$\overline{P_1} = P \times R$ $\quad$ $\overline{P_2} = R \times P$ $\quad$ $I \times I = \mathfrak{O}_1 + \mathfrak{O}_2$

4. THE RING $R \bowtie E$ WHEN E IS A CANONICAL IDEAL OF R

Next goal is to investigate the construction $R \bowtie I$, in case I is an m–canonical ideal of an arbitrary ring R (not necessarily a domain) [definition recalled later].

Note that, given an R–module H, for each R–module F, we can consider the R–module:

$$F^{*H} := \mathrm{Hom}_R(F, H)\,.$$

We have the following canonical homomorphism:

$$\rho_F : F \to (F^{*H})^{*H}\,, \quad a \mapsto \rho_F(a)\,, \text{where } \rho_F(a)(f) := f(a)\,,$$

for all $f \in F^{*H}$ ($= \mathrm{Hom}_R(F, H)$), $a \in F$. We say that the R–module F is *H–reflexive* (respectively, *H–torsionless*) if ρ_F is an isomorphism (respectively, monomorphism) of R–modules.

Given a regular ideal I of the ring R and a R–submodule F of $T(R)$, set:

$$(I : F) := \{z \in T(R) \mid zF \subseteq I\} .$$

*If $F =: J$ is a regular fractional ideal of R then $(I : J)$ is also a regular fractional ideal of R,*and it is not hard to prove that *there exists a canonical isomorphism:*

$$(I : (I : J)) \xrightarrow{\sim} (I : J)^{*_I} \xrightarrow{\sim} (J^{*_I})^{*_I} = \mathrm{Hom}_R(\mathrm{Hom}_R(J, I), I) ,$$

and so, in this situation, we can identify the map $\rho_J : J \to (J^{*_I})^{*_I}$ with the inclusion $J \subseteq (I : (I : J))$, thus *each regular fractional ideal J is I–torsionless.*

We say that a regular ideal I of a ring R is *an m–canonical ideal of R* if each regular fractional ideal J of R is I–reflexive, i.e. the map $\rho_J : J \to (J^{*_I})^{*_I}$ is an isomorphism or, equivalently, $J = (I : (I : J))$ (cf. [M-1973] and [HHP-1998]).

Recall that a regular fractional ideal J of a ring R is called *a divisorial ideal of R* if $(R : (R : J)) = J$.
Clearly an invertible ideal of R is a divisorial ideal.

Let I be a regular ideal of a ring R and set:

$$\mathcal{F}(R,I) := \{F \mid \ F \text{ is a regular } I\text{–torsionless } R\text{–submodule of } R^n, \text{ for some } n \geq 1\},$$

$$\mathcal{F}_1(R,I) := \{F \mid \ F \text{ is a regular } I\text{–torsionless } R\text{–submodule of } R\}.$$

We say that

• *the ring R* is *$\mathcal{F}(R,I)$–reflexive* if every $F \in \mathcal{F}(R,I)$ is I–reflexive (i.e. the canonical monomorphism $\rho_F : F \to \text{Hom}_R(\text{Hom}_R(F,I),I)$ is an isomorphism of R–modules).

Similarly, we say that

• *the ring R* is *$\mathcal{F}_1(R,I)$–reflexive* if each F in $\mathcal{F}_1(R,I)$ is I–reflexive.

On the other hand, when R is an integral domain and I is a nonzero ideal of R, Bazzoni and Salce [BS-1996] introduced the following notion:

▶ the integral domain R is said to be *I–reflexive* (respectively, *I–divisorial*), if each I–torsionless $\text{Hom}_R(I, I)$–module of finite rank (respectively, of rank 1) is I–reflexive.

Proposition 12 *Let R be an integral domain and I a nonzero ideal of R. Then R is $\mathcal{F}(R, I)$–reflexive (respectively, $\mathcal{F}_1(R, I)$–reflexive) if and only if R is I–reflexive (respectively, I–divisorial) and $R = (I : I)$.*

□

Note that *if R is $\mathcal{F}_1(R,I)$–reflexive (in particular, if R is $\mathcal{F}(R,I)$–reflexive), then I is an m–canonical ideal of R*, since each regular ideal J of R belongs to $\mathcal{F}_1(R,I)$.

[We have already observed that each regular ideal J in R is I–torsionless. Moreover, if J' is a regular fractional ideal of R, then for some regular element $d \in R$, $dJ' =: J$ is a regular ideal in R and $J' = d^{-1}J = d^{-1}(I : (I : J)) =$
$= (I : d(I : J)) = (I : (I : d^{-1}J)) = (I : (I : J'))$.]

Theorem 13 *Let* R *be a ring admitting an regular ideal* I *such that* R *is* $\mathcal{F}(R,I)$*–reflexive.*
Set $T := R\bowtie I$ *and* $I_T := \mathrm{Hom}_R(T, I)$,
then T *is* $\mathcal{F}(T, I_T)$*–reflexive and* I_T *is isomorphic as* T*-module to* T*.* □

As a consequence we obtain:

Corollary 14 *Let* R *be a ring admitting a regular ideal* I *such that* R *is* $\mathcal{F}(R,I)$*–reflexive.*
Then every regular fractional ideal of T $(= R\bowtie I)$ *is divisorial.* □

Corollary 15 *Let R be a Noetherian local integral domain and let I be an m–canonical ideal of R and set $T := R\bowtie I$.*
Then T is a local reduced Noetherian ring, $\dim(R) = \dim(T)$ *and every regular fractional ideal of T is divisorial.*

Final Remark 16 Marco D'Anna in the context of Cohen-Macaulay rings has proved the following:

Let R be a Cohen-Macaulay local ring and let I be a proper ideal of R.
Then $R\bowtie I$ is Gorenstein if and only if R has a canonical module ω_R such that $I \cong \omega_R$.

Example 17 Let k be a field and let $R := k[\![X^4, X^6, X^{11}, X^{13}]\!]$ and let $I := (X^{10}, X^{12}, X^{17})$.

Then it can be shown that the numerical semigroup

$$S := \{0, 4, 6, 8, 10 \rightarrow \cdots\}$$

canonically associated to R has a canonical semigroup

$$C := \{x \in \mathbb{Z} \mid 9 - x \notin S\} = \{0, 2, 4, 6, 7, 8, 10 \rightarrow \cdots\}$$

which gives rise to a proper canonical ideal in S by considering:

$$10 + C = \{10, 12, 14, 16, 17, 18, 20 \rightarrow \cdots\}.$$

Since this canonical ideal in S is generated by $\{10, 12, 17\}$, then we deduce that I is a canonical ideal of R (since the value semigroup $v(I) = 10 + C$, Jäger [J-1977]).
In this case, the ring:

$$R\bowtie I = \left\{(f, g) \mid f, g \in k[\![X^4, X^6, X^{11}, X^{13}]\!] \text{ and } f - g \in I\right\}$$

is a 1-dimensional local reduced Gorenstein ring, dominating R.
(Note that one can reach the same conclusion by showing that the subsemigroup U of $\mathbb{N} \times \mathbb{N}$ associated to $R\bowtie I$ is symmetric.)

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