Chapter 8

LOCALIZING SYSTEMS AND SEMISTAR OPERATIONS

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INTRODUCTION

In 1994 A. Okabe and R. Matsuda [22] introduced the notion of semistar operation; see also, [21] and [20]. This concept extends the classical concept of star operation, as developed in Gilmer's book [12], and hence the related classical theory of ideal systems based on the works of W. Krull, E. Noether, H. Prüfer, and P. Lorenzen from the 1930's. For a systematic treatment of these ideas, see the books by P. Jaffard [17] and F. Halter-Koch [14], where a complete and updated bibliography is available.

The purpose of the present work is to establish a natural bridge between localizing systems and semistar operations that are stable under finite intersections. Localizing systems are also called topologizing systems or Gabriel's filters; Bourbaki [4, Ch.2, §2, Ex. 17-25] and [9, §5.1]; see also, Stenström [25]. Some of the main results here extend those obtained by Garcia, Jara, and Santos in [11], where the case of star operations stable under finite intersections were first investigated. (They called these operations half-centered star operations, emphasizing the relation with the half-centered hereditary torsion theories.)

The semistar operations stable under finite intersections have an important role in the theory of semistar operations. As a matter of fact, to each semistar operation $\star$ we can associate in a natural way a semistar operation $\overline{\star}$, stable under finite intersections. It turns out that $\overline{\star}$ is the largest semistar opera ion with the finite intersection property that precedes $\star$. Furthermore, the semistar operations that arise naturally by taking the intersections of localizations over a fixed ring R satisfy this stability property.

We extend several results obtained for star operations by D.D. Anderson [1], D.D. Anderson and D.F. Anderson [2], and D.D. Anderson and S.J. Cook [3] to semistar operations.

This paper is the first step in developing a systematic foundation for the theory of semistar operations linked to that of localizing systems. In this setting, more work needs to be done along the lines of the papers by R. Matsuda [19], and A. Okabe and R. Matsuda [23] before a general theory of Kronecker function rings can be developed. This approach would permit a relaxation of the classical restrictions on R (not necessarily integrally closed) and on $\star$ (not necessarily endlich arithmetisch brauchbar) and would fit nicely with the characterization of Kronecker function rings recently developed by Halter-Koch [13].

1. SEMISTAR OPERATIONS

Let R be an integral domain with quotient field K. Let $\overline{\mathbf{F}}(R)$ denote the set of all nonzero R-submodules of K and let $\mathbf{F}(R)$ be the set of all nonzero fractional ideals of R, i.e., all $E \in \overline{\mathbf{F}}(R)$ such that there exists a nonzero $d \in R$ with $dE \subseteq R$. Let $\mathbf{f}(R)$ be the set of nonzero finitely generated R-submodules of K. Then $\mathbf{f}(R) \subseteq \mathbf{F}(R) \subseteq \overline{\mathbf{F}}(R)$.

A mapping

$$\overline{\mathbf{F}}(R) \to \overline{\mathbf{F}}(R), \quad E \mapsto E^\star$$

is called a *semistar operation on* R if for all $x \in K$, $x \neq 0$, and $E, F \in \overline{\mathbf{F}}(R)$:

($\star_1$) $(xE)^\star = xE^\star$;

($\star_2$) $E \subseteq E^\star$, and $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

($\star_3$) $E^{\star\star} = E^\star$.

If $E \in \overline{\mathbf{F}}(R)$, then $E^\star \in \overline{\mathbf{F}}(R^\star) \subseteq \overline{\mathbf{F}}(R)$. The R-submodules of K belonging to

$$\overline{\mathbf{F}}^\star(R) = \{E^\star : E \in \overline{\mathbf{F}}(R)\}$$

are called *semistar R-modules of* K. Similarly, we can consider

$$\mathbf{F}^\star(R) = \{I^\star : I \in \mathbf{F}(R)\} \quad \text{and} \quad \mathbf{f}^\star(R) = \{J^\star : J \in \mathbf{f}(R)\}.$$

It is easy to see that $\mathbf{F}^\star(R) \subseteq \mathbf{F}(R^\star)$, but in general $\mathbf{F}(R^\star) \not\subseteq \mathbf{F}(R)$, since $(R : R^\star)$ could be the zero ideal. A (fractionary) ideal I of R is called a *(fractionary) semistar ideal of* R, if $I \in \mathbf{F}^\star(R)$.

Remark 1.0. (a) Let $F \in \overline{\mathbf{F}}(R)$ and $J \in \mathbf{F}(R)$. Then F (respectively, J) is a semistar R-module (respectively, ideal) of K (respectively, R) if and only if $F = F^\star$ (respectively, $J = J^\star$).

(b) Note that in general $\overline{\mathbf{F}}^\star(R) \subsetneqq \overline{\mathbf{F}}(R^\star)$ and $\mathbf{F}^\star(R) \subsetneqq \mathbf{F}(R^\star)$. Let (V, M) be a 1-dimensional nondiscrete valuation domain with quotient field K. Let $\star$ be the canonical v operation as defined in [12]. Note that any valuation domain is a conducive domain; i.e., for each V-submodule E of K, $(V : E) \neq 0$, [6, Theorem 4.5] or [5, Theorem 1]. Therefore $\overline{\mathbf{F}}(V) - \mathbf{F}(V) = K$. Also, the nondivisorial ideals of V are of the form xM, where $0 \neq x \in K$, [9, Proposition 4.2.5]. Thus $M \neq M_v = V$ and $V = V_v$. Clearly $\overline{\mathbf{F}_\mathbf{v}}(V) \subsetneqq \overline{\mathbf{F}}(V_v) = \overline{\mathbf{F}}(V)$ and $\mathbf{F}_\mathbf{v}(V) \subsetneqq \mathbf{F}(V_v) = \mathbf{F}(V)$.

Remark 1.1. Let $\star$ be a semistar operation on R. Assume that $R = R^\star$. Then for each $E \in \mathbf{F}(R)$, $E^\star \in \mathbf{F}(R)$. In fact, if $0 \neq d \in R$ such that $dE \subseteq R$, then $dE^\star = (dE)^\star \subseteq R^\star = R$. In this case the semistar operation $\star$, restricted to $\mathbf{F}(R)$, defines a star operation [12, §32].

A semistar operation $\star$ on R is *proper* if $R \subsetneqq R^\star$.

In order to give some examples of proper semistar operations, we need to establish some basic properties of these operations. It is apparent from the definition that semistar operations may have many of the properties of "classical" star c; erations. On the other hand, since we do not assume that $R^\star = R$, we will see that they have other interesting properties.

The proof of the next theorem is similar to the proof given in [12, Proposition 32.2 and Theorem 32.5].

Theorem 1.2. *(A) Let $\star$ be a semistar operation on R. Then for all $E, F \in \overline{\mathbf{F}}(R)$ and for every family $\{E_i : i \in I\}$ of elements in $\overline{\mathbf{F}}(R)$:*

(1) $(\sum_{i \in I} E_i)^\star = (\sum_{i \in I} E_i^\star)^\star$;

(2) $\cap_{i \in I} E_i^\star = (\cap_{i \in I} E_i^\star)^\star$, *if* $\cap_{i \in I} E_i^\star \neq (0)$;

(3) $(EF)^\star = (E^\star F^\star)^\star = (EF^\star)^\star = (E^\star F)^\star$.

(B) If S is an overring of R, then $S^\star$ is an overring of $R^\star$. In particular, $R^\star$ is an overring of R.

(C) Let $\mathcal{R} = \{R_\alpha : \alpha \in A\}$ be a family of overrings of R. Then $E \mapsto E^\star$, where $E^\star = \cap_{\alpha \in A} ER_\alpha$ is a semistar operation on R. Moreover, $E^\star R_\alpha = ER_\alpha$ for each $\alpha \in A$. □

The semistar operation given by Theorem 1.2(C) is called the *semistar operation defined by* $\mathcal{R}$ and will be denoted by $\star_{\mathcal{R}}$. It is obvious that $\star_{\mathcal{R}}$ is a proper semistar operation if and only if $\cap_{\alpha \in A} R_\alpha \supsetneqq R$. In particular, if S is a proper overring of R and $\mathcal{R} = \{S\}$ we write $\star_{\{S\}}$ instead of $\star_{\mathcal{R}}$.

We list some other examples of semistar operations.

Example 1.3. (a) Set $E^e = K$, for each $E \in \overline{\mathbf{F}}(R)$, then the map $E \mapsto E^e$ defines a trivial semistar operation that is proper when $R \neq K$, called the *e-operation*. It is obvious that $e = \star_{\{K\}}$.

(b) The map $E \mapsto E^d = E$, for each $E \in \overline{F}(R)$ defines a trivial semistar operation called the *d-operation*. Clearly $d = \star_{\{R\}}$.

(c) For each $E \in \overline{\mathbf{F}}(R)$, set $E^{-1} = (R : E) = \{x \in K : xE \subseteq R\}$. The map $E \mapsto E_v = (E^{-1})^{-1}$ is a semistar operation, called the *v-operation on* R, such that $R_v = R$. By Remark 1.1, this operation when restricted to $\mathbf{F}(R)$ is the classical (star) v-operation on R.

(d) If $\mathcal{R} = \{R_\alpha : \alpha \in A\}$ is a family of overrings of R and if for each $\alpha \in A$, $\star_\alpha$ is a semistar operation on R_α, then $E \mapsto E^{\star_A}$, where $E^{\star_A} = \cap_{\alpha \in A}(ER_\alpha)^{\star_\alpha}$, defines a semistar operation on R. It is easy to see that $(E^{\star_A}R_\alpha)^{\star_\alpha} = (ER_\alpha)^{\star_\alpha}$, for each $E \in \overline{\mathbf{F}}(R)$ and for each $\alpha \in A$. Note that this example generalizes the construction of Theorem 1.2(C) (take $\star_\alpha = d$, for each $\alpha \in A$). This construction, in the case of star operations, was considered by D.D. Anderson [1, Theorem 2].

A semistar operation $\star$ on R is said to be of *finite type*if, for each $E \in \overline{\mathbf{F}}(R)$, $E^\star = \cup\{F^\star : F \in \mathbf{f}(R), F \subseteq E\}$. The e-operation and the d-operation are semistar operations of finite type.

PROBLEM: Find conditions for $\star_A$ to be of finite type.

For each semistar operation $\star$ on R, a semistar operation of finite type can be defined in the following way: $E \mapsto E^{\star_f}$, where $E^{\star_f} = \cup\{F^\star : F \in \mathbf{f}(R)$ with $F \subseteq E\}$, for each $E \in \overline{\mathbf{F}}(R)$. The operation $\star_f$ is called *the finite semistar operation associated to* $\star$. Obviously, $\star$ is a semistar operation of finite type $\Leftrightarrow \star = \star_f$.

Example 1.4. If we consider the (semistar) v-operation, then the finite semistar operation associated to v is called the (semistar) *t-operation*, where for each $E \in \overline{\mathbf{F}}(R)$ $E_t = \cup\{F_v : F \in \mathbf{f}(R)$ with $F \subseteq E\}$.

Note that if $E \in \overline{\mathbf{F}}(R) - \mathbf{F}(R)$, $E^{-1} = (0)$ then, since $E \in \overline{F}(R) - F(R)$ if and only if, $E^{-1} = 0$, we have $E_v = K$. Hence the (semistar) v-operation is an example of a semistar operation extended "trivially" by the star operation v. More precisely:

Remark 1.5. (a) Let $\star$ be a star operation on an integral domain R. For each $E \in \overline{\mathbf{F}}(R)$, let

$$E^{\star_e} = E^\star, \quad \text{if } E \in \mathbf{F}(R),$$
$$E^{\star_e} = K, \quad \text{if } E \in \overline{\mathbf{F}}(R) - \mathbf{F}(R).$$

Then the map $E \mapsto E^{\star_e}$ defines a semistar operation on R such that $R^{\star_e} = R$. This is called *the trivial semistar extension of* $\star$. The mapping $\star \mapsto \star_e$ determines a canonical embedding of the set of all star operations on R to the set of all semistar operations on R.

(b) Note that $R \neq K$ if and only if $K \in \overline{\mathbf{F}}(R) - \mathbf{F}(R)$. It is obvious that for each semistar operation $\star$ on R, $K^\star = K$. An example of a semistar operation that is not trivally extended by a star operation is given by $\star_{\{S\}}$ where S is an overring of R such that $(R : S) = 0$ with $S \neq K$.

Define a partial ordering on the set of semistar operations that are defined on R in the following way:

$$\star_1 \leq \star_2 \Leftrightarrow E^{\star_1} \subseteq E^{\star_2} \text{ for each } E \in \overline{\mathbf{F}}(R).$$

We say that $\star_1$ *is equivalent to* $\star_2$ and write $\star_1 \sim \star_2$, if $(\star_1)_f = (\star_2)_f$; i.e., $F^{\star_1} = F^{\star_2}$ for each $F \in \mathbf{f}(R)$.

The proof of the following is straightforward (cf. with [2, p. 1623] and [22, Propositions 13 and 15]).

Proposition 1.6. *Let* $\star, \star_1$, *and* $\star_2$ *be semistar operations on* R.

(1) If $S \subseteq T$ *are overrings of* R, *then* $d = \star_{\{R\}} \leq \star_{\{S\}} \leq \star_{\{T\}} \leq \star_{\{K\}} = e$.

(2) $\star_f \leq \star$ *and* $\star_f \sim \star$.

(3) $\star_1 \leq \star_2 \Rightarrow (\star_1)_f \leq (\star_2)_f$.

(4) The following are equivalent:

(i) $\star_1 \leq \star_2$;

(ii) $(E^{\star_1})^{\star_2} = E^{\star_2}$, *for each* $E \in \overline{\mathbf{F}}(R)$;

(iii) $(E^{\star_2})^{\star_1} = E^{\star_2}$, *for each* $E \in \overline{\mathbf{F}}(R)$;

(iv) $\overline{\mathbf{F}}^{\star_2}(R) \subseteq \overline{\mathbf{F}}^{\star_1}(R)$.

(5) When $R^\star = R$, *then* $(E^\star)^{-1} = E^{-1}$ *for each* $E \in \overline{\mathbf{F}}(R)$, *hence* $\star \leq v$ *and* $\star_f \leq t$. □

The previous proposition shows that for each overring S of R, $\star_{\{S\}}$ is the smallest semistar operation on R such that $R^\star = S$; and v (respectively, t)

is the largest semistar operation (respectively, semistar operation of finite type) on R such that $R = R^\star$.

Mutatis mutandis, the statements (2), (3), (4), and (5) of Proposition 1.6 hold for star operations (cf. [12, p. 395 and Theorem 34.1], [2, p. 1623]).

If $\star$ is a semistar operation on R, then it is easy to see that

$(\star_4)$ $\quad (E \cap F)^\star \subseteq E^\star \cap F^\star$, for each $E, F \in \overline{\mathbf{F}}(R)$;

$(\star_5)$ $\quad (E : F)^\star \subseteq (E^\star : F^\star) = (E^\star : F)$, for each $E, F \in \overline{\mathbf{F}}(R)$.

We say that the *semistar operation $\star$ is stable* if

$$(\star_{\mathbf{st}}) \quad (E \cap F)^\star = E^\star \cap F^\star, \text{ for each } E, F \in \overline{\mathbf{F}}(R).$$

Remark 1.7. Let $\star$ be a semistar operation on R.

(a) If $(E :_R F) = \{x \in R : xF \subseteq E\}$, then $(E :_R F)^\star \subseteq (E^\star :_{R^\star} F^\star)$.

(b) If $\star$ is stable, then

$$(\star_{\mathbf{st}}{}') \quad (E :_R F)^\star = (E^\star :_{R^\star} F^\star), \text{ for each } E \in \overline{\mathbf{F}}(R) \text{ and for each } F \in \mathbf{f}(R).$$

Proof. The proof of (a) is straightforward, and (b) follows easily by showing that $(E : F)^\star = (E^\star : F^\star)$. □

We will show in the next section that the converse also holds; i.e., $(\star_{\mathbf{st}}) \Leftrightarrow (\star_{\mathbf{st}}{}')$ (Theorem 2.10(B)). Therefore we will have a characterization of stable semistar operations analogous to that proved by D.D. Anderson and S.J. Cook [3] for star operations.

Example 1.8. Let $\mathcal{R} = \{R_\alpha : \alpha \in A\}$ be a family of flat overrings of an integral domain. Then $\star_{\mathcal{R}}$ is a stable semistar operation; since for each $E, F \in \overline{\mathbf{F}}(R)$ and for each $\alpha \in A$, $(E \cap F)R_\alpha = ER_\alpha \cap FR_\alpha$ [4, Ch.1,§2,N.6].

Note that the v-operation is not stable. As a matter of fact, even if R is Noetherian and I, J are two integral ideals of R, it can happen that $(I \cap J)_v \subsetneq I_v \cap J_v$. For instance, if k is a field and $R = k[[x^3, x^4, x^5]]$, then R is a 1-dimensional Noetherian local domain with maximal ideal $M = (x^3, x^4, x^5)$. Let $I = (x^3, x^4)$ and $J = (x^3, x^5)$. Then $I_v = J_v = M_v = M$, but $I \cap J = (x^3)$ and $(I \cap J)_v = (I \cap J) = (x^3) \subsetneq (I_v \cap J_v) = M$.

2. LOCALIZING SYSTEMS AND SEMISTAR OPERATIONS

An hereditary torsion theory for a commutative ring R is characterized by the family $\mathcal{F}$ of the ideals I of R for which R/I is a torsion module (for more details cf. [25, Ch. VI]). It turns out that such a family $\mathcal{F}$ of ideals is the family of the neighborhoods of 0 for a certain linear topology of R; the notion of localizing system (or topologizing system) was introduced (in a more general context) by P. Gabriel in order to characterize such topologies from an ideal-theoretic point of view (cf. for instance with [4, Ch. II, §2, Exercises 17-25]).

A *localizing system $\mathcal{F}$ of an integral domain R* is a family of integral ideals of R such that

(LS1) If $I \in \mathcal{F}$ and J is an ideal of R such that $I \subseteq J$, then $J \in \mathcal{F}$;

(LS2) If $I \in \mathcal{F}$ and J is an ideal of R such that $(J :_R iR) \in \mathcal{F}$ for each $i \in I$, then $J \in \mathcal{F}$.

Note that axioms **(LS1)** and **(LS2)** ensure, in particular, that $\mathcal{F}$ is a filter. Moreover, axiom **(LS2)** is linked to the fact that the linear topology corresponding to a hereditary torsion theory has the property that the class of discrete modules is closed under extensions [25, Ch. VI, §5]. More precisely, from an ideal-theoretic point of view, when considering the exact sequence

$$0 \to \frac{I}{J \cap I} \to \frac{R}{J} \to \frac{R}{I + J} \to 0,$$

(LS2) ensures that if $R/(I+J)$ and $I/(J \cap I)$ belong to the torsion class $\mathcal{T}$ associated to $\mathcal{F}$ then also R/J belongs to $\mathcal{T}$.

To avoid uninteresting cases, assume that a *localizing system $\mathcal{F}$ is nontrivial*, i.e., $(0) \notin \mathcal{F}$ and $\mathcal{F}$ is nonempty. It is easy to see that if $I, J \in \mathcal{F}$, then $IJ \in \mathcal{F}$ (and, thus, $I \cap J \in \mathcal{F}$). If K is the quotient field of R, then

$$R_{\mathcal{F}} = \{x \in K : (R :_R xR) \in \mathcal{F}\} = \cup\{(R : I) : I \in \mathcal{F}\},$$

and $R_{\mathcal{F}}$ is an overring of R called *the ring of fractions with respect to $\mathcal{F}$*. If $E \in \overline{\mathbf{F}}(R)$, then $E_{\mathcal{F}} = \{x \in K : (E :_R xR) \in \mathcal{F}\} = \cup\{(E : I) : I \in \mathcal{F}\}$ belongs to $\overline{F}(R_{\mathcal{F}})$. For instance, if S is a multiplicative subset of R, then $\mathcal{F} = \{I \text{ ideal of } R : I \cap S = \emptyset\}$ is a localizing system of R and $R_{\mathcal{F}} = S^{-1}R$. In particular, let P be a prime ideal of R and $\mathcal{F}(P) = \{I : I \text{ ideal of } R, I \nsubseteq P\}$. Then $\mathcal{F}(P)$ is a localizing system and $R_{\mathcal{F}(P)} = R_P$.

Since the intersection of a family $\{\mathcal{F}_\alpha : \alpha \in A\}$ of localizing systems of R is still a localizing system, $\mathcal{F} = \cap\{\mathcal{F}_\alpha : \alpha \in A\}$ is a localizing system and it follows that $R_{\mathcal{F}} = \cap\{R_{\mathcal{F}_\alpha} : \alpha \in A\}$. Specializing this idea, we see that if $\Delta \subseteq \operatorname{Spec}(R)$, then $\mathcal{F}(\Delta) = \cap\{\mathcal{F}(P) : P \in \Delta\}$ is a localizing system and $R_{\mathcal{F}(\Delta)} = \cap\{R_P : P \in \Delta\}$, [9, Lemma 5.1.2 and Proposition 5.1.4].

If $\mathcal{F}' \subseteq \mathcal{F}''$ are two localizing systems of R, then $R_{\mathcal{F}'} \subseteq R_{\mathcal{F}''}$ [9, Lemma 5.1.3]; but it may happen that $\mathcal{F}' \subsetneq \mathcal{F}''$ and $R_{\mathcal{F}'} = R_{\mathcal{F}''}$.

Example 2.1. If V is a valuation domain and P is a nonzero idempotent prime ideal of V, then $\hat{\mathcal{F}}(P) = \{I : I \text{ ideal of } V \text{ and } I \supseteq P\}$ is a localizing system of V and

$$\hat{\mathcal{F}}(P) \supsetneq \{I : I \text{ ideal of } V \text{ and } I \supsetneq P\} = \mathcal{F}(P).$$

Moreover, $V_{\hat{\mathcal{F}}(P)} = V_P = V_{\mathcal{F}(P)}$, [9, Proposition 5.1.12].

Remark 2.2. Prüfer domains R for which $\mathcal{F}' \neq \mathcal{F}''$ implies $R_{\mathcal{F}'} \neq R_{\mathcal{F}''}$ coincide with generalized Dedekind domians, which were introduced by Popescu [24]; see also [9, §§5.2 and 5.4].

Lemma 2.3. *If $\mathcal{F}$ is a localizing system of an integral domain R, then*

(1) $(E \cap F)_{\mathcal{F}} = E_{\mathcal{F}} \cap F_{\mathcal{F}}$, for each $E, F \in \overline{\mathbf{F}}(R)$;

(2) $(E : F)_{\mathcal{F}} = (E_{\mathcal{F}} : F_{\mathcal{F}})$, for each $E \in \overline{\mathbf{F}}(R)$ and for each $F \in \mathbf{f}(R)$;

(3) $(E_{\mathcal{F}} : F) = (E_{\mathcal{F}} : F_{\mathcal{F}})$, for each $E, F \in \overline{\mathbf{F}}(R)$.

Proof. (1) This follows from the fact that $((E \cap F) : I) = (E : I) \cap (F : I)$.

(2) By the proof of Remark 1.7(b), (1)⇒(2).

(3) This is straightforward. □

Proposition 2.4. *Let $\mathcal{F}$ be a localizing system of an integral domain R. For each $E \in \overline{\mathbf{F}}(R)$, the map $E \mapsto E_{\mathcal{F}} = \cup_{J \in \mathcal{F}}(E : J)$ is a stable semistar operation on R.*

Proof. It is obvious that the map $E \mapsto E_{\mathcal{F}}$ satisfies properties $(\star_1)$ and $(\star_2)$. Let $x \in (E_{\mathcal{F}})_{\mathcal{F}}$, then $xJ \subseteq E_{\mathcal{F}}$ for some $J \in \mathcal{F}$. Therefore, for each $j \in J$, there exists $K_j \in \mathcal{F}$ such that $xj \in (E : K_j)$, hence $((E :_R xR) :_R jR) \supseteq K_j \in \mathcal{F}$. From **(LS2)**, $(E :_R xR) \in \mathcal{F}$; that is $x \in E_{\mathcal{F}}$. □

Remark 2.5. (a) In general $E_{\mathcal{F}} \supsetneq ER_{\mathcal{F}}$ even if E is a proper integral ideal of R. For instance, let V be a valuation domain with idempotent maximal ideal M, of the type $V = K + M$, where K is a field. Let k be a proper subfield of K and define $R = k + M$. Since M is idempotent it is easy to see that $\mathcal{F} = \{M, R\}$ is a localizing system of R. Then $M_{\mathcal{F}} = R_{\mathcal{F}} = (M : M) = V$ and $MR_{\mathcal{F}} = MV = M$.

(b) Recall that if I is a nonzero ideal contained in R, then $I_{\mathcal{F}} = R_{\mathcal{F}}$ if and only if $I \in \mathcal{F}$ [10, Lemma 1.1(a)].

Denote the semistar operation $E \mapsto E_{\mathcal{F}}$ by $\star_{\mathcal{F}}$. By Remark 2.5(a), $\star_{\mathcal{F}}$ is in general different from $\star_{\{R_{\mathcal{F}}\}}$; more precisely $\star_{\{R_{\mathcal{F}}\}} \leq \star_{\mathcal{F}}$. The following result characterizes when the equality holds.

Proposition 2.6. *Let $\mathcal{F}$ be a localizing system of an integral domain R. The following are equivalent:*

(i) $\star_{\{R_{\mathcal{F}}\}} = \star_{\mathcal{F}}$;

(ii) $IR_{\mathcal{F}} = I_{\mathcal{F}}$ for each integral ideal I of R;

(iii) $R_{\mathcal{F}}$ is R-flat and $\mathcal{F} = \{I : I \text{ ideal of } R \text{ and } IR_{\mathcal{F}} = R_{\mathcal{F}}\}$.

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii). If $I \in \mathcal{F}$, then $I_{\mathcal{F}} = R_{\mathcal{F}}$ and hence $IR_{\mathcal{F}} = R_{\mathcal{F}}$. This implies that $R_{\mathcal{F}}$ is R-flat, [9, Remark 5.1.11(b)].

(iii)⇒(i). Since $ER_{\mathcal{F}} \subseteq E_{\mathcal{F}}$, for each $E \in \overline{\mathbf{F}}(R)$, we need to show the opposite inclusion. Because $R_{\mathcal{F}}$ is R-flat, we know that $\mathcal{F}_0 = \{I \text{ ideal of } R : IR_{\mathcal{F}} = R_{\mathcal{F}}\}$ is a localizing system of $R, R_{\mathcal{F}_0} = R_{\mathcal{F}}$ and $\mathcal{F}_0 \subseteq \mathcal{F}$, [9, Proposition 5.1.10 and Remark 5.1.11(a)]. By flatness, we also have:

$$\begin{aligned} x \in E_{\mathcal{F}_0} \quad &\Leftrightarrow \quad (E :_R xR) \in \mathcal{F}_0 \Leftrightarrow (E :_R xR)R_{\mathcal{F}_0} = R_{\mathcal{F}_0} \\ &\Leftrightarrow \quad (ER_{\mathcal{F}_0} :_{R_{\mathcal{F}_0}} xR_{\mathcal{F}_0}) = R_{\mathcal{F}_0} \Leftrightarrow x \in ER_{\mathcal{F}_0}, \end{aligned}$$

for each $E \in \overline{\mathbf{F}}(R)$. Since we are also assuming that $\mathcal{F} = \mathcal{F}_0$, we conclude that $E^{\star_{\mathcal{F}}} = E_{\mathcal{F}} = ER_{\mathcal{F}} = E^{\star_{\{R_{\mathcal{F}}\}}}$. □

Remark 2.7. The condition that $R_{\mathcal{F}}$ is R-flat is not equivalent to (i) and (ii) in the previous result. Let V, P, and $\hat{\mathcal{F}}(P)$ be as in Example 2.1, then $V_{\hat{\mathcal{F}}(P)} = V_P$ and $PV_{\hat{\mathcal{F}}(P)} = PV_P = P$. Moreover, $P_{\hat{\mathcal{F}}(P)} = (P : P) = V_P$, since $P \in \hat{\mathcal{F}}(P)$ (Remark 2.5(b)).

Proposition 2.8. *If $\star$ is a semistar operation on R, then $\mathcal{F}^\star = \{I : I \text{ ideal of } R \text{ with } I^\star \cap R = R\}$ is a localizing system of R.*

Proof. The property **(LS1)** is obviously true for $\mathcal{F}^\star$. Let I be an ideal of R and $J \in \mathcal{F}^\star$ such that $(I :_R xR)^\star \cap R = R$ for each $x \in J$. Then

$$\begin{aligned} R \quad &= \quad (I :_R xR)^\star \cap R = (I : xR)^\star \cap R \subseteq (I : xR)^\star \cap R^\star \cap R \\ &\subseteq \quad (I^\star : xR^\star) \cap R = (I^\star :_{R^\star} xR^\star) \cap R \end{aligned}$$

Thus $x \in I^\star$, for each $x \in J$. Therefore $J \subseteq I^\star \cap R$ and hence $I^\star \cap R \in \mathcal{F}^\star$. This implies that $I^\star \cap R = R$; i.e., $I \in \mathcal{F}^\star$. □

$\mathcal{F}^\star$ is called the *localizing system associated to* $\star$.

Remark 2.9. If $E = I$ is a nonzero integral ideal of R, then the following statements are equivalent:

(i) $I^\star \cap R \subsetneq R$;

(ii) $I^\star \subsetneq R^\star$.

In particular, we have:

$$\begin{aligned}\mathcal{F}^\star &= \{I : I \text{ ideal of } R \text{ with } I^\star \cap R = R\}\\ &= \{I : I \text{ ideal of } R \text{ with } I^\star = R^\star\}.\end{aligned}$$

Theorem 2.10. *(A) Let $\mathcal{F}$ be a localizing system of an integral domain R and let $\star_{\mathcal{F}}$ be the semistar operation on R associated with $\mathcal{F}$. Then*

$$\mathcal{F} = \mathcal{F}^{\star_{\mathcal{F}}} = \{I \text{ ideal of } R : I_{\mathcal{F}} \cap R = R\}.$$

(B) Let $\star$ be a semistar operation on R and let $\mathcal{F}^\star$ be the localizing system associated with $\star$. Then

$$\star_{\mathcal{F}^\star} \leq \star.$$

Moreover, the following are equivalent:

(i) $\star_{\mathcal{F}^\star} = \star$;

(ii) $\star$ is a stable semistar operation;

(iii) $(E :_R F)^\star = (E^\star :_{R^\star} F^\star)$, for each $E \in \overline{\mathbf{F}}(R)$ and for each $F \in \mathbf{f}(R)$;

(iv) $(E :_R xR)^\star = (E^\star :_{R^\star} xR^\star)$ for each $E \in \overline{\mathbf{F}}(R)$ and for each $0 \neq x \in K$.

Proof. (A) Note that for each nonzero ideal I of R, $I_{\mathcal{F}} \cap R = R \Leftrightarrow I_{\mathcal{F}} = R_{\mathcal{F}} \Leftrightarrow I \in \mathcal{F}$, see Remark 2.5(b).

(B) Let $E \in \overline{\mathbf{F}}(R)$ and $x \in E^{\star_{\mathcal{F}^\star}} = E_{\mathcal{F}^\star}$. Then $(E :_R xR) \in \mathcal{F}^\star$, hence:

$$\begin{aligned}R &= R \cap (E :_R xR)^\star = R \cap ((E : xR) \cap R)^\star\\ &\subseteq R \cap (E : xR)^\star \cap R^\star \subseteq R \cap (E^\star :_{R^\star} xR^\star).\end{aligned}$$

Thus, $1 \in (E^\star :_{R^\star} xR^\star)$ which implies that $x \in E^\star$.

For the second part of (B), (i)$\Rightarrow$(ii) by Proposition 2.4, (ii)$\Rightarrow$(iii) by Remark 1.7(b), and (iii)$\Rightarrow$(iv) is trivial. To show that (iv)$\Rightarrow$(i), we need only prove that $E^\star \subseteq E^\star_{\mathcal{F}^\star}$, for each $E \in \overline{\mathbf{F}}(R)$. Let $x \in E^\star$, then

$$R = R \cap (E^\star :_{R^\star} xR^\star) = R \cap (E :_R xR)^\star.$$

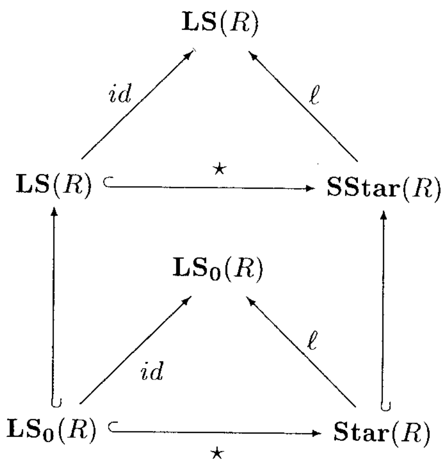
Therefore, $(E :_R xR) \in \mathcal{F}^\star$; i.e., $x \in E_{\mathcal{F}^\star}$. □

Let $\mathbf{SStar}(R)$ (respectively, $\mathbf{Star}(R)$) be the set of all the semistar (respectively, star) operations defined on R and let $\mathbf{LS}(R)$ (respectively, $\mathbf{LS}_0(R)$),

denote the set of localizing systems of R (respectively, localizing systems $\mathcal{F}$ of R such that $R_{\mathcal{F}} = R$).

Corollary 2.11. *The canonical map $\star : \mathbf{LS}(R) \to \mathbf{SStar}(R)$ (respectively, $\star : \mathbf{LS}_0(R) \to \mathbf{Star}(R)$), $\mathcal{F} \mapsto \star_{\mathcal{F}}$ is injective and order preserving. The image of this map is the set of all stable semistar (respectively, stable star) operations.*

Proof. Consider the following diagram



where $\star : \mathbf{LS}(R) \to \mathbf{SStar}(R)$ is defined by $\mathcal{F} \mapsto \star_{\mathcal{F}}$, $\ell : \mathbf{SStar}(R) \to \mathbf{LS}(R)$ is defined by $\star \mapsto \mathcal{F}^\star$, *id* denotes the identity map and the second vertical embedding is defined in Remark 1.5(a). It is easy to see that if $\mathcal{F} \in \mathbf{LS}_0(R)$, then $\star_{\mathcal{F}} \in \mathbf{Star}(R)$ (when restricted to $\mathbf{F}(R)$); and if $\star \in \mathbf{Star}(R)$, then $\mathcal{F}^\star \in \mathbf{LS}_0(R)$ (because $x \in R_{\mathcal{F}^\star} \Rightarrow (R :_R xR) \in \mathcal{F}^\star \Rightarrow (R :_R xR)^\star = R \Rightarrow (R :_R xR) = R \Rightarrow x \in R$). By Theorem 2.10(A) the diagram commutes and by Theorem 2.10(B) the map $\star$ defines a bijection with the set $\mathbf{SStar}_{st}(R)$ of all stable semistar operations defined on R. It is straightforward to see that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ implies $\star_{\mathcal{F}_1} \leq \star_{\mathcal{F}_2}$ and $\star_1 \leq \star_2$ implies that $\mathcal{F}^{\star_1} \subseteq \mathcal{F}^{\star_2}$. □

The relation between hereditary torsion theories (or, equivalently, localizing systems) and star operations was first noted by J.M. Garcia Hernandez in [16]. The previous corollary generalizes [11, Theorem 2].

3. FINITENESS CONDITIONS

A *localizing system of finite type* $\mathcal{F}$, defined on an integral domain R, is a localizing system such that for each $I \in \mathcal{F}$ there exists a nonzero finitely generated ideal $J \in \mathcal{F}$ with $J \subseteq I$. For instance, if T is an overring of R, $\mathcal{F}(T) = \{I : I \text{ ideal of } R, IT = T\}$ is a localizing system of finite type; in particular, if T is R-flat, then $R_{\mathcal{F}(T)} = T$ [9, Proposition 5.1.10]. Example 2.1 is not of finite type.

Lemma 3.1. *Given a localizing system $\mathcal{F}$ of an integral domain R, then*

$$\mathcal{F}_f = \{I \in \mathcal{F} : I \supseteq J, \text{ for some nonzero finitely generated ideal } J \in \mathcal{F}\}$$

is a localizing system of finite type of R.

Proof. Since $\mathcal{F}_f$ obviously satisfies **(LS1)**, we concentrate on **(LS2)**. Let I be an ideal of R such that for some nonzero finitely generated ideal $J \in \mathcal{F}$, $(I :_R xR) \in \mathcal{F}_f$ for each $x \in J$. In particular, if $J = \xi_1 R + \cdots + \xi_n R$, then there exists a finitely generated $H_i \in \mathcal{F}$ such that $H_i \subseteq (I :_R \xi_i R)$ for $i = 1, 2, \cdots, n$. If $H = \Pi_{i=1}^n H_i$, then $H \in \mathcal{F}_f$ and $H\xi_i \subseteq I$ for each i. Therefore $HJ \subseteq I$ with $HJ \in \mathcal{F}_f$. This implies that $I \in \mathcal{F}_f$. □

Proposition 3.2. *Let $\mathcal{F}$ be a localizing system and $\star$ a semistar operation defined on an integral domain R.*

(1) If $\mathcal{F}$ is of finite type, then $\star_{\mathcal{F}}$ is of finite type.

(2) If $\star$ is of finite type, then $\mathcal{F}^\star$ is of finite type.

Proof. (1) Let $E \in \overline{\mathbf{F}}(R)$ and $x \in E^{\star_{\mathcal{F}}} = E_{\mathcal{F}}$. Then there exists a finitely generated ideal $J \in \mathcal{F}$ such that $xJ \subseteq E$. Thus $x \in (xJ : J) \subseteq (xJ)_{\mathcal{F}} \subseteq E_{\mathcal{F}}$, where $xJ \in \mathbf{f}(R)$, and hence $E_{\mathcal{F}} = \cup\{F_{\mathcal{F}} : F \in \mathbf{f}(R)\}$.

(2) Let I be an ideal of $\mathcal{F}^\star$, then $I^\star \cap R = R$. Since $\star$ is of finite type, for some finitely generated ideal $J \subseteq I$, $J^\star \cap R = R$. Therefore $J \in \mathcal{F}^\star$. □

We denote by $\mathbf{LS_f}(R)$ (respectively, $\mathbf{LS_{0f}}(R)$) the set of localizing systems of R of finite type (respectively, of finite type $\mathcal{F}$ such that $R_{\mathcal{F}} = R$). Denote by $\mathbf{SStar_f}(R)$ (respectively, $\mathbf{Star_f}(R)$) the set of semistar (respectively, star) operations of R of finite type.

Proposition 3.3. *Let R be an integral domain, let $\star : \mathbf{LS}(R) \to \mathbf{SStar}(R)$ be the map defined in Corollary 2.11, and let f be defined by the following two maps*

$$f \;:\; \mathbf{LS}(R) \to \mathbf{LS_f}(R), \quad \mathcal{F} \mapsto \mathcal{F}_f$$
$$f \;:\; \mathbf{SStar}(R) \to \mathbf{SStar_f}(R), \quad \star \mapsto \star_f.$$

Consider the following diagrams:

$$\begin{array}{ccc} \mathbf{LS}(R) & \xrightarrow{\star} & \mathbf{SStar}(R) \\ f\downarrow & & \downarrow f \\ \mathbf{LS_f}(R) & \xrightarrow{\star} & \mathbf{SStar_f}(R) \end{array} \qquad \begin{array}{ccc} \mathcal{F} & \longmapsto & \star_{\mathcal{F}} \\ \downarrow & & \downarrow \\ \mathcal{F}_f & \longmapsto & \star_{\mathcal{F}_f}, (\star_{\mathcal{F}})_f. \end{array}$$

Then $\star_{\mathcal{F}_f} \leq (\star_{\mathcal{F}})_f$, for each $\mathcal{F} \in \mathbf{LS}(R)$.

Proof. Since $\mathcal{F}_f \subseteq \mathcal{F}$, then $\star_{\mathcal{F}_f} \leq \star_{\mathcal{F}}$, by Corollary 2.11. Moreover, $\star_{\mathcal{F}_f}$ is a semistar operation of finite-type by Proposition 3.2. Hence $\star_{\mathcal{F}_f} = (\star_{\mathcal{F}_f})_f \leq (\star_{\mathcal{F}})_f$. □

For sake of simplicity, if we retain the same notation for the maps $\star$ and f, when restricted to $\mathbf{LS_0}(R)$ and $\mathbf{Star}(R)$, then:

Corollary 3.4. *Consider the diagram*

$$\begin{array}{ccc} \mathbf{LS_0}(R) & \xrightarrow{\star} & \mathbf{Star}(R) \\ f\downarrow & & \downarrow f \\ \mathbf{LS_{0f}}(R) & \xrightarrow{\star} & \mathbf{Star_f}(R). \end{array}$$

Then, for each $\mathcal{F} \in \mathbf{LS_0}(R)$, $\star_{\mathcal{F}_f} \leq (\star_{\mathcal{F}})_f$. □

PROBLEM. Characterize $\mathcal{F} \in \mathbf{LS}(R)$ such that $\star_{\mathcal{F}_f} = (\star_{\mathcal{F}})_f$. (Obviously $\mathcal{F} = \mathcal{F}_f \Rightarrow \star_{\mathcal{F}_f} = (\star_{\mathcal{F}})_f$.)

We note that, in general $\star_{\mathcal{F}_f} \lneqq (\star_{\mathcal{F}})_f$ as the following example shows.

Example 3.5. Let R, V, M, and $\mathcal{F}$ be as in Remark 2.5(a). Then $\mathcal{F}_f = \{R\}$ and $I^{\star_{\mathcal{F}_f}} = (I : R) = I$, for each ideal I of R. On the other hand, $I^{(\star_{\mathcal{F}})_f} = \cup\{J^{\star_{\mathcal{F}}} : J \text{ finitely generated ideal}, J \subseteq I\} = \cup\{(J : M) : J \text{ finitely generated ideal}, J \subseteq I\}$. For instance, if $I = xR$, for some nonzero $x \in R$, then $(xR)^{\star_{\mathcal{F}_f}} = xR \subsetneqq (xR)^{(\star_{\mathcal{F}})_f} = x(R : M) = xV$.

If we start with a semistar operation $\star$ on R, then we can consider the following semistar operations associated with $\star$:

$$\bar{\star} = \star_{\mathcal{F}^\star}, \quad \tilde{\star} = (\star)_{(\mathcal{F}^\star)_f}$$

where for each $E \in \overline{\mathbf{F}}(R)$,

$$\begin{aligned} E^{\bar{\star}} &= E_{\mathcal{F}^\star} = \cup\{(E : I) : I \in \mathcal{F}^\star\}, \\ E^{\tilde{\star}} &= E_{(\mathcal{F}^\star)_f} = \cup\{(E : J) : J \in \mathcal{F}^\star, J \text{ finitely generated}\}. \end{aligned}$$

It follows from Corollary 2.11, that $\bar{\bar{\star}} = \bar{\star}$, $\tilde{\tilde{\star}} = \tilde{\star}$, and $\tilde{\star} \leq \bar{\star}$; moreover, $\star_1 \leq \star_2$ implies that $\bar{\star}_1 \leq \bar{\star}_2$ and $\tilde{\star}_1 \leq \tilde{\star}_2$.

Note that when $\star$ is a star operation, then $\bar{\star}$ and $\tilde{\star}$ are star operations that coincide respectively with the star operations $\bar{\star}$ and $\star_w$ introduced by D.D. Anderson and S.J. Cook [3, §2]. In particular, the notion of $\star_w$ star operation construction generalizes the w-operation by Wang Fanggui, and R.L. McCasland in [7].

Proposition 3.6. *Let $\star$ be a semistar operation on an integral domain R, then:*

(a) $(\bar{\star})_f \leq \star_f \leq \star$ *and* $(\bar{\star})_f \leq \bar{\star} \leq \star$.

(b) $\widetilde{(\star_f)} = \overline{(\star_f)} = \tilde{\star}$.

Proof. Statement (a) follows from Propostion 1.6(2) and (3), and Theorem 2.10(B). We already have observed that $\tilde{\star} \leq \bar{\star}$. By Proposition 3.2(1), $\tilde{\star}$ is of finite type, hence $\tilde{\star} \leq (\bar{\star})_f$. By the previous considerations, $\widetilde{(\star_f)} \leq \overline{(\star_f)}$ and $\widetilde{(\star_f)} \leq \tilde{\star}$. Let $E \in \overline{\mathbf{F}}(R)$ and $x \in E^{\tilde{\star}} = E^{\star(\mathcal{F}^\star)_f}$. Then there exists a finitely generated ideal J of R such that $xJ \subseteq E$ and $J^\star \cap R = R$. Since J is finitely generated, $J^{\star_f} = J^\star$ and thus $J \in \mathcal{F}^{\star_f}$, hence $x \in E^{\widetilde{(\star_f)}}$; i.e., $\tilde{\star} \leq \widetilde{(\star_f)}$. If $y \in E^{\overline{(\star_f)}}$, then $yI \subseteq E$ for some I such that $I^{\star_f} \cap R = R$. Since $I^{\star_f} = \cup\{J^\star : J \text{ is a finitely generated ideal}, J \subseteq I\}$, then necessarily for some finitely generated $J, J^\star \cap R = R$. Thus $yJ \subseteq E$ with J finitely generated, $J \subseteq I$, and $J \in \mathcal{F}^{\star_f}$. Hence $I \in (\mathcal{F}^{\star_f})_f$ and whence $y \in E^{\widetilde{(\star_f)}}$; i.e., $\overline{(\star_f)} \leq \widetilde{(\star_f)}$. □

Proposition 3.7. *Let $\star$ be a semistar operation on an integral domain R. Then*

(1) $\bar{\star}$ is the largest stable semistar operation on R that precedes $\star$; in particular $\star$ is stable if and only if $\star = \bar{\star}$;

(2) if $\star'$ is a semistar operation such that $\bar{\star} \leq \star' \leq \star$, then $\mathcal{F}^{\bar{\star}} = \mathcal{F}^{\star'} = \mathcal{F}^\star$.

Proof. (1) Since $\bar{\star}$ is associated to a localizing system, $\bar{\star}$ is stable by Proposition 2.4. Moreover, if $\star'$ is a stable semistar operation preceding $\star$, then $\star' = \overline{(\star')} \leq \bar{\star}$.

(2) Clearly $\mathcal{F}^{\bar{\star}} \subseteq \mathcal{F}^{\star'} \subseteq \mathcal{F}^\star$. The conclusion follows from Theorem 2.10(A), because $\mathcal{F}^{\bar{\star}} = \mathcal{F}^{\star_{\mathcal{F}^\star}} = \mathcal{F}^\star$. □

Corollary 3.8. *Let $\star$ be a semistar operation on an integral domain R. Then the diagram of inclusions of semistar operations described in Proposition 3.6 gives rise to the following inclusions of localizing systems:*

$$\mathcal{F}^{\star_f} = \mathcal{F}^{\tilde{\star}} = \mathcal{F}^{\widetilde{(\star_f)}} = \mathcal{F}^{\overline{(\star_f)}} = \mathcal{F}^{(\bar{\star})_f} \subseteq \mathcal{F}^{\bar{\star}} = \mathcal{F}^\star.$$

Moreover, $\mathcal{F}^{\star_f} = (\mathcal{F}^\star)_f$.

Proof. By Proposition 3.7 and Proposition 3.6, we need only prove that $\mathcal{F}^{\star_f} = (\mathcal{F}^\star)_f$. From Proposition 3.2, $\mathcal{F}^{\star_f}$ is a localizing system of finite type. Thus $\mathcal{F}^{\star_f} = (\mathcal{F}^{\star_f})_f \subseteq (\mathcal{F}^\star)_f$. If $I \in (\mathcal{F}^\star)_f$, then there exists a finitely generated ideal $J \subseteq I$ such that $R = J^\star \cap R = J^{\star_f} \cap R$. Hence $J \in \mathcal{F}^{\star_f}$, which implies that $I \in \mathcal{F}^{\star_f}$. □

Corollary 3.9. *Let $\star$ be a semistar operation on an integral domain R. Then:*

(1) $\bar{\star} = \tilde{\star} \Leftrightarrow \bar{\star}$ *is a semistar operation of finite type;*

(2) $\star = \tilde{\star} \Leftrightarrow \star$ *is a stable semistar operation of finite type;*

(3) $\tilde{\bar{\star}} = \tilde{\star} = \bar{\tilde{\star}}$;

(4) $\star_f = \tilde{\star} \Leftrightarrow \star_f = \overline{(\star_f)}$.

Proof. (1) Note that $\bar{\star}$ and $\tilde{\star}$ are stable semistar operations by Proposition 2.4. Thus, it follows from Corollary 3.8, that $\bar{\star} = \star_{\mathcal{F}^{\bar{\star}}} = \star_{\mathcal{F}^\star}$ and $\tilde{\star} = \star_{\mathcal{F}^{\tilde{\star}}} = \star_{\mathcal{F}^{\star_f}} = \star_{\mathcal{F}^{(\bar{\star})_f}}$. By Corollary 2.11, Proposition 3.3, and Corollary 3.8, we conclude that $\bar{\star} = \tilde{\star}$ if and only if $\mathcal{F}^\star = \mathcal{F}^{\star_f} = (\mathcal{F}^\star)_f$; i.e., $\bar{\star}$ is of finite type.

(2) This follows from (1), since $\star = \tilde{\star}$ if and only if $\star = \bar{\star}$ and $\bar{\star} = \tilde{\star}$.

(3) This is a consequence of Corollary 3.8 and Corollary 2.11.

(4) This follows from Proposition 3.6. □

It is easy to show, *mutatis mutandis*, Proposition 3.6 and 3.7, and Corollaries 3.8 and 3.9 hold also when restricted to star operations, and localizing systems $\mathcal{F}$ of R with the property that $R_{\mathcal{F}} = R$.

With the notation introduced above consider the following diagram (cf. Corollary 2.11 and Proposition 3.3).

$$\textbf{(3.10)}\quad \begin{array}{ccccccccc} \mathbf{SStar}(R) & \xrightarrow{\ell} & \mathbf{LS}(R) & \xrightarrow{\star} & \mathbf{SStar}(R) & \star & \longmapsto & \mathcal{F}^\star & \longmapsto & \bar{\star} \\ \downarrow f & & \downarrow f & & \downarrow f & \downarrow & & \downarrow & & \downarrow \\ \mathbf{SStar_f}(R) & \xrightarrow{\ell} & \mathbf{LS_f}(R) & \xrightarrow{\star} & \mathbf{SStar_f}(R) & \star_f & \longmapsto & \mathcal{F}^{\star_f} & \longmapsto & \tilde{\star} \leq (\bar{\star})_f \end{array}$$

Since the left square is always commutative (Corollary 3.8), by Corollary 3.9 *the following properties are equivalent*:

(i) *diagram (3.10) commutes;*

(ii) *the right side of diagram (3.10) commutes;*

(iii) $\bar{\tilde{\star}} = \tilde{\star}$, *for each* $\star \in$ **SStar**(R);

(iv) $\bar{\tilde{\star}} \in$ **SStar**$_{\mathfrak{f}}(R)$.

The following example (cf. with [3]) will show that, in general, $\tilde{\star} = \overline{(\star_f)} \lneqq (\bar{\star})_f$.

Example 3.11. A semistar operation $\star$ such that $\overline{(\star_f)} \lneqq (\bar{\star})_f$. Let R be an integrally closed domain and $\{V_\alpha : \alpha \in A\}$ be the family of all valuation overrings of R. For each $E \in \overline{\mathbf{F}}(R)$, set $E_b = \cap\{EV_\alpha : \alpha \in A\}$. The map $E \mapsto E_b$ defines a semistar operation on R with $R_b = R$, Theorem 1.2(C). If R is an essential domain; i.e., if each V_α is a localization of R at its center in R, then the b-operation is stable, Example 1.8. By Proposition 1.6(5), $b \le v$. We want to show that, as in the star case [12, Proposition 44.13], $b \sim v$. Let $F = x_1R + \cdots + x_nR \in \mathbf{f}(R)$ and let $\xi \in F_v$. We show that $\xi \in F_b$. Assume $\xi \notin F_b$, then $\xi \notin FV_\alpha = FR_{P_\alpha}$ for some $\alpha \in A$, where $P_\alpha = V_\alpha \cap R$. Then $FR_{P_\alpha} = xR_{P_\alpha}$ for some $x \in F$, because FR_{P_α} is an invertible ideal of the valuation domain R_{P_α}. Hence $xR_{P_\alpha} \subsetneqq \xi R_{P_\alpha}$ [12, Theorem 16.3]. For each generator x_i of F, write $x_i = xr_it^{-1}$ where $r_i \in R$ and $t \in R\backslash P_\alpha$, for every α. Thus $F \subseteq xt^{-1}R$ and therefore $F_v \subseteq xt^{-1}R$. On the other hand $\xi \notin xt^{-1}R$ and thus $\xi \notin F_v$. This contradiction proves that $\xi \in F_b$.

By assuming that R is essential, we have deduced that $b_f = v_f = t$ and that $b = \tilde{b}$. By Kang's theorem of characterizations of the integrally closed domains that are $PvMD$-domains ([18, Chapter 2] or [1, Theorem 6]), we know that if R is an integrally closed, but not a $PvMD$-domain, then t is not stable; i.e., $t \gneqq \tilde{t}$. Therefore, if R is an essential domain, but not a $PvMD$-domain, then $b \ge (\bar{b})_f = b_f = t \gneqq \tilde{t} = \overline{(b_f)} = \tilde{b}$. (An explicit example of an integral domain of this type was given by Heinzer and Ohm [15].)

4. SPECTRAL LOCALIZING SYSTEMS AND SEMISTAR OPERATIONS

Let Δ be a nonempty set of prime ideals of an integral domain R. For each $E \in \overline{\mathbf{F}}(R)$, define $E^{\star_\Delta} = \cap\{ER_P : P \in \Delta\}$. In the following to avoid trivial cases the set Δ is always assumed to be a nonempty set.

Lemma 4.1. *(1) The mapping $E \mapsto E^{\star_\Delta}$, for each $E \in \overline{\mathbf{F}}(R)$, defines a semistar operation on R.*

(2) For each $E \in \overline{\mathbf{F}}(R)$ and for each $P \in \Delta$, $ER_P = E^{\star_\Delta}R_P$.

(3) $\star_\Delta$ is a stable semistar operation.

(4) For each $P \in \Delta$, $P^{\star_\Delta} \cap R = P$.

(5) For each integral ideal I of R such that $I^{\star_\Delta} \cap R \neq R$, there exists a prime ideal $P \in \Delta$ such that $I \subseteq P$.

Proof. (1) and (2) are particular cases of Theorem 1.2 (C).

(3) If $E, F \in \overline{\mathbf{F}}(R)$, then by flatness:

$$\begin{aligned} E^{\star_\Delta} \cap F^{\star_\Delta} &= (\cap_{P\in\Delta} ER_P) \cap (\cap_{P\in\Delta} FR_P) = \cap_{P\in\Delta}(ER_P \cap FR_P) \\ &= \cap_{P\in\Delta}(E\cap F)R_P = (E \cap F)^{\star_\Delta}. \end{aligned}$$

(4) This is obvious, since $P^{\star_\Delta} = \cap\{PR_Q : Q \in \Delta \text{ and } Q \supseteq P\}$.

(5) If $I^{\star_\Delta} \cap R \neq R$, then $I^{\star_\Delta} \subsetneqq R^{\star_\Delta}$ and $I^{\star_\Delta}R_P \neq R_P$, for some $P \in \Delta$. Therefore, $I^{\star_\Delta}R_P \subseteq PR_P$ and hence $I \subseteq I^{\star_\Delta} \subseteq I^{\star_\Delta}R_P \cap R \subseteq PR_P \cap R = P$. □

Note that, *mutatis mutandis*, the properties stated in Lemma 4.1 hold for star operations [2, Theorem 1].

A *semistar operation* $\star$, defined on an integral domain R, is *spectral* if there exists $\Delta \subseteq \text{Spec}(R)$ such that $\star = \star_\Delta$. In this case, we will say that $\star$ is the *spectral semistar operation associated with* Δ. Say that $\star$ *possesses enough primes*, or $\star$ is a *quasi-spectral semistar operation*, if for each integral ideal I of R such that $I^\star \cap R \neq R$, there exists a prime ideal P of R with $I \subseteq P$ and $P^\star \cap R = P$. Lemma 4.1(5) shows that a spectral semistar operation is quasi-spectral. We will see in Example 4.16 that the converse is not true in general.

Note that the map $\Delta \mapsto \star_\Delta$ is contravariant; i.e., $\Delta_1 \subseteq \Delta_2 \Rightarrow \star_{\Delta_2} \le \star_{\Delta_1}$.

Recall that a *localizing system* $\mathcal{F}$ of R is called *spectral* if there exists a set of prime ideals Δ of R such that $\mathcal{F} = \mathcal{F}(\Delta) = \cap\{\mathcal{F}(P) : P \in \Delta\}$, where $\mathcal{F}(P)$ is the localizing system $\{I : I \text{ ideal of } R, I \nsubseteq P\}$, [9, p. 127 and Proposition 5.1.7]. Note that, as in the semistar operation case, $\Delta_1 \subseteq \Delta_2 \Rightarrow \mathcal{F}(\Delta_2) \subseteq \mathcal{F}(\Delta_1)$.

Given $\mathcal{F}$, we can consider the following subset of Spec(R) associated to $\mathcal{F}$:

$$\Phi = \Phi_{\mathcal{F}} = \{P \in \text{Spec}(R) : P \notin \mathcal{F}\}.$$

If $\mathcal{F}$ is nontrivial, Φ is nonempty and we can consider $\mathcal{F}_{sp} = \mathcal{F}(\Phi)$ which is called the *spectral localizing system associated to* $\mathcal{F}$. It is easy to see that

$$\mathcal{F} \subseteq \mathcal{F}_{sp}, \quad \text{and} \quad \mathcal{F}_1 \subseteq \mathcal{F}_2 \Rightarrow (\mathcal{F}_1)_{sp} \subseteq (\mathcal{F}_2)_{sp}.$$

Lemma 4.2. *Let Δ be a nonempty set of prime ideals of an integral domain. Then*

$$\star_\Delta = \star_{\mathcal{F}(\Delta)} \text{ and } \mathcal{F}^{\star_\Delta} = \mathcal{F}(\Delta);$$

i.e., the (spectral) semistar operation $\star_\Delta$ is associated to the (spectral) localizing system $\mathcal{F}(\Delta)$ and conversely.

Proof. Since $\star_\Delta$ is stable (Lemma 4.1(3)),

$$\star_\Delta = \overline{(\star_\Delta)} = \star_{\mathcal{F}^{\star_\Delta}}.$$

Moreover,

$$\begin{aligned} \mathcal{F}^{\star_\Delta} &= \{I : I \text{ ideal of } R \text{ such that } (\cap_{P\in\Delta} IR_P) \cap R = R\} \\ &= \{I : I \text{ ideal of } R \text{ such that } IR_P = R_P, \text{ for each } P \in \Delta\} \\ &= \{I : I \text{ ideal of } R \text{ such that } I \not\subseteq P \text{ for each } P \in \Delta\} \\ &= \mathcal{F}(\Delta). \end{aligned}$$

□

Proposition 4.3. *Let $\mathcal{F}$ be a nontrivial localizing system of an integral domain R.*

(A) The following properties are equivalent:

- *(i) $\mathcal{F}$ is a spectral localizing system;*
- *(ii) $\mathcal{F} = \mathcal{F}_{sp}$;*
- *(iii) for each ideal I of R, with $I \notin \mathcal{F}$, there exists a prime ideal P of R such that $I \subseteq P$ and $P \notin \mathcal{F}$.*

(B) The following properties are equivalent:

- *(j) $\mathcal{F}$ is a localizing system of finite type;*
- *(jj) there exists a quasi-compact subspace Δ of* Spec(R) *such that $\mathcal{F} = \mathcal{F}(\Delta)$;*
- *(jjj) $\Phi_{\max} = \{P \in \text{Spec}(R) : P \notin \mathcal{F}$ and it is maximal with respect to this property $\}$ is quasi-compact and $\mathcal{F} = \mathcal{F}(\Phi_{\max})$.*

Proof. [9, (5.1f), and Propositions 5.1.7 and 5.1.8]. □

Corollary 4.4. *(1) If $\star$ is a spectral semistar operation, then $\mathcal{F}^\star$ is a spectral localizing system.*

(2) If $\mathcal{F}$ is a spectral localizing system, then $\star_{\mathcal{F}}$ is a spectral semistar operation.

Proof. This is a consequence of Lemma 4.2. □

Remark 4.5. If Δ is a nonempty subset of prime ideals of an integral domain R and if

$$\Delta^\downarrow = \{Q \in \text{Spec}(R) : Q \subseteq P, \text{ for some } P \in \Delta\},$$

then it is easy to see that for each Λ such that $\Delta \subseteq \Lambda \subseteq \Delta^\downarrow$, $\mathcal{F}(\Delta) = \mathcal{F}(\Lambda)$ (cf. [8, Lemma 1.3]). In particular, from Lemma 4.2, $\star_\Delta = \star_{\Delta^\downarrow}$. Furthermore, if $\Delta_{\max}$ is the set of maximal elements of Δ and if for each $P \in \Delta$, there exists $Q \in \Delta_{\max}$ with $P \subseteq Q$; then $\star_\Delta = \star_{\Delta_{\max}}$.

Corollary 4.6. *Let $\star$ be the spectral semistar operation defined on an integral domain R that is associated to a nonempty subspace Δ of* Spec(R). *Let $\nabla = \{P \in \text{Spec}(R) : P \notin \mathcal{F}(\Delta)\}$ and let $\nabla_{\max}$ be the set of maximal elements in ∇. Then*

(1) $\star_\Delta = \star = \star_\nabla$.

(2) The following are equivalent:

- *(i) $\star_\Delta$ is a semistar operation of finite type;*
- *(ii) $\mathcal{F}(\Delta)$ is a localizing system of finite type;*
- *(iii) $\nabla_{\max}$ is quasi-compact and $\star = \star_{\nabla_{\max}}$;*
- *(iv) there exists a quasi-compact subspace F of* Spec(R) *such that $\star_\Delta = \star_F$.*

Proof. (1) This follows from the fact that $\mathcal{F}(\Delta) = \mathcal{F}(\nabla)$, [9, 5.1f, p. 128], and Lemma 4.2.

(2) The equivalence (i)⇔(ii) is a consequence of Lemma 4.2 and Proposition 3.2.

(ii)⇔(iii) By Proposition 4.3(B), $\mathcal{F}(\Delta)$ is of finite type if and only if $\mathcal{F}(\Delta) = \mathcal{F}(\nabla_{\max})$ and $\nabla_{\max}$ is quasi-compact. The conclusion follows by Lemma 4.2.

(iii)⇒(iv) Trivial.

(iv)⇒(ii) Since $\star_\Delta = \star_F$, $\star_{\mathcal{F}(\Delta)} = \star_{\mathcal{F}(F)}$ (Lemma 4.2), and so $\mathcal{F}(\Delta) = \mathcal{F}(F)$, Theorem 2.10(A). The conclusion follows from Proposition 4.3(B). □

Remark 4.7. Note that $\star_\Delta$ is a semistar operation of finite type provided that the representation $R^{\star_\Delta} = \cap\{R_P : P \in \Delta\}$ is locally finite; i.e., each nonzero $z \in R^{\star_\Delta}$ is a unit in almost all $R_P, P \in \Delta$. As a matter of fact, this condition implies that Δ is quasi-compact. To see this, let $\{y_\lambda : \lambda \in \Lambda\}$ be a family of elements of R such that $\Delta \subseteq \cup_{\lambda\in\Lambda} D(y_\lambda)$. By assumption, for a given $y_{\overline{\lambda}}$, there exists at most a finite set $\{P_1, \ldots, P_t\} \subseteq \Delta$ such that $\Delta - \{P_1, \cdots, P_t\} \subseteq D(y_{\overline{\lambda}})$. If $P_i \in D(y_i)$ for $1 \le i \le t$, then clearly $\Delta \subseteq D(y_{\overline{\lambda}}) \cup D(y_1) \cup \ldots \cup D(y_t)$.

In case $R = R^{\star_\Delta} = \cap\{R_P : P \in \Delta\}$ is locally finite, D.D. Anderson [1, Theorem 1(6)] proved that $\star_\Delta$ defines on R a star operation of finite type.

As we did for localizing systems, we can try to associate, in some canonical way, to each a semistar operation a spectral semistar operation.

Given a semistar operation $\star$ defined on an integral domain R, consider the set
$$\Pi^\star = \{P \in \mathrm{Spec}(R) : P \neq 0 \text{ and } P^\star \cap R \neq R\}.$$
If $\Pi^\star$ is nonempty, we can consider the spectral semistar operation
$$\star_{sp} = \star_{\Pi^\star}$$
called *the spectral semistar operation associated to* $\star$. By Lemma 4.2, $\mathcal{F}^{\star_{sp}} = \mathcal{F}(\Pi^\star)$.

It is easy to see that:
$$\star_1 \leq \star_2 \Rightarrow \Pi^{\star_2} \subseteq \Pi^{\star_1} \Rightarrow (\star_1)_{sp} \leq (\star_2)_{sp};$$
in particular, $(\bar{\star})_{sp} \leq \star_{sp}$.

Our next goal is to study the relation between $\star$ and $\star_{sp}$.

Proposition 4.8. *Let $\star$ be a semistar operation defined on an integral domain R. Assume that $\Pi^\star \neq \emptyset$. The following statements are equivalent:*

(i) $\star_{sp} \leq \star$;

(ii) $\star$ *is quasi-spectral;*

(iii) $E^\star = \cap\{E^\star R_P : P \in \Pi^\star\}$, *for each* $E \in \overline{\mathbf{F}}(R)$.

Proof. (i)$\Rightarrow$(ii). Suppose that for each ideal I of R, with $I^\star \cap R \neq R$, we have $I \not\subseteq P$ for each $P \in \Pi^\star$. Then $IR_P = R_P$, for each $P \in \Pi^\star$. Thus $I^{\star_{sp}} = \cap\{IR_P : P \in \Pi^\star\} = \cap\{R_P : P \in \Pi^\star\} = R^{\star_{sp}}$. By assumption $I^{\star_{sp}} \subseteq I^\star$, hence $I^\star \cap R = R$, a contradiction.

(ii)$\Rightarrow$(iii). For each $P \in \Pi^\star$, let $z = xy^{-1} \in E^\star R_P$ where $x \in E^\star$ and $y \in R - P$. Whence, $z^{-1}E^\star \cap R \not\subseteq P$, since $y \in z^{-1}E^\star \cap R$. Furthermore, it is easy to see that $J = z^{-1}E^\star \cap R$ is a semistar ideal of R; i.e., $J^\star = J$. This leads to a contradiction, since $\star$ is quasi-spectral.

(iii)$\Rightarrow$(i). This follows since $E^{\star_{sp}} = \cap\{ER_P : P \in \Pi^\star\}$ for each $E \in \overline{\mathbf{F}}(R)$. □

Remark 4.9. If $\star$ is a quasi-spectral semistar operation defined on an integral domain R such that $R^\star$ is not a field, then $\Pi^\star \neq \emptyset$. To see this choose $x \in R$ that is not a unit of $R^\star$. Then $xR^\star \cap R \neq R$, hence there exists a prime ideal P of R with $P^\star \cap R = P$ (hence $P \in \Pi^\star$) and $xR^\star \cap R \subseteq P$.

Corollary 4.10. *Let $\star$ be a semistar operation defined on an integral domain R. Then $\star$ is spectral if and only if $\star = \star_{sp}$.*

Proof. It is obvious that if $\star = \star_{sp}$, then $\star$ is spectral.

Conversely, if $\star = \star_\Delta$ for some $\Delta \subseteq \mathrm{Spec}(R)$, then by Lemma 4.2, $\star_\Delta = \star_{\mathcal{F}(\Delta)}$ and $\star_{sp} = \star_{\mathcal{F}(\Pi^\star)}$. Since it is easy to see that $\Delta \subseteq \Pi^\star$, then $\mathcal{F}(\Pi^\star) \subseteq \mathcal{F}(\Delta)$. Let $I \notin \mathcal{F}(\Pi^\star)$, then $I \subseteq P$ for some $P \in \Pi^\star$. Thus, $I \subseteq I^\star \cap R \subseteq P^\star \cap R \subsetneq R$. We claim that $I \notin \mathcal{F}(\Delta)$. If $I \in \mathcal{F}(\Delta)$, then for each $Q \in \Delta$, $I \not\subseteq Q$ and thus $IR_Q = R_Q$. Hence $I^\star = I^{\star_\Delta} = R^{\star_\Delta}$ and so $I^\star \cap R = R$, a contradiction.

By the previous considerations, $\mathcal{F}(\Pi^\star) = \mathcal{F}(\Delta)$ and thus $\star_{\mathcal{F}(\Pi^\star)} = \star_{\mathcal{F}(\Delta)}$. □

Recall that given a semistar operation $\star$ on R, we have introduced the following localizing systems on R associated with $\star$:
$$\mathcal{F}^\star = \{I : I \text{ ideal of } R \text{ with } I \neq 0 \text{ and } I^\star \cap R = R\}$$
and when $\Pi^\star \neq \emptyset$,
$$\mathcal{F}(\Pi^\star) = \{I : I \text{ ideal of } R \text{ with } I \not\subseteq P, \text{ for each } P \in \Pi^\star\}.$$
The semistar operations associated to these localizing systems are, respectively, $\bar{\star}$ and $\star_{sp}$.

Proposition 4.11. *Let $\star$ be a semistar operation defined on an integral domain R. Assume that $\Pi^\star \neq \emptyset$, then:*

(1) $\mathcal{F}^\star \subseteq \mathcal{F}(\Pi^\star)$ *and* $(\mathcal{F}^\star)_{sp} = \mathcal{F}^{\star_{sp}}$*;*

(2) $\bar{\star} \leq \star_{sp}$*;*

(3) If $\star$ is spectral, then $\mathcal{F}^\star = \mathcal{F}(\Pi^\star)$ (and hence $\bar{\star} = \star_{sp}$).

Proof. (1) If $I \in \mathcal{F}^\star$, then $I^\star \cap R = R$. Thus, for each $P \in \Pi^\star$, $I^\star \not\subseteq P^\star$ and hence $I \not\subseteq P$; i.e., $I \in \mathcal{F}(\Pi^\star)$. Note that $\mathcal{F}^{\star_{sp}} = \mathcal{F}^{\star_{\Pi^\star}} = \mathcal{F}^{\star_{\mathcal{F}(\Pi^\star)}} = \mathcal{F}(\Pi^\star)$ (Lemma 4.2) and $(\mathcal{F}^\star)_{sp} = \mathcal{F}(\Phi^\star)$ where
$$\Phi^\star = \{P \in \mathrm{Spec}(R) : P \notin \mathcal{F}^\star\}.$$
It is obvious that $\Pi^\star \subseteq \Phi^\star$, hence $(\mathcal{F}^\star)_{sp} \subseteq \mathcal{F}^{\star_{sp}}$.

If $I \notin \mathcal{F}(\Phi^\star) = (\mathcal{F}^\star)_{sp}$, then $I \subseteq Q$ for some prime ideal $Q \notin \mathcal{F}^\star$, hence $Q^\star \cap R \neq R$, so $Q \in \Pi^\star$. We conclude that $I \notin \mathcal{F}(\Pi^\star) = \mathcal{F}^{\star_{sp}}$, whence $(\mathcal{F}^\star)_{sp} = \mathcal{F}^{\star_{sp}}$.

(2) This follows from (1) because $\bar{\star} = \star_{\mathcal{F}^\star}$ and $\star_{sp} = \star_{\mathcal{F}(\Pi^\star)}$.

(3) We know by Corollary 4.10 that $\star$ is spectral if and only if $\star = \star_{sp}$, and hence $\mathcal{F}^\star = \mathcal{F}^{\star_{sp}} = \mathcal{F}(\Pi^\star)$. □

In the next result, among other facts, we will show that the converse of Proposition 4.11(3) does not hold in general.

Theorem 4.12. *Let $\star$ be a semistar operation defined on an integral domain R and assume that $\Pi^\star \neq \emptyset$.*

(1) $\star_{sp} = \overline{(\star_{sp})}$.

(2) The following properties are equivalent:

(i) $\star$ *is quasi-spectral;*

(ii) $\star_{sp} = \overline{\star}$*;*

(iii) $\overline{\star}$ *is spectral;*

(iv) $\mathcal{F}^\star = \mathcal{F}(\Pi^\star)$*;*

(v) $\mathcal{F}^\star$ *is spectral.*

(3) The semistar operation $\star$ is spectral if and only if it is quasi-spectral and stable.

Proof. (1) Since $\star_{sp} = \star_{\mathcal{F}(\Pi^\star)}$, apply Proposition 2.4 and Proposition 3.7(1).

(2) (i)$\Rightarrow$(ii) If $\star$ is quasi-spectral, then Proposition 4.8 implies that $\star_{sp} \leq \star$. On the other hand, $\overline{\star} \leq \star_{sp}$ by Proposition 4.11 (2). Using (1) we have $\star_{sp} = \overline{(\star_{sp})} \leq \overline{\star} \leq \star_{sp}$.

(ii)$\Rightarrow$(iii) and (iv)$\Rightarrow$(v) are trivial.

(ii)$\Rightarrow$(iv) Since $\star_{sp} = \overline{\star}$, then by Proposition 3.7, Lemma 4.2, and Theorem 2.10(A), $\mathcal{F}^\star = \mathcal{F}^{\overline{\star}} = \mathcal{F}^{\star_{sp}} = \mathcal{F}^{\star_{\mathcal{F}(\Pi^\star)}} = \mathcal{F}(\Pi^\star)$.

(v)$\Rightarrow$(iii) Apply Lemma 4.2.

(iii)$\Rightarrow$(i) Let I be an integral ideal of R such that $I^\star \cap R \neq R$. By Propositions 3.7 and 4.11(3), $\mathcal{F}^\star = \mathcal{F}^{\overline{\star}} = \mathcal{F}(\Pi^{\overline{\star}})$. Since $\mathcal{F}^\star$ and $\mathcal{F}^{\overline{\star}}$ are spectral, Proposition 4.3(A) implies that $\mathcal{F}^\star = \mathcal{F}^{\overline{\star}} = \mathcal{F}(\Phi^\star)$, where $\Phi^\star = \{P \in \text{Spec}(R) : P \notin \mathcal{F}^\star\}$. Since $I \notin \mathcal{F}^\star = \mathcal{F}(\Phi^\star)$, there exists $P \in \Phi^\star$ with $I \subseteq P$. Because $P \notin \mathcal{F}^\star$, $P^\star \cap R \neq R$ and hence $P \in \Pi^\star$. Therefore $\star$ is quasi-spectral.

(3) By Lemma 4.1((3) and (5)), $\star$ is quasi-spectral and stable. Conversely, by the previous statement (2) and Proposition 3.7, $\star = \overline{\star} = \star_{sp}$. The conclusion follows from Corollary 4.10. □

From a historical point of view concerning the statement (3) of the previous theorem, [1, Theorem 4] gives necessary and sufficient conditions for a star operation to be spectral; [22, Theorem 22] gives the result for semistar operations.

Corollary 4.13. *(A) Let $\mathcal{F}$ be a nontrivial localizing system defined on an integral domain R, then*

$$(\star_{\mathcal{F}})_{sp} = \star_{\mathcal{F}_{sp}}.$$

In other words, the following diagram

$$(sp - \mathbf{LS}) \qquad \begin{array}{ccc} \mathbf{LS}(R) & \xrightarrow{\ \star\ } & \mathbf{SStar}(R) \\ {\scriptstyle sp}\downarrow & & \downarrow{\scriptstyle sp} \\ \mathbf{LS}(R) & \xrightarrow{\ \star\ } & \mathbf{SStar}(R) \end{array}$$

commutes.

(B) Let $\star$ be a semistar operation on R such that $\Pi^\star \neq \emptyset$, then

$$(\mathcal{F}^\star)_{sp} = \mathcal{F}^{\star_{sp}}.$$

In other words, the following diagram

$$(sp - \mathbf{SS}) \qquad \begin{array}{ccc} \mathbf{SStar}(R) & \xrightarrow{\ \ell\ } & \mathbf{LS}(R) \\ {\scriptstyle sp}\downarrow & & \downarrow{\scriptstyle sp} \\ \mathbf{SStar}(R) & \xrightarrow{\ \ell\ } & \mathbf{LS}(R) \end{array}$$

commutes when restricted to semistar operations $\star$, where $\Pi^\star \neq \emptyset$.

Proof. (A) Note that

$$\Pi^{\star_{\mathcal{F}}} = \{P \in \text{Spec}(R) : P_{\mathcal{F}} \cap R \neq R\} \subseteq \Phi = \{P \in \text{Spec}(R) : P \notin \mathcal{F}\},$$

because $P \in \mathcal{F}$ if and only if $P_{\mathcal{F}} = R_{\mathcal{F}}$ (Remark 2.5(b)). Conversely, if $P \in \Phi$, then $P_{\mathcal{F}} \neq R_{\mathcal{F}}$ and so $P \subseteq P_{\mathcal{F}} \cap R \subsetneq R$. In fact, $P_{\mathcal{F}} \cap R = P$; for if $x \in (P_{\mathcal{F}} \cap R) - P$, then $xI \subseteq P$ for some $I \in \mathcal{F}$ and so $I \subseteq P$, a contradiction. Now use Lemma 4.2 to get $(\star_{\mathcal{F}})_{sp} = \star_{\Pi^{\star_{\mathcal{F}}}} = \star_{\Phi} = \star_{\mathcal{F}(\Phi)} = \star_{\mathcal{F}_{sp}}$.

Part (B) was already proved in Proposition 4.11(1). □

Remark 4.14. Note that from the commutativity of diagrams $(sp - \mathbf{LS})$ and $(sp - \mathbf{SS})$ we deduce that, when $\Pi^\star \neq \emptyset$, $(\overline{\star})_{sp} = \overline{(\star_{sp})}$.

Example 4.15. *A stable semistar operation which is not quasi-spectral (hence, not spectral).* Let (V, M) be a 1-dimensional nondiscrete valuation domain with quotient field K. From Remark 1.0(b) we have $\overline{\mathbf{F}}(V) - \mathbf{F}(V) = K$, the nondivisorial ideals of V are of the form xM where $0 \neq x \in K$, and $M_v = V$. Hence $\Pi^v = \emptyset$. In particular v is not a quasi-spectral (semi-)star operation on V and $\mathcal{F}^v = \{M, V\}$. We claim that v is stable. If $I, J \in \mathbf{F}(V)$, then there exists $0 \neq d \in V$ such that $dI \subseteq V$ and $dJ \subseteq V$. Without loss of generality, assume that $dJ \subseteq dI$; hence $J \subseteq I$. It is obvious that $(I \cap J)_v = J_v = I_v \cap J_v$. From Theorem 2.10(B), we deduce that for each $E \in \overline{\mathbf{F}}(V)$, $E_v = E_{\mathcal{F}^v} = (E : M)$.

Example 4.16. *A quasi-spectral semistar operation which is not stable (hence, not spectral).* Let R be as in Example 3.11. If R is essential but not a $PvMD$, we proved that $b_f = v_f = t \gneqq \bar{t}$, thus t is not stable. However, t is quasi-spectral. To see this let I be an ideal of R such that $I_t \subsetneqq R$, then there exists a maximal t-ideal P of R, which is a prime ideal such that $I_t \subseteq P_t = P$ and thus $I \subseteq P$, ([17] or the following Lemma 4.20).

Our final goal is to study the following diagrams:

$$\begin{array}{ccc} \grave{\mathbf{L}}\mathbf{S}(R) & \xrightarrow{sp} & \mathbf{LS}(R) \\ f\downarrow & & \downarrow f \\ \mathbf{LS}(R) & \xrightarrow{sp} & \mathbf{LS}(R) \end{array} \qquad \begin{array}{ccc} \mathbf{SStar}(R) & \xrightarrow{sp} & \mathbf{SStar}(R) \\ f\downarrow & & \downarrow f \\ \mathbf{SStar}(R) & \xrightarrow{sp} & \mathbf{SStar}(R) \end{array}$$

In general these diagrams do not commute.

Proposition 4.17. *Let $\mathcal{F}$ be a localizing system of an integral domain R. Then $(\mathcal{F}_f)_{sp} = \mathcal{F}_f \subseteq (\mathcal{F}_{sp})_f$. Moreover, $(\mathcal{F}_f)_{sp} = (\mathcal{F}_{sp})_f$ if and only if, for each finitely generated ideal J of R, with $J \notin \mathcal{F}$, there exists a prime ideal P of R such that $J \subseteq P$ and $P \notin \mathcal{F}$. (When $\mathcal{F}$ satisfies this property we say that $\mathcal{F}$ is* finitely spectral.*)*

Proof. By Proposition 4.3(A), a localizing system of finite type is spectral, hence $(\mathcal{F}_f)_{sp} = \mathcal{F}_f$.

For the second part, we first assume that $\mathcal{F}$ is finitely spectral. Let $I \in (\mathcal{F}_{sp})_f$, then there exists a finitely generated ideal $J \subseteq I$ such that $J \nsubseteq P$, for each prime ideal $P \notin \mathcal{F}$. Then $J \in \mathcal{F}$; for otherwise if $J \notin \mathcal{F}$, we could find a prime ideal P of R such that $J \subseteq P$ and $P \notin \mathcal{F}$, which is a contradiction. So $J \in \mathcal{F}$, J is finitely generated and $J \subseteq I$ implies that $I \in \mathcal{F}_f$.

Conversely, if $(\mathcal{F}_{sp})_f = \mathcal{F}_f$, then $\mathcal{F}$ is finitely spectral. So if for some finitely generated ideal J of R with $J \notin \mathcal{F}$, we would have that $J \nsubseteq P$ for each $P \notin \mathcal{F}$, then $J \in (\mathcal{F}_{sp})_f$ with $J \notin \mathcal{F}_f$, which is a contradiction. □

Corollary 4.18. *If $\mathcal{F}$ is a spectral localizing system of an integral domain R, then*

$$(\mathcal{F}_f)_{sp} = \mathcal{F}_f = (\mathcal{F}_{sp})_f. \qquad \square$$

Example 4.19. *A non-finitely spectral localizing system.* Let V, P and $\hat{\mathcal{F}}(P)$ be as in Example 2.1. Suppose that P is the maximal ideal of V. For the sake of simplicity, we denote simply by $\mathcal{F}$ the localizing system $\hat{\mathcal{F}}(P)$.

Assume that P is idempotent and branched and set $P_0 = \cap_{n \geq 1} H^n$, where H is a P-primary ideal, $H \neq P$. Then,

$$\mathcal{F} = \{V, P\}, \quad \mathcal{F}_f = \{V\}, \quad \text{and } \mathcal{F}_{sp} = \mathcal{F}(P_0) = \{I : I \text{ ideal of } V, P_0 \subsetneqq I\}.$$

Therefore, $\mathcal{F}$ is not finitely spectral since

$$(\mathcal{F}_{sp})_f = \mathcal{F}_{sp} \supsetneqq \mathcal{F}_f = \mathcal{F}(P) = (\mathcal{F}_f)_{sp}.$$

If we assume that P is unbranched, then $\mathcal{F}$ is spectral because

$$\mathcal{F} = \mathcal{F}_{sp} = \cap\{\mathcal{F}(Q) : Q \in \operatorname{Spec}(R), Q \subsetneqq P\} = \{V, P\}.$$

In this case we have

$$\mathcal{F}(P) = (\mathcal{F}_f)_{sp} = \mathcal{F}_f = (\mathcal{F}_{sp})_f \subsetneqq \mathcal{F} = \mathcal{F}_{sp}.$$

PROBLEM: Find an example of a finitely spectral non-spectral localizing system.

Finally, we want to examine the diagram

$$\begin{array}{ccc} \mathbf{SStar}(R) & \xrightarrow{sp} & \mathbf{LS}(R) \\ f\downarrow & & \downarrow f \\ \mathbf{SStar}(R) & \xrightarrow{sp} & \mathbf{LS}(R) \end{array}$$

We start with the following preliminary results:

Lemma 4.20. *Let $\star$ be a semistar operation of finite type defined on an integral domain R, with $R^\star \neq K$, where K is the quotient field of R. If I is a proper integral semistar ideal of R; i.e., $0 \neq I^\star \cap R = I \subsetneqq R$, then I is contained in a proper integral maximal semistar ideal of R. Furthermore, a proper maximal semistar ideal of R is a prime ideal.*

Proof. The set of proper integral semistar ideals of R is nonempty and inductive. For instance, if x is a nonzero element of R and a nonunit in $R^\star$, then $xR^\star \cap R$ is a proper semistar ideal of R. Let $\{I_\alpha : \alpha \in A\}$ be a chain of proper integral semistar ideals of R. Then

$$(\cup_{\alpha \in A} I_\alpha)^\star \supseteq \cup_{\alpha \in A} I_\alpha^\star.$$

On the other hand, since $\star$ is of finite type, if $x \in (\cup_{\alpha \in A} I_\alpha)^\star$ then $x \in J^\star$ for some $J \in \mathbf{f}(R)$ where $J \subseteq \cup_{\alpha \in A} I_\alpha$. Clearly, $J \subseteq I_\alpha$ for an appropriate $\alpha \in A$. Thus $x \in J^\star \subseteq I_\alpha^\star \subseteq \cup_{\alpha \in A} I^\star$. Therefore

$$(\cup_{\alpha \in A} I_\alpha)^\star \cap R = (\cup_{\alpha \in A} I_\alpha^\star) \cap R = \cup_{\alpha \in A}(I_\alpha^\star \cap R) = \cup_{\alpha \in A} I_\alpha.$$

From Zorn's Lemma, we deduce that each proper integral semistar ideal I of R is contained in a proper integral maximal semistar ideal Q of R.

In order to prove that Q is a prime ideal of R, take $x, y \in R - Q$ and suppose that $xy \in Q$. By the maximality of Q, $(Q, x)^\star = R^\star$. By the finitness of $\star$, we can find a finitely generated ideal $J \subseteq Q$ such that $(J, x)^\star = R^\star$. Consider the ideal $y(J, x) = (yJ, yx)$. Then $y(J, x) \subseteq Q$, hence $y \in yR^\star \cap R \subseteq y(J, x)^\star \cap R = (yJ, yx)^\star \cap R \subseteq Q^\star \cap R = Q$. This contradicts the assumption that $y \notin Q$. □

Corollary 4.21. *A semistar operation of finite type is quasi-spectral.* □

Remark 4.22. If $\star$ is a semistar operation of finite type and if

$$\mathcal{L} = \{E \in \overline{\mathbf{F}}(R) : E^\star \subsetneq R^\star\}$$

then it is easy to prove that each element $E \in \mathcal{L}$ is contained in a maximal member of $\mathcal{L}$. Moreover, each maximal member N of $\mathcal{L}$ is such that $N = N^\star$ and is a prime ideal of $R^\star$.

If I is a nonzero ideal of R and $I \notin \mathcal{F}^\star$, then $I^\star \cap R \neq R$ and hence $I \in \mathcal{L}$. It is straightforward to see that:

$$\begin{aligned}\Pi^\star_{\max} &= \{M : M \notin \mathcal{F}^\star \text{ where } M \text{ is an ideal of } R, \text{ and it is maximal} \\ &\qquad \text{with respect to this property}\} \\ &= \{N \cap R : N \in \mathcal{L} \text{ and } N \text{ is maximal in } \mathcal{L}\}.\end{aligned}$$

Proposition 4.23. *Let $\star$ be a semistar operation defined on an integral domain and let $\Pi^\star \neq \emptyset$. Then the following hold.*

(1) $(\star_f)_{sp} = \overline{(\star_f)} \leq \star_f$ and hence $((\star_f)_{sp})_f = (\star_f)_{sp}$.

(2) The following conditions are equivalent:

(i) $(\star_f)_{sp} = \star_f$;

(ii) $\star_f$ is stable;

(iii) $\star_f$ is spectral.

Proof. (1) By Corollary 4.21, $\star_f$ is quasi-spectral and hence $(\star_f)_{sp} = \overline{(\star_f)}$ (Theorem 4.12(2)). Moreover, $\overline{(\star_f)} \leq \star_f$ by Theorem 2.10(B). Furthermore, since $\overline{(\star_f)} = \tilde{\star}$ is of finite type, $(\overline{(\star_f)})_f = \overline{(\star_f)}$ (Proposition 3.6).

(2) This follows from (1) and from Proposition 3.7(1) and Theorem 4.12(3). □

Proposition 4.24. *Let $\star$ be a semistar operation defined on an integral domain R. Assume that $\Pi^\star \neq \emptyset$. Then:*

(1) $(\bar{\star})_f \leq (\star_{sp})_f$, hence $(\star_f)_{sp} \leq (\star_{sp})_f$;

(2) If $\star$ is quasi-spectral, then $(\bar{\star})_f = (\star_{sp})_f$;

(3) $(\star_f)_{sp} = (\star_{sp})_f$ if and only if $(\mathcal{F}^\star)_f = \mathcal{F}(\Pi^\star)_f$ and $(\star_{sp})_f$ is stable.

Proof. (1) Note that $\bar{\star} \leq \star_{sp}$ by Proposition 4.11(2), hence $(\bar{\star})_f \leq (\star_{sp})_f$. By Proposition 4.23, we know that $(\star_f)_{sp} = \overline{(\star_f)}$. Since $\overline{(\star_f)} \leq (\bar{\star})_f$ (Proposition 3.6), we conclude that $(\star_f)_{sp} \leq (\star_{sp})_f$.

(2) By Theorem 4.12(2), if $\star$ is quasi-spectral then $\bar{\star} = \star_{sp}$, which implies the conclusion.

(3) Since $(\star_f)_{sp} = \overline{(\star_f)}$, Proposition 3.7(2) and Corollary 3.8 imply $\mathcal{F}^{(\star_f)_{sp}} = \mathcal{F}^{\overline{(\star_f)}} = \mathcal{F}^{\star_f} = (\mathcal{F}^\star)_f$.

On the other hand, since $\mathcal{F}^{\star_{sp}} = \mathcal{F}(\Pi^\star)$ (Lemma 4.2),

$$\begin{aligned}\mathcal{F}^{(\star_{sp})_f} &= \{I : I \text{ is an ideal of } R \text{ such that } I \supseteq J \\ &\qquad \text{with } J \text{ finitely generated and } J \in \mathcal{F}^{\star_{sp}} = \mathcal{F}(\Pi^\star)\} \\ &= \mathcal{F}(\Pi^\star)_f.\end{aligned}$$

It is clear that $(\star_f)_{sp} = (\star_{sp})_f$ implies that $(\mathcal{F}^\star)_f = (\mathcal{F}(\Pi^\star))_f$ and that $(\star_{sp})_f$ is stable, because it is spectral (Theorem 4.12(3) and Corollary 4.10).

The converse follows from Theorem 2.10 and the fact that a spectral semistar operation is stable. □

PROBLEM: If $\star$ is quasi-spectral, is $(\star_f)_{sp} = (\star_{sp})_f$?

Let $\star$ be a semistar operation defined on an integral domain R. An ideal I of R is *$\star$-invertible* if these exists an ideal J of R such that $(IJ)^\star = R^\star$.

Proposition 4.25. *Let $\star$ be a quasi-spectral semistar operation defined on an integral domain R and let I be an ideal of R. Then I is $\star$-invertible if and only if I is $\bar{\star}$-invertible. In particular, if $\mathcal{F}^\star = \{R\}$, then I is $\star$-invertible if and only if I is invertible.*

Proof. Since $\star$ is quasi-spectral, $\mathcal{F}^\star = \mathcal{F}^{\bar{\star}} = \mathcal{F}(\Pi^\star)$, Proposition 3.7(2) and Theorem 4.12(2).

($\Rightarrow$) Assume that $(IJ)^\star = R^\star$ and $(IJ)^{\bar{\star}} \subsetneq R^{\bar{\star}}$. Then $IJ \subseteq P$ for some $P \in \Pi^\star$. Hence, $(IJ)^\star \subseteq P^\star \subsetneq R^\star$, a contradiction.

($\Leftarrow$) Since $(IJ)^{\bar{\star}} = R^{\bar{\star}}$ for some ideal J of R and $\bar{\star} \leq \star$, then necessarily $1 \in (IJ)^\star$; i.e., $(IJ)^\star = R^\star$.

For the last statement of the Proposition, note that if $\mathcal{F}^\star = \{R\}$, then $E^{\bar\star} = E_{\mathcal{F}^\star} = E$, for each $E \in \overline{\mathbf{F}}(R)$. □

Set $\text{Inv}_\star(R) = \{I : I$ ideal of R and I $\star$-invertible$\}$. It is easy to see that $\text{Inv}_\star(R)$ forms a group under the product defined by $I \cdot J = (IJ)^\star$. The subset $\text{Princ}(R) = \{xR : x \in K, x \neq 0\}$ is a subgroup of $\text{Inv}_\star(R)$, and the quotient group

$$Cl_\star(R) = \text{Inv}_\star(R)/\text{Princ}(R)$$

is called the *$\star$-class group of R.*

Corollary 4.26. *If $\star$ is a semistar operation of an integral domain R, then $Cl_{\star_f}(R) = Cl_{\tilde\star}(R)$. In particular $Cl_t(R) = Cl_{\tilde v}(R)$.*

Proof. Note that $\star_f$ is a quasi-spectral semistar operation on R and $\overline{(\star_f)} = \tilde\star$. Now apply the previous proposition. The second part of the Corollary follows from the first, when $\star = v$. □

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References

[1] D.D. Anderson, *Star operations induced by overrings*, Comm. Algebra **16** (1988), 2535-2553.

[2] D.D. Anderson and D.F. Anderson, *Some remarks on star operations and the class group*, J. Pure Appl. Algebra **51** (1988), 27-33.

[3] D.D. Anderson and S.J. Cook, *Star operations and their induced lattices*, preprint.

[4] N. Bourbaki, Algèbre Commutative, Hermann, Paris, 1961-1965.

[5] V. Barucci, D. Dobbs, and M. Fontana, *Conducive integral domains as pullbacks*, Manuscripta Math. **54** (1986), 261-277.

[6] E. Bastida and R. Gilmer, *Overrings and divisorial ideals of rings of the form "D + M"*, Michigan Math. J. **20** (1973), 79-95.

[7] Wang Fanggui and R.L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra **25** (1997), 1285-1306.

[8] M. Fontana, *Kaplansky ideal transform: a survey*, M. Dekker Lect. Notes **205** (1999), 271-306.

[9] M. Fontana, J. Huckaba, and I. Papick, Prüfer Domains, M. Dekker, New York, 1997.

[10] S. Gabelli, *Prüfer (##) domains and localizing systems of ideals*, M. Dekker Lect. Notes, **205** (1999), 391-410.

[11] J. Garcia, P. Jara, and E. Santos, *Prüfer $\star$-multiplication domains and torsion theories*, Comm. Algebra, **27** (1999), 1275-1295.

[12] R. Gilmer, Multiplicative Ideal Theory, M. Dekker, New York, 1972.

[13] F. Halter-Koch, *Kronecker function rings and generalized integral closures*, preprint 1999.

[14] F. Halter-Koch, Ideal Systems: An Introduction to Multiplicative Ideal Theory, M. Dekker, New York, 1998.

[15] W. Heinzer and J. Ohm, *An essential ring which is not a v-multiplication ring*, Can. J. Math., **25** (1973), 856-861.

[16] J.M. Garcia Hernandez, *Radicales de anillos y modulos noetherianos relativos*, Ph.D. Thesis, Univ. Granada (1995).

[17] P. Jaffard, Les Systèmes d'Idéaux, Dunod, Paris, 1960.

[18] B. Kang, $\star$-Operations On Integral Domains, Ph.D. Dissertation, The University of Iowa, 1987.

[19] R. Matsuda, *Kronecker function rings of semistar operations on rings*, Algebra Colloquium, **5** (1998), 241-254.

[20] R. Matsuda and I. Sato, *Note on star-operations and semistar operations*, Bull. Fac. Sci. Ibaraki Univ., **28** (1996), 155-161.

[21] R. Matsuda and T. Sugatani, *Semistar operations on integral domains*, II. Math. J. Toyama Univ., **18** (1995), 155-161.

[22] A. Okabe and R. Matsuda, *Semistar-operations on integral domains*, Math. J. Toyama Univ. **17** (1994), 1-21.

[23] A. Okabe and R. Matsuda, *Kronecker function rings of semistar operations*, Tsukuba J. Math., **21** (1997), 529-548.

[24] N. Popescu, *A characterization of generalized Dedekind domains*, Rev. Roumaine Math. Pures Appl. **29** (1984), 777-786.

[25] B. Stenström, Rings of Quotients, Springer, Berlin 1975.