Polynomial extensions of star and semistar operations

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§0. Notation and Basic Definitions

Let D be an integral domain with quotient field K. Let

- $\overline{\mathfrak{F}}(D)$ be the set of all nonzero D-submodules of K,
- $\mathfrak{F}(D)$ be the set of all nonzero fractional ideals of D, and
- $\mathbf{f}(D)$ be the set of all nonzero finitely generated D–submodules of K.

Then, obviously,

$$\mathbf{f}(D) \subseteq \mathfrak{F}(D) \subseteq \overline{\mathfrak{F}}(D).$$

In 1994, Okabe and Matsuda introduced the notion of semistar operation $\star$ of an integral domain D, as a natural generalization of the Krull's notion of star operation (allowing $D \neq D^\star$).

• A mapping $\star : \overline{\mathfrak{F}}(D) \to \overline{\mathfrak{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $0 \neq z \in K$ and for all $E, F \in \overline{\mathfrak{F}}(D)$, the following properties hold:

($\star_1$) $(zE)^\star = zE^\star$;

($\star_2$) $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Note that J. Elliott (2010) has recently developed a very general theory on closure operations related to semistar operations and he has shown for instance that, for a closure operation on $\overline{\mathfrak{F}}(D)$, condition **($\star_1$)** is equivalent to

$$EF \subseteq G^\star \Rightarrow E^\star F^\star \subseteq G^\star \text{ for all } E, F, G \in \overline{\mathfrak{F}}(D).$$

• When $D^\star = D$, we say that $\star$ restricted to $\mathfrak{F}(D)$ defines *a star operation of D*

i.e., $\star : \mathfrak{F}(D) \to \mathfrak{F}(D)$ verifies the properties $(\star_2)$, $(\star_3)$ and

$(\star\star_1)$ $(zD)^\star = zD\,, \quad (zE)^\star = zE^\star$.

• A *semistar operation of finite type* $\star$ is an operation such that

$$E^\star = E^{\star_f} := \bigcup\{F^\star \mid F \subseteq E, F \in \mathbf{f}(D)\} \quad \text{for all } E \in \overline{\mathfrak{F}}(D).$$

§1. Introduction

One of the first attempt of relating star operations defined on an integral domain D with star operations defined on the polynomial extension $D[X]$ is due to Houston-Malik-Mott [HMM, 1984].
Note also that recently A. Mimouni [M, 2008] worked at similar problems.
The following are among the main results obtained in [HMM, 1984].

Under some technical assumptions, given $*$ is a star operation of finite type on $D[X]$, it is possible to induce in a "natural way" a star operation $*_0$ on D in such a way

$$D[X] \text{ is a } P{*}MD \Leftrightarrow D \text{ is a } P{*_0}MD\,.$$

In particular,

$$D[X] \text{ is a } Pv_{D[X]}MD \Leftrightarrow D \text{ is a } Pv_D MD\,.$$

• In 2007 in a joint work with G.W. Chang [CF1], we started to study the problem of the possibility of extending in a "canonical way" a semistar (or a star) operation $\star$ defined on D to a semistar (or a star) operation $\star_1$ defined on $D[X]$, having in view, among various questions, a sort of "ascending version" of the previous result:

$$D \text{ is a P}\star\text{MD} \;\Leftrightarrow\; D[X] \text{ is a P}\star_1\text{MD}.$$

• At the same time, in 2007 G. Picozza investigated various problems on semistar Noetherian domains and, in particular, the possibility of a semistar version of Hilbert Basis Theorem: i.e., given a semistar (or a star) operation $\star$ defined on D determine a semistar (or a star) operation $\star'$ defined on $D[X]$ such that

$$D \text{ is } \star\text{–Noetherian} \;\Leftrightarrow\; D[X] \text{ is } \star'\text{–Noetherian}.$$

Picozza's motivations were related to the following facts:

- Noetherian = d–Noetherian; Mori = v–Noetherian = t–Noetherian; strong Mori = w–Noetherian.
- D is d_D–Noetherian $\Leftrightarrow$ $D[X]$ is $d_{D[X]}$–Noetherian (Hilbert, 1888)
  D is w_D–Noetherian $\Leftrightarrow$ $D[X]$ is $w_{D[X]}$–Noetherian;
  (F.G. Wang - McCasland, 1999);
  but D is t_D–Noetherian $\not\Rightarrow$ $D[X]$ is $t_{D[X]}$–Noetherian,
  (Roitman, 1990).

Picozza investigated the natural problem: what is the "star-theoretic" reason of the different behaviour of the previous star operations when passing to the polynomial extensions ?

There are several other reasons for investigating the problem of ascending star and semistar operations in polynomial extension (e.g., star (or semistar) Krull dimensions, star (or semistar) class groups, etc.), but I have no time to go more in details with other preliminaries in this talk.

§2. Stable star and semistar operations in polynomial extensions

The problem of ascending in a canonical way a star or a semistar operation to a polynomial domains is not easy in general. We have recently obtained a satisfactory solution only for *stable star or semistar operations of finite type* (Chang-Fontana, J. Algebra 2007).

However, this case was sufficiently general to lead us to give a complete answer to the problem of ascending for instance the Prüfer star (or, semistar)-multiplication property from a domain D to the polynomial extension $D[X]$.

The starting point was based on a series of results obtained in a joint paper with J. Huckaba (2000), where we established a close connection between stable star or semistar operations and localizing systems of ideals (in the sense of Gabriel-Popescu).

Given a semistar operation $\ast$ on $D[X]$, for each $E \in \overline{\mathfrak{F}}(D)$ set

$$E^{\ast_0} := (E[X])^\ast \cap K.$$

Lemma 1

(1) *$\ast_0$ is a semistar operation on D called* the semistar operation canonically induced by $\ast$ on D. *In particular, if $\ast$ is a (semi)star operation on $D[X]$, then $\ast_0$ is a (semi)star operation on D.*

(2) $(E^{\ast_0}[X])^\ast = (E[X])^\ast$ *for all* $E \in \overline{\mathfrak{F}}(D)$.

(3) $(\ast_{_f})_0 = (\ast_0)_{_f}$ *and* $(\widetilde{\ast})_0 = \widetilde{\ast_0}$. *In particular, if $\ast$ is a semistar operation of finite type (respectively, stable), then $\ast_0$ is a semistar operation of finite type (respectively, stable).*

(4) *If $\ast'$ and $\ast''$ are two semistar operations on $D[X]$ and $\ast' \leq \ast''$, then* $\ast'_0 \leq \ast''_0$.

(5) $(d_{D[X]})_0 = d_D$, $(w_{D[X]})_0 = w_D$, $(t_{D[X]})_0 = t_D$, $(v_{D[X]})_0 = v_D$, *and* $(b_{D[X]})_0 = b_D$.

Note that from $E^{*_0} = (E[X])^* \cap K$, by tensoring with the D-algebra $D[X]$, we have $E^{*_0}[X] = (E[X])^* \cap K[X]$, for all $E \in \overline{\mathfrak{F}}(D)$. Moreover, it may happen that $E^{*_0}[X] \subsetneq (E[X])^*$ for some $E \in \overline{\mathfrak{F}}(D)$.

Example A

Let P be a given nonzero prime ideal of an integral domain D. Let $\star$ be the finite type stable semistar operation defined by $E^\star := ED_P$, for all $E \in \overline{\mathfrak{F}}(D)$. Let $*$ be the semistar operation on $D[X]$ defined by $A^* := AD_P(X)$, for all $A \in \overline{\mathfrak{F}}(D[X])$. Clearly, for each $E \in \overline{\mathfrak{F}}(D)$, $(E[X])^* \cap K = E[X]D_P(X) \cap K = ED_P = E^\star$, i.e., $*_0 = \star$.
On the other hand, $E^{*_0}[X] = E^\star[X] = ED_P[X] \subsetneq E[X]D_P(X) = (E[X])^*$ (even if
$E^{*_0}[X] = (ED_P(X) \cap K)[X] = E[X]D_P(X) \cap K[X] = (E[X])^* \cap K[X]$).
Note that, in this example, $\widetilde{\star} = \star$ and $* = \widetilde{*}$.

In order to better investigate this situation, we introduce the following definitions.

A semistar operation $\ast$ on the polynomial domain $D[X]$ is called

- *an extension of a semistar operation* $\star$ defined on D if $E^\star = (E[X])^\ast \cap K$, for all $E \in \overline{\mathfrak{F}}(D)$.
- *a strict extension of a semistar operation* $\star$ defined on D if $E^\star[X] = (E[X])^\ast$, for all $E \in \overline{\mathfrak{F}}(D)$.

Clearly, a strict extension is an extension.
By Lemma 1, a semistar operation $\ast$ on $D[X]$ is an extension of $\star := \ast_0$ and, by Example 10, in general is not a strict extension of $\star$.

Given two semistar operations $*'$ and $*''$ on the polynomial domain $D[X]$, we say that

• $*'$ *and* $*''$ *are equivalent over* D, for short $*' \sim *''$, if $(E[X])^{*'} \cap K = (E[X])^{*''} \cap K$, for each $E \in \overline{\mathfrak{F}}(D)$.

• $*'$ *and* $*''$ *are strictly equivalent over* D, for short $*' \approx *''$, if $(E[X])^{*'} = (E[X])^{*''}$, for each $E \in \overline{\mathfrak{F}}(D)$.

Clearly, two extensions (respectively, strict extensions) $*'$ and $*''$ on $D[X]$ of the same semistar operation defined on D are equivalent (respectively, strictly equivalent). In particular, we have:

$$*' \approx *'' \ \Rightarrow \ *' \sim *'' \ \Leftrightarrow \ *'_0 = *''_0 \,.$$

We will see that the converse of the first implication above does not hold in general. In order to construct some counterexamples, we need a deeper study of the problem of "raising" semistar operations from D to $D[X]$; i.e., given a semistar operation $\star$ on D, finding all the semistar operations $*$ on $D[X]$ such that $\star = *_0$.

Recall that, given a family of semistar operations $\{\star_\lambda \mid \lambda \in \Lambda\}$ on an integral domain D, *the semistar operation $\wedge\star_\lambda$ on D is defined for all $E \in \overline{\mathfrak{F}}(D)$ by setting $E^{\wedge\star_\lambda} := \bigcap\{E^{\star_\lambda} \mid \lambda \in \Lambda\}$.*
The following statement is an easy consequence of the definitions.

Proposition 2

Given a family of semistar operations $\{*_\lambda \mid \lambda \in \Lambda\}$ on $D[X]$, assume that each $*_\lambda$ is an extension (respectively, a strict extension) of a given semistar operation $\star$ defined on D, then $\wedge *_\lambda$ is also an extension (respectively, a strict extension) of $\star$.

From the previous proposition, we deduce that, if a semistar operation on D admits an extension (respectively, a strict extension) to $D[X]$, then it admits a unique minimal extension (respectively, a unique minimal strict extension).

At this point, it is natural to ask the following questions:

Q1. *Given a semistar operation $\star$ defined on D, is it possible to find "in a canonical way" an extension (respectively, a strict extension) of $\star$ on $D[X]$?*

Q2. *Given an extension $*$ on $D[X]$ of a semistar operation $\star$ defined on D. Is it possible to define a strict extension $*'$ on $D[X]$ of $\star$ (and thus $*' \sim *$) ?*

(In the statement of the previous question, we do not require that $*' \approx *$, since this condition would imply that the extension $*$ on $D[X]$ was already a strict extension of $\star$.)

We start the investigation of the previous questions, by considering semistar operations on D defined by families of overrings.

In this particular, but rather important setting (#), we will provide positive answers to both questions.

(#) This setting is enough general to include the case of stable semistar operations, hence the results obtained in the present context generalize the previous results (obtained by totally different methods) for the case of stable semistar operations of finite type.

Let $\mathcal{T} := \{T_\lambda \mid \lambda \in \Lambda\}$ be a set of overrings of D, and set

$$E^{\wedge_{\mathcal{T}}} := \bigcap_\lambda ET_\lambda, \text{ for each } E \in \overline{\mathfrak{F}}(D)\,.$$

Then, $\wedge_{\mathcal{T}}$ is a semistar operation on D, and $\wedge_{\mathcal{T}}$ is (semi)star if and only if $D = \bigcap_\lambda T_\lambda$.
It is easy to see that, for each $E \in \overline{\mathfrak{F}}(D)$ and for each $\lambda \in \Lambda$,

$$E^{\wedge_{\mathcal{T}}} T_\lambda = ET_\lambda\,.$$

Let $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\}$ be a family of overrings of an integral domain D with quotient field K. Let X be an indeterminate over K and denote by $T_\lambda(X)$ the Nagata ring of T_λ. For each $A \in \overline{\mathfrak{F}}(D[X])$, we set:

$$\begin{aligned} A^{(\wedge_\mathcal{T})} &:= \bigcap\nolimits_\lambda AT_\lambda(X)\,, \\ A^{\langle\wedge_\mathcal{T}\rangle} &:= A^{(\wedge_\mathcal{T})} \cap AK[X]\,, \\ A^{[\wedge_\mathcal{T}]} &:= \bigcap\nolimits_\lambda AT_\lambda[X]\,. \end{aligned}$$

Clearly, $A^{[\wedge_\mathcal{T}]} \subseteq A^{\langle\wedge_\mathcal{T}\rangle} \subseteq A^{(\wedge_\mathcal{T})}$ for all $A \in \overline{\mathfrak{F}}(D[X])$, hence

$$[\wedge_\mathcal{T}] \leq \langle\wedge_\mathcal{T}\rangle \leq (\wedge_\mathcal{T})\,.$$

Moreover, if $\mathcal{T}$ is nonempty, $(D[X])^{\langle\wedge_\mathcal{T}\rangle} \subseteq K[X]$.
On the other hand, $1/(1+X) \in \bigcap_\lambda T_\lambda(X) = (D[X])^{(\wedge_\mathcal{T})}$.
Hence, $(D[X])^{\langle\wedge_\mathcal{T}\rangle} \subsetneq (D[X])^{(\wedge_\mathcal{T})}$ and so

$$\langle\wedge_\mathcal{T}\rangle \lneq (\wedge_\mathcal{T})\,.$$

Let $\mathcal{T} = \{(T_\lambda \mid \lambda \in \Lambda\}$ be a family of overrings of an integral domain D with quotient field K. Set $\boldsymbol{(\mathcal{T})} := \{T_\lambda(X) \mid \lambda \in \Lambda\}$, $\langle\mathcal{T}\rangle := \{T_\lambda(X) \mid \lambda \in \Lambda\} \cup \{K[X]\}$, and $\boldsymbol{[\mathcal{T}]} := \{T_\lambda[X] \mid \lambda \in \Lambda\}$. Clearly,

$$\boldsymbol{(}\wedge_{\mathcal{T}}\boldsymbol{)} = \wedge_{\boldsymbol{(}\mathcal{T}\boldsymbol{)}}, \quad \langle\wedge_{\mathcal{T}}\rangle = \wedge_{\langle\mathcal{T}\rangle}, \quad \text{and } \boldsymbol{[}\wedge_{\mathcal{T}}\boldsymbol{]} = \wedge_{\boldsymbol{[}\mathcal{T}\boldsymbol{]}}; \quad \text{moreover:}$$

Proposition 3

(1) *If $\mathcal{T}$ is not empty and $\mathcal{T} \neq \{K\}$, then $\boldsymbol{[}\wedge_{\mathcal{T}}\boldsymbol{]} \lneq \langle\wedge_{\mathcal{T}}\rangle$.*

(2) *For each $E \in \overline{\mathfrak{F}}(D)$,*

$$(E[X])^{\boldsymbol{[}\wedge_{\mathcal{T}}\boldsymbol{]}} = E^{\wedge_{\mathcal{T}}}[X] = (E[X])^{\langle\wedge_{\mathcal{T}}\rangle} \text{ and } (E[X])^{\boldsymbol{(}\wedge_{\mathcal{T}}\boldsymbol{)}} = E^{\wedge_{\mathcal{T}}}(X),$$
$$E^{\wedge_{\mathcal{T}}} = (E[X])^{\boldsymbol{[}\wedge_{\mathcal{T}}\boldsymbol{]}} \cap K = (E[X])^{\langle\wedge_{\mathcal{T}}\rangle} \cap K = (E[X])^{\boldsymbol{(}\wedge_{\mathcal{T}}\boldsymbol{)}} \cap K.$$

(3) *$\boldsymbol{[}\wedge_{\mathcal{T}}\boldsymbol{]}$, $\langle\wedge_{\mathcal{T}}\rangle$, and $\boldsymbol{(}\wedge_{\mathcal{T}}\boldsymbol{)}$ (respectively, $\boldsymbol{[}\wedge_{\mathcal{T}}\boldsymbol{]}$ and $\langle\wedge_{\mathcal{T}}\rangle$) are distinct extensions (respectively, distinct strict extensions) of $\wedge_{\mathcal{T}}$.*

(4) *$\boldsymbol{[}\wedge_{\mathcal{T}}\boldsymbol{]} \sim \langle\wedge_{\mathcal{T}}\rangle \sim \boldsymbol{(}\wedge_{\mathcal{T}}\boldsymbol{)}$ and, moreover, $\boldsymbol{[}\wedge_{\mathcal{T}}\boldsymbol{]} \approx \langle\wedge_{\mathcal{T}}\rangle$, but neither $\boldsymbol{[}\wedge_{\mathcal{T}}\boldsymbol{]}$ nor $\langle\wedge_{\mathcal{T}}\rangle$ are strictly equivalent to $\boldsymbol{(}\wedge_{\mathcal{T}}\boldsymbol{)}$.*

Next, I will discuss some applications of the previous results, starting from the case of the *b*-operation, which can be obtained by using the family of all valuation overrings. First, some notation.

Let $\boldsymbol{\mathcal{W}} := \{W_\lambda \mid \lambda \in \Lambda\}$ be a family of valuation overrings of D and let $\wedge_{\boldsymbol{\mathcal{W}}}$ be the ab semistar operation on D defined by the family of valuation overrings $\boldsymbol{\mathcal{W}}$ of D (i.e., $E^{\wedge_{\mathcal{W}}} := \bigcap\{EW \mid W \in \boldsymbol{\mathcal{W}}\}$ for all $E \in \overline{\boldsymbol{\mathfrak{F}}}(D)$).

Example B

With the previous notation,

(a) for each $A \in \overline{\boldsymbol{\mathfrak{F}}}(D[X])$, $A^{(\wedge_{\mathcal{W}})} = \bigcap_\lambda AW_\lambda(X)$;

(b) for each $E \in \overline{\boldsymbol{\mathfrak{F}}}(D)$, $E^{(\wedge_{\mathcal{W}})_0}$ $(= \bigcap_\lambda EW_\lambda(X) \cap K) = E^{\wedge_{\mathcal{W}}}$;

(c) for each $F \in \mathbf{f}(D)$,

$$F^{\wedge_{\mathcal{W}}} = F\mathrm{Kr}(D, \wedge_{\boldsymbol{\mathcal{W}}}) \cap K = F^{(\wedge_{\mathcal{W}})_0}.$$

In particular,

$$(\wedge_{\boldsymbol{\mathcal{W}}})_{0,f} = (\wedge_{\boldsymbol{\mathcal{W}}})_{f,0} = (\wedge_{\boldsymbol{\mathcal{W}}})_{a,0} = \wedge_{\boldsymbol{\mathcal{W}},a} = \wedge_{\boldsymbol{\mathcal{W}},f}\,.$$

We study now the important case in which the family of valuation overrings $\boldsymbol{\mathcal{W}}$ of D coincides with the family $\boldsymbol{\mathcal{V}}$ of all valuation overrings of D. Set

$$[b_D] := [\wedge_{\boldsymbol{\mathcal{V}}}], \quad \langle b_D \rangle := \langle \wedge_{\boldsymbol{\mathcal{V}}} \rangle, \quad (b_D) := (\wedge_{\boldsymbol{\mathcal{V}}}).$$

Proposition 4

(1) *$[b_D]$, $\langle b_D \rangle$, and (b_D) are semistar operations on $D[X]$ with $[b_D] \leq \langle b_D \rangle \leq (b_D)$. If D is integrally closed then $[b_D]$ and $\langle b_D \rangle$ are (semi)star operations on $D[X]$. In general, (b_D) is not a (semi)star operation on $D[X]$ even if D is integrally closed .*

(2) *If $D \neq K$, i.e. if D has at least one nontrivial valuation overring, then $[b_D]$, $\langle b_D \rangle$, and (b_D) (respectively, $[b_D]$ and $\langle b_D \rangle$) are distinct extensions (respectively, distinct strict extensions) of b_D.*

(3) *$\langle b_D \rangle$ and (b_D) are $\mathcal{ab}$ semistar operations such that $[b_D] \leq b_{D[X]} = [b_D]_a \leq \langle b_D \rangle \leq (b_D)$, but in general $[b_D]$ is not an $\mathcal{eab}$ semistar operation.*

Example C

We next construct an integral domain D such that $[\mathbf{b}_D]$ is not an eab semistar operation. Let $D := \mathbb{R} + T\mathbb{C}[\![T]\!]$, i.e., D is a pseudo-valuation domain with canonically associated valuation overring $V := \mathbb{C}[\![T]\!]$ and quotient field $K := \mathbb{C}(\!(T)\!)$. Since $\mathbb{R} \subset \mathbb{C}$ is a finite field extension the valuation overrings of D are just V and K, thus it is straightforward to see that that $d_D \lneq b_D = \wedge_{\{V\}}$ and

$$
\begin{aligned}
[\mathbf{b}_D] = \wedge_{\{V[X],\ K[X]\}} \lneq \ & [\mathbf{b}_D]_a = b_{D[X]} \leq \langle b_D \rangle = \wedge_{\{V(X),\ K[X]\}} \\
\lneq \ & (\mathbf{b}_D) = \wedge_{\{V(X)\}}.
\end{aligned}
$$

Clearly, $[b_D]$ is not an eab semistar operation on $D[X]$, because if $[\mathbf{b}_D]$ $(= \wedge_{\{V[X],\ K[X]\}} = \star_{\{V[X]\}})$ was an eab semistar operation on $D[X]$, since it is of finite type, then $b_{D[X]} = (d_{D[X]})_a \leq [\mathbf{b}_D] \leq b_{D[X]}$, which is a contradiction (since $V[X]$ is not a Prüfer domain).

The two trivial semistar operations the identity, d_D, and the constant extension to K, e_D, can be described by (trivial) set of overrings.

Example D

The identity (semi)star operation d_D on an integral domain D, is defined by the family of a single overring $\boldsymbol{\mathcal{D}} := \{D\}$ of D, i.e., $d_D = \wedge_{\boldsymbol{\mathcal{D}}}$. Set

$$[d_D] := [\wedge_{\boldsymbol{\mathcal{D}}}], \ \langle d_D \rangle := \langle \wedge_{\boldsymbol{\mathcal{D}}} \rangle, \ (d_D) := (\wedge_{\boldsymbol{\mathcal{D}}}).$$

Clearly, if D is not a field,

- $d_{D[X]} = [d_D] \lneq \langle d_D \rangle \lneq (d_D)$ and
- $\langle d_D \rangle$ (and $[d_D]$) is a (semi)star operation on $D[X]$, but in general (d_D) is not a (semi)star operation on $D[X]$.

Moreover,

- $\langle d_D \rangle$, (d_D) (and $[d_D]$) are stable semistar operations of finite type, since $\langle d_D \rangle$ is defined by the two flat overrings $D(X)$ and $K[X]$ of $D[X]$ and (d_D) is defined by a unique flat overring $D(X)$ of $D[X]$ (Proposition 3((3) and (4))).

Example E

The trivial semistar operation e_D on an integral domain D, with quotient field K, is defined by $E^{e_D} := K$ for each $E \in \overline{\mathfrak{F}}(D)$.

Clearly, e_D can be also defined by the family of a single overring $\mathcal{K} := \{K\}$ of D, i.e., $e_D = \wedge_{\mathcal{K}}$. Set

$$[e_D] := [\wedge_{\mathcal{K}}]\,, \ \ \langle e_D \rangle := \langle \wedge_{\mathcal{K}} \rangle\,, \ \ (e_D) := (\wedge_{\mathcal{K}})\,.$$

Clearly,

$$[e_D] = \langle e_D \rangle \lneq (e_D) = e_{D[X]},$$

where $[e_D]$ $(= \langle e_D \rangle)$ is the stable semistar operation of finite type on $D[X]$, defined by the flat overring $K[X]$, i.e., $[e_D] = \langle e_D \rangle = \wedge_{\{K[X]\}}$.