Amalgamated algebras along an ideal: a class of ring extensions related to Nagata's idealization

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Preliminary report, joint work with Marco D'Anna and Carmelo Finocchiaro

§1. The Genesis

- Let A be a commutative ring with identity and let $\mathcal{R}$ be a ring without identity which is an A-module.

Following the construction described by D.D. Anderson in 2006 (A Tribute to the Work of Robert Gilmer), we can define a multiplicative structure in the A–module $A \oplus \mathcal{R}$, by setting
$(a, x)(a', x') := (aa', ax' + a'x + xx')$ for all $a, a' \in A$ and $x, x' \in \mathcal{R}$.

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Lemma 1

With the notation introduced above, we have:

(1) $A\dot{\bigoplus}\mathcal{R}$ *is a ring with identity* $(1,0)$*, which has an A–algebra structure induced by the canonical ring embedding* $\iota_A : A \hookrightarrow A\dot{\bigoplus}\mathcal{R}$*, defined by* $a \mapsto (a,0)$ *for all* $a \in A$*.*

(2) *If we identify* $\mathcal{R}$ *with its canonical image* $(0) \times \mathcal{R}$ *under the canonical (A–module) embedding* $\iota_{\mathcal{R}} : \mathcal{R} \hookrightarrow A\dot{\bigoplus}\mathcal{R}$*, defined by* $x \mapsto (0,x)$ *for all* $x \in \mathcal{R}$*, then* $\mathcal{R}$ *becomes an ideal in* $A\dot{\bigoplus}\mathcal{R}$*.*

(3) *If we identify* A *with* $A \times (0)$ *(respectively,* $\mathcal{R}$ *with* $(0) \times \mathcal{R}$*) inside* $A\dot{\bigoplus}\mathcal{R}$*, then the ring* $A\dot{\bigoplus}\mathcal{R}$ *is an A–module generated by* $(1,0)$ *and* $\mathcal{R}$*, i.e.,* $A(1,0) + \mathcal{R} = A\dot{\bigoplus}\mathcal{R}$*. Moreover, if* $p_A : A\dot{\bigoplus}\mathcal{R} \twoheadrightarrow A$ *is the canonical projection (defined by* $(a,x) \mapsto a$ *for all* $a \in A$ *and* $x \in \mathcal{R}$*), then*

$$0 \to \mathcal{R} \xrightarrow{\iota_{\mathcal{R}}} A\dot{\bigoplus}\mathcal{R} \xrightarrow{p_A} A \to 0$$

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(2) *If we identify* $\mathcal{R}$ *with its canonical image* $(0) \times \mathcal{R}$ *under the canonical (*A*–module) embedding* $\iota_{\mathcal{R}} : \mathcal{R} \hookrightarrow A\dot{\oplus}\mathcal{R}$, *defined by* $x \mapsto (0, x)$ *for all* $x \in \mathcal{R}$, *then* $\mathcal{R}$ *becomes an ideal in* $A\dot{\oplus}\mathcal{R}$.

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The previous construction takes its roots in the classical construction, introduced by Dorroh in 1932, for embedding a ring $\mathcal{R}$ (with or without identity, possibly without regular elements) in a ring with identity.

• Following Dorroh's ideas, we can consider in any case $\mathcal{R}$ as a $\mathbb{Z}$-module and we can construct the ring

$$\mathtt{Dh}(\mathcal{R}) := \mathbb{Z}\dot{\bigoplus}\mathcal{R} \quad (\mathtt{Dh}, \text{ in Dorroh's honour}).$$

• Note that $\mathtt{Dh}(\mathcal{R})$ is a commutative ring with identity $\mathbf{1}_{\mathtt{Dh}(\mathcal{R})} := (1, 0)$, $\mathtt{Dh}(\mathcal{R}) = \mathbb{Z}\cdot\mathbf{1}_{\mathtt{Dh}(R)} + \mathcal{R}$ and $\mathtt{Dh}(\mathcal{R})/\mathcal{R}$ is naturally isomorphic to $\mathbb{Z}$.

• On the bad side, note that if the ring $\mathcal{R} = R$ *has an identity* 1_R, then the canonical embedding of R into $\mathtt{Dh}(R)$ (defined by $x \mapsto (0, x)$ for all $x \in R$) does not preserve the identity, since $(0, 1_R) \neq \mathbf{1}_{\mathtt{Dh}(R)}$.

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In order to overcome this difficulty, in 1935 Dorroh gave a variation of the previous construction, which can be described now as a particular case of the general construction introduced above.

More precisely, if $\mathcal{R}$ has positive characteristic n (whenever $\mathcal{R}$ is a ring with or without identity), then $\mathcal{R}$ can be considered as a $\mathbb{Z}/n\mathbb{Z}$-module, so

$$\mathtt{Dh}_n(\mathcal{R}) := (\mathbb{Z}/n\mathbb{Z}) \dot{\bigoplus} \mathcal{R}$$

is a ring with identity $\mathbf{1}_{\mathtt{Dh}_n(R)} := (\overline{1}, 0)$, having characteristic n.

Moreover, as above,

$$\mathtt{Dh}_n(\mathcal{R}) = (\mathbb{Z}/n\mathbb{Z}) \cdot \mathbf{1}_{\mathtt{Dh}_n(R)} + \mathcal{R}$$

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§2. The amalgamation of an algebra along an ideal

A natural situation in which we can apply the previous general construction (Lemma 1) is the following.

• Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. Note that f induces on J a natural structure of A–module by setting

$$a \cdot j := f(a)j \quad \text{for all } a \in A \text{ and } j \in J.$$

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The following properties follow from Lemma 1.

Lemma 2

With the notation introduced above, we have:

(1) $A\dot{\bigoplus}J$ *is a ring.*

(2) *The map* $f^{\bowtie} : A\dot{\bigoplus}J \to A \times B$*, defined by* $(a, j) \mapsto (a, f(a) + j)$ *for all* $a \in A$ *and* $j \in J$*, is an injective ring homomorphism.*

(3) *The map* $\iota_A : A \to A\dot{\bigoplus}J$ *(respectively,* $\iota_J : J \to A\dot{\bigoplus}J$*), defined by* $a \mapsto (a, 0)$ *for all* $a \in A$ *(respectively, by* $j \mapsto (0, j)$ *for all* $j \in J$*), is an injective ring homomorphism (respectively, an injective A–module homomorphism).*

(4) *Let* $p_A : A\dot{\bigoplus}J \to A$ *be the canonical projection (defined by* $(a, j) \mapsto a$ *for all* $a \in A$ *and* $j \in J$*), then the following is a split exact sequence of A–modules:*

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- We set

$$A\bowtie^f J := f^{\bowtie}(A\dot{\oplus}J) = \{(a, f(a)+j) \mid a \in A, j \in J\} \subseteq A \times B\,.$$

Clearly, $A \cong \Gamma(f) := \{(a, f(a)) \mid a \in A\} \subseteq A\bowtie^f J \ (\subseteq A \times B)$.

The motivation for replacing $A\dot{\oplus}J$ with its canonical image $A\bowtie^f J$ inside $A \times B$ is related to the fact that the multiplicative structure defined in $A\dot{\oplus}J$, which looks somewhat *"artificial"*, becomes the restriction to $A\bowtie^f J$ of the natural multiplication defined *componentwise* in the direct product $A \times B$.

- The ring $A\bowtie^f J$ will be called *the amalgamation of the A–algebra B along J, with respect to $f : A \to B$.*

In very different contexts, particular cases of such construction were also considered by A.L.S. Corner (1969) and T.S. Shores (1974).

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The motivation for replacing $A \dot{\bigoplus} J$ with its canonical image $A \bowtie^f J$ inside $A \times B$ is related to the fact that the multiplicative structure defined in $A \dot{\bigoplus} J$, which looks somewhat *"artificial"*, becomes the restriction to $A \bowtie^f J$ of the natural multiplication defined *componentwise* in the direct product $A \times B$.

- The ring $A \bowtie^f J$ will be called *the amalgamation of the A−algebra B along J, with respect to $f : A \longrightarrow B$*.

In very different contexts, particular cases of such construction were also considered by A.L.S. Corner (1969) and T.S. Shores (1974).

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§3. Nagata's idealization.

• Let A be a commutative ring and $\mathcal{M}$ a A–module. Recall that, in 1955, Nagata introduced the ring extension of A called *the idealization of $\mathcal{M}$ in A*, denoted here by $A \ltimes \mathcal{M}$, as the A–module $A \oplus \mathcal{M}$ endowed with a multiplicative structure defined by:

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If we identify $\mathcal{M}$ and A with their canonical images in $A \ltimes \mathcal{M}$, then $\mathcal{M}$ becomes an ideal in $A \ltimes \mathcal{M}$ which is *nilpotent of index* 2 (i.e., $\mathcal{M}^2 = 0$) and the following

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• We can apply the construction of Lemma 1 by taking $\mathcal{R} := \mathcal{M}$, where $\mathcal{M}$ is a A–module, and considering $\mathcal{M}$ as a (commutative) ring without identity, endowed with a trivial multiplication (defined by $x \cdot y := 0$ for all $x, y \in \mathcal{M}$).

In this way, we have that the Nagata's idealization is a particular case of the construction considered in Lemma 1, since $A \ltimes \mathcal{M} = A \dot{\bigoplus} \mathcal{M}$.

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Although this, the Nagata idealization and the constructions of the type $A \bowtie^f J$ can be very different from an algebraic point of view.

In fact, for example, if $\mathcal{M}$ is a nonzero A–module, the ring $A \ltimes \mathcal{M}$ is always non-reduced (the element $(0, x)$ is nilpotent for all $x \in \mathcal{M}$), but the amalgamation $A \bowtie^f J$ can even be an integral domain, as we will see in a moment.

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§4 The constructions $A + \mathbf{X}B[\mathbf{X}]$ and $A + \mathbf{X}B[\![\mathbf{X}]\!]$

Let $A \subset B$ be an extension of commutative rings and $\mathbf{X} := \{X_1, X_2, ..., X_n\}$ a finite set of indeterminates over B.

- In the polynomial ring $B[\mathbf{X}]$, we can consider the following subring

$$A + \mathbf{X}B[\mathbf{X}] := \{h \in B[\mathbf{X}] \mid h(\mathbf{0}) \in A\},$$

where $\mathbf{0}$ is the $n-$tuple whose components are 0.

This is a particular case of the general construction introduced above.

- In fact, if $\sigma' : A \hookrightarrow B[\mathbf{X}] =: B'$ is the natural embedding and $J' := \mathbf{X}B[\mathbf{X}]$, then it is easy to check that

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• More generally, if J is an ideal of B and if consider the ideal $\mathcal{J}'' := \mathbf{X}J[\![\mathbf{X}]\!]$ of B'' $(= B[\![\mathbf{X}]\!])$ then

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where $\mathbf{0}$ is the $n-$tuple whose components are 0.

This is also a particular case of the general construction introduced above.

• In fact, if $\sigma'' : A \hookrightarrow B[\![\mathbf{X}]\!] =: B''$ is the natural embedding and $J'' := \mathbf{X}B[\![\mathbf{X}]\!]$, then it is easy to check that

$$A + \mathbf{X}B[\![\mathbf{X}]\!] \quad \textit{is isomorphic to} \quad A \bowtie^{\sigma''} J''.$$

• More generally, if J is an ideal of B and if consider the ideal $\mathcal{J}'' := \mathbf{X}J[\![\mathbf{X}]\!]$ of B'' $(= B[\![\mathbf{X}]\!])$ then

$$A + \mathbf{X}J[\![\mathbf{X}]\!] \quad \textit{is isomorphic to} \quad A \bowtie^{\sigma''} \mathcal{J}''.$$

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For a survey on the use of examples built with the previous constructions, see T. Lucas (2000).

§5 The $D + M$ construction

Let M be a maximal ideal of a ring (usually, an integral domain) T and let D be a subring of T such that $M \cap D = (0)$. The ring $D + M$ $(:= \{x + m \mid x \in D,\ m \in M\})$ is canonically isomorphic to $D \bowtie^{\iota} M$, where $\iota : D \hookrightarrow T$ is the natural embedding.

• More generally, let $\{M_\lambda \mid \lambda \in \Lambda\}$ be a subset of the set of the maximal ideals of T such that $M_\lambda \cap D = (0)$ for some $\lambda \in \Lambda$, and set $J := \bigcap_{\lambda \in \Lambda} M_\lambda$, then $D + J := \{x + j \mid x \in D,\ j \in J\}$ is canonically isomorphic to $D \bowtie^{\iota} J$.

• In particular, if $D := K$ is a field contained in T and $J := \mathrm{Jac}(T)$ is the Jacobson ideal of (the K–algebra) T, then $K + \mathrm{Jac}(T)$ is canonically isomorphic to $K \bowtie^{\iota} \mathrm{Jac}(T)$, where $\iota : K \hookrightarrow T$ is the natural embedding.

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§6 Iteration of the construction $A \bowtie I$

We start recalling an "ancestor" of the construction $A \bowtie^f J$.

- If A is a ring and I is an ideal of A, we can consider *the amalgamated duplication of the ring A along its ideal I* (= *the simple amalgamation of A along I*), i.e.,

$$A \bowtie I := \{(a, a+i) \mid a \in A,\ i \in I\}\ (:= A \bowtie^{\mathrm{id}_A} I)\,.$$

For the sake of simplicity, set $A' := A \bowtie I$. It is immediately seen that $I' := \{0\} \times I$ is an ideal of A', and thus we can consider again the simple amalgamation of A' along I', i.e., the ring

$$A'' := A' \bowtie I' \ (= (A \bowtie I) \bowtie (\{0\} \times I)).$$

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However, we can show that A'' can be interpreted as an amalgamation of algebras, giving in this way an answer to a problem posed by B. Olberding in 2006 at Padova's Conference in honour of L. Salce.

As a matter of fact, more generally, we have proved that if we iterate an amalgamation of algebras we still obtain an amalgamation of algebras. Instead of giving the details of this result, I will mention in a moment an example for showing the interest in iterating the amalgamation of algebras.

Note that the previous question is very natural since, when we consider the Nagata's idealization $A' := A \ltimes \mathcal{M}$ (where, as usual, A is a commutative ring and $\mathcal{M}$ a A–module), we can iterate this construction with respect to the A'-module $\mathcal{M}' := \{0\} \times \mathcal{M}$ and it is not hard to see that the iterated Nagata's idealization $A' \ltimes \mathcal{M}'$ is canonically isomorphic to the (classical) Nagata's idealization $A \ltimes (\mathcal{M} \times \mathcal{M})$.

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Example 1

We can apply the (iterated simple) amalgamation to curve singularities.
Let A be the ring of an algebroid curve with h branches (i.e., A is a one-dimensional reduced ring of the form $K[\![X_1, X_2, \ldots, X_r]\!]/\bigcap_{i=1}^{h} P_i$, where K is an algebraically closed field, $X_1, X_2, \ldots, X_r$ are indeterminates over K and P_i is an height $r-1$ prime ideal of $K[\![X_1, X_2, \ldots, X_r]\!]$, for $1 \leq i \leq r$).
If I is a regular and proper ideal of A, then, with an argument similar to that used by D'Anna (in the proof of Theorem 14, J. Algebra 2006, where the case of a simple amalgamation of the ring of the given algebroid curve is investigated), it can be shown that n-iterated amalgamation of A along the ideal I, denoted by $A \bowtie^n I$ is still a ring of an algebroid curve.
Moreover, in this case, $A \bowtie^n I$ has exactly $(n+1)h$ branches.
More precisely, for each of the h branches of A, there are exactly $n+1$ branches of $A \bowtie^n I$ isomorphic to it under the canonical surjective map $\mathrm{Spec}(A \bowtie^n I) \twoheadrightarrow \mathrm{Spec}(A)$.

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Remark 3

One of the motivations for considering the "amalgamation construction" is strictly related to a previous joint work with M. D'Anna (2007). One of the main results of this paper is the following:

Let A be a Noetherian local integral domain and let I be a m(ultiplicative)–canonical ideal of A and set $R := A \bowtie I := A \bowtie^{id_A} I$. Then R is a Noetherian local reduced ring, with $\dim(R) = \dim(A)$, such that every regular fractional ideal of R is divisorial.

More precisely, if A is a 1-dimensional Noetherian local integral domain and $I := \omega$ is a canonical ideal of A, then $R := A \bowtie I$ is a 1-dimensional reduced Gorenstein local ring.

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We say that a regular ideal I of a ring R is *a multiplicative-canonical ideal of R* (or simply a *m–canonical ideal)* if each regular fractional ideal J of R is I–reflexive, i.e., $J = (I : (I : J))$.
Note that this definition is a natural extension of the concept introduced in the integral domain case by W. Heinzer, J. Huckaba and I. Papick (1998) and of the notion of canonical ideal given by J. Herzog and E. Kunz (1971) and by E. Matlis (1973) for 1–dimensional Cohen-Macaulay rings.
In general, given a Cohen-Macaulay local ring (R, M, k) of dimension d, *a canonical module* of R is an R–module ω such that the k–dimension of $\mathrm{Ext}^i_R(k, \omega)$ is 1 for $i = d$ and 0 for $i \neq d$. If R is not local, *a canonical module for R* is an R–module ω such that all the localizations ω_M at the maximal ideals M of R are canonical modules of R_M.
When a canonical module ω exists and it is isomorphic to an ideal I of R, I is called *a canonical ideal of R.*

In higher dimension, the notions of canonical ideal and m–canonical ideal do not coincide. (HHP have shown that a Noetherian domain with dimension bigger than 1 does not admit a m–canonical ideal, while there exist (Noetherian) Cohen-Macaulay domains of dimension bigger than 1 with canonical ideal (e.g., a Noetherian factorial domain D of dimension ≥ 2; in this case, D is a Gorenstein domain).)

§7 Amalgamation and pullbacks

• We recall that, if $\alpha : A \to C$, $\beta : B \to C$ are ring homomorphisms, the subring $D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the *pullback* (or *fiber product*) of α and β.

Proposition 4

Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Set $C := A \times (B/J)$ and consider the canonical ring homomorphisms $u : A \to C$ and $v : A \times B \to C$ defined by

$$u(a) := (a, f(a)+J) \quad \text{and} \quad v((a, b)) := (a, b+J) \quad \text{for all } a \in A \text{ and } b \in B.$$

Then, the ring $A \bowtie^f J$ is canonically isomorphic to the pullback $u \times_C v$.

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i.e., the following diagrams of canonical homomorphisms are pullbacks:

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(where $\beta : b \mapsto b+J$, $\forall b \in B$, and $\check{f} : a \mapsto f(a)+J$, $\forall a \in A$) has the property that the canonical surjective map $\beta' : A\bowtie^f J \twoheadrightarrow A$ is *a retraction* (i.e., $A \hookrightarrow A\bowtie^f J \twoheadrightarrow A$ is the identity map of A).

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More precisely:

Given the ring homomorphisms $\alpha : A \to C$ and $\beta : B \to C$, where β is surjective, consider the pullback

$$\begin{array}{ccc} D & \xrightarrow{\beta'} & A \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow{\beta} & C\,. \end{array}$$

If β' is a retraction, with $\gamma' : A \to D$ such that $\beta' \circ \gamma' = \mathrm{id}_A$, then D is canonically isomorphic to the amalgamation $A \bowtie^{\varphi} J$, where $\varphi := \alpha' \circ \gamma' : A \to B$ and $J := \mathtt{Ker}(\beta)$.

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We know now several basic properties of the rings of the type $A\bowtie^f J$, for instance:

- Characterizations of when $A\bowtie^f J$ is a reduced ring;
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- Characterizations of when $A\bowtie^f J$ is a Noetherian ring;
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§8. The Noetherianity of the ring $A \bowtie^f J$

Proposition 5

The following conditions are equivalent.

(i) *$A \bowtie^f J$ is a Noetherian ring.*

(ii) *A and $B_\diamond := f(A) + J$ are Noetherian rings.*

The previous proposition has a moderate interest, because the Noetherianity of $A \bowtie^f J$ is not directly related to the data (i.e., A, B, f and J), but to the ring $B_\diamond = f(A) + J$ which, when $f^{-1}(J) = \{0\}$, is canonically isomorphic $A \bowtie^f J$.

However, if $f_\diamond : A \to B_\diamond$ is the canonical map obtained by composing $A \twoheadrightarrow f(A)$ with $f(A) \hookrightarrow f(A) + J = B_\diamond$, it is easy to verify that $A \bowtie^f J = A \bowtie^{f_\diamond} J$.

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Proposition 6

Assume that at least one of the following conditions holds:

(a) *J is a finitely generated A–module (with the structure naturally induced by f).*

(b) *f is a finite homomorphism.*

(c) *B is Noetherian and $\breve{f} : A \to B/J$, defined by $a \mapsto f(a) + J$ for all $a \in A$, is a finite homomorphism.*

Then $A \bowtie^f J$ is Noetherian if and only if A is Noetherian. In particular, if A is a Noetherian ring and B is a Noetherian A–module (e.g., if f is a finite homomorphism) then $A \bowtie^f J$ is a Noetherian ring for all ideal J of B.

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As a consequence of the previous proposition, we obtain a characterization of Noetherianity of the rings of the form $A + XB[X]$ and $A + XB[\![X]\!]$.

Note that S. Hizem and A. Benhissi in 2005 have already given a characterization of the Noetherianity of the power series rings of the type $A + XB[\![X]\!]$.

The next corollary provides a simple proof of Hizem and Benhissi's Theorem and enlarges this characterization to the polynomial case (in several indeterminates).

Corollary 7

Let $A \subseteq B$ be a ring extension and $\mathbf{X} := \{X_1, X_2, \ldots, X_n\}$ a finite set of indeterminates over B. Then the following conditions are equivalent.

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For instance $\mathbb{R} + X\mathbb{C}[X]$ ($\cong \mathbb{R} \bowtie^{\sigma'} X\mathbb{C}[X]$ (where $\sigma' : \mathbb{R} \hookrightarrow \mathbb{C}[X]$ is the natural embedding) is a Noetherian ring (Corollary 7), but $X\mathbb{C}[X]$ is not finitely generated as an $\mathbb{R}$–vector space (nor, σ' is a finite homomorphism).

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Therefore, the Noetherianity of the ring $A \bowtie^f J$ does not imply that J is finitely generated as an A–module (with the structure induced by f).

For instance $\mathbb{R} + X\mathbb{C}[X]$ ($\cong \mathbb{R} \bowtie^{\sigma'} X\mathbb{C}[X]$ (where $\sigma' : \mathbb{R} \hookrightarrow \mathbb{C}[X]$ is the natural embedding) is a Noetherian ring (Corollary 7), but $X\mathbb{C}[X]$ is not finitely generated as an $\mathbb{R}$–vector space (nor, σ' is a finite homomorphism).

This fact shows that condition **(a)** (or, **(b)**) of Proposition 6 is not necessary for the Noetherianity of $A \bowtie^f J$.

Example 2

Let $A \subseteq B$ be a ring extension, J an ideal of B and $\mathbf{X} := \{X_1, X_2, \ldots, X_r\}$ a finite set of indeterminates over B.

We set $B' := B[\mathbf{X}]$, $\mathcal{J}' := \mathbf{X}J[\mathbf{X}]$ and we denote by σ' the canonical embedding of A into B', then we already observed that the ring $A \bowtie^{\sigma'} \mathcal{J}'$ is naturally isomorphic to the ring $A + \mathbf{X}J[\mathbf{X}]$.

In this case, we can characterize the Noetherianity of the ring $A + \mathbf{X}J[\mathbf{X}]$, without assuming a finiteness condition on the inclusion $A \subseteq B$ (as in Corollary 7 **(iii)**) or on the inclusion $A + \mathbf{X}J[\mathbf{X}] \subseteq B[\mathbf{X}]$.
More precisely, *the following conditions are equivalent.*

(i) *$A + \mathbf{X}J[\mathbf{X}]$ is a Noetherian ring.*

(ii) *A is a Noetherian ring, J is an idempotent ideal of B and it is finitely generated as an A–module.*

Note that, if $A + \mathbf{X}J[\mathbf{X}]$ is Noetherian and B is not Noetherian, then $A \subseteq B$ and $A + \mathbf{X}J[\mathbf{X}] \subseteq B[\mathbf{X}]$ are necessarily not finite.

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