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On integral domains whose overrings are Kaplansky ideal transforms

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Abstract

Let R be an integral domain with quotient field K. The *Kaplansky transform* of an ideal I of R is given by $\Omega(I) = \{z \in K | \operatorname{rad}((R :_R zR)) \supseteq I\}$. For finitely generated ideals, this agrees with the Nagata transform. We attempt to characterize Ω-domains, that is, domains each of whose overrings is a Kaplansky transform. We obtain a particularly satisfactory characterization when we restrict to the class of Prüfer domains: a Prüfer domain R is an Ω-domain if and only if for each nonzero branched prime ideal P of R the set $P^{\downarrow} = \{Q \in \operatorname{Spec}(R) | Q \subseteq P\}$ is open in the Zariski topology.

MSC: 13A15; 13F05; 13B22

1. Introduction and preliminary results

Let R be an integral domain with quotient field K, and let I be an ideal of R. We call the following overring of R the *Nagata transform of I with respect to R*:

$$T_R(I) = \bigcup_{n \geq 0} (R : I^n) = \{x \in K \,|\, xI^n \subseteq R \text{ for some } n \geq 0\}.$$

In [13] (see also [9]) Kaplansky introduced a more general notion of ideal transform, which we call the *Kaplansky ideal transform of I with respect to R*:

$$\begin{aligned}\Omega_R(I) &= \{z \in K \,|\, \text{for each } a \in I \text{ there is an integer } n(a) \geq 1 \text{ such that } za^{n(a)} \in R\} \\ &= \{z \in K \,|\, \operatorname{rad}((R :_R zR)) \supseteq I\}.\end{aligned}$$

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When there is no danger of ambiguity, we shall use $T(I)$ instead of $T_R(I)$ and $\Omega(I)$ instead of $\Omega_R(I)$. The Kaplansky transform was studied by Hays [9] (where it was called the S-transform).

It is straightforward to check that $T(I) \subseteq \Omega(I)$ and that we have equality when I is finitely generated.

Following Brewer and Gilmer [1], we say that the domain R is a *T-domain* (respectively an *FT-domain*) if each overring of R is the Nagata transform of an ideal (respectively, a finitely generated ideal) of R. In [1], Brewer and Gilmer obtained a complete characterization of FT-domains but only partial results for T-domains, for which they posed several questions. We begin by recalling some of their main results.

Theorem 1.1 (Brewer and Gilmer [1, Theorem 1.5]). *For an integral domain R, the following statements are equivalent.*

(1) *R is an FT-domain.*
(2) *Each overring of R is the Nagata transform of a principal ideal of R.*
(3) *Each valuation overring of R is the Nagata transform of a finitely generated ideal of R.*
(4) *R is a semilocal Prüfer domain with the following property: if $\{P_n\}_{n \geq 0}$ is a strictly descending infinite sequence of prime ideals of R and if P is a prime ideal of R, then $P_n \subseteq P$ for some $n \geq 0$.*

For the case of T-domains, one of the principal results of [1] is the following:

Theorem 1.2 (Brewer and Gilmer [1, Theorem 2.15]). *Let R be an integral domain with no idempotent proper prime ideals. Then the following statements are equivalent:*

(1) *R is a T-domain;*
(2) *each valuation overring of R is the Nagata transform of an ideal of R;*
(3) *R is an FT-domain.*

Corollary 1.3. *For a Noetherian domain R, the following conditions are equivalent:*

(1) *R is a T-domain;*
(2) *R is an FT-domain;*
(3) *R is a semilocal PID.*

According to [1, Corollary 2.5], the conditions of Corollary 1.3 are also equivalent for Krull domains, and Hedstrom [10] generalized this to domains of Krull type. We observe that these conditions are also equivalent for a Mori domain. Indeed, conditions (1) and (2) are equivalent, since, for each ideal I in a Mori domain, there is a finitely generated ideal $J \subseteq I$ such that $I_v = J_v$, whence $T(I) = T(I_v) = T(J_v) = T(J)$. For the equivalence of (2) and (3), it suffices to recall that a Mori Prüfer domain must be Dedekind.

In general, a T-domain need not be integrally closed. For example, let (V,M) be a one-dimensional valuation domain with $M=M^2$ and with quotient field K, let $\varphi: V\to k=V/M$ be the canonical projection, let $k_0\subseteq k$ be a minimal extension of fields, and let $R=\varphi^{-1}(k_0)$. It is easy to see that the only proper overrings of R are V and K. Moreover, $K=T_R(0)$ and $V=(R:M)=\bigcup_{n\geq 0}(R:M^n)=T_R(M)$. Thus R is a T-domain, and the integral closure $\bar{R}$ of R coincides with V, since $k_0\subseteq k$ is an algebraic extension.

In their paper, Brewer and Gilmer posed the following questions:

(Q.1) If R is a T-domain, does it follow that each overring of R is a T-domain?
(Q.2) If R is a T-domain, is the integral closure $\bar{R}$ of R necessarily a Prüfer domain?
(Q.3) If R is a T-domain, is R necessarily semilocal?

Note that by Theorem 1.1, these questions all have positive answers in the case of FT-domains.

In the spirit of [4], when considering the non-Noetherian case, it seems preferable to replace the Nagata transform with the Kaplansky transform. Let us define an *Ω-domain* to be a domain each of whose overrings is a Kaplansky transform. It is then natural to ask whether the questions above have positive answers when "T-domain" is replaced by "Ω-domain".

In this work, we show that the "Ω" versions of questions (Q.1) and (Q.2) above have positive answers; as for (Q.3), we give an example (Section 4) of a non-semilocal Ω-domain. We also attempt to obtain a satisfactory characterization of Ω-domains. It is not difficult to show that an integrally closed Ω-domain is a Prüfer domain, and we show in Theorem 2.11 that a Prüfer domain R is a Ω-domain if and only if, for each nonzero prime P of R, either the set $P^{\downarrow}=\{Q\in \mathrm{Spec}(R)\mid Q\subseteq P\}$ is open in the Zariski topology or P is unbranched (meaning, in the context of a Prüfer domain, that P is the union of the (chain of) primes properly contained in P). We also obtain a reasonably good description of semilocal (not necessarily integrally closed) Ω-domains.

In the remainder of the present section, we collect some of the ideas and results which we shall need in the sequel.

The following lemma gives some of the properties of the Kaplansky transform.

Lemma 1.4. (1) *If I is an ideal of R, then $\Omega(I)=\bigcap_{P\not\supseteq I}R_P=\bigcap_{f\in I}R_f$. Hence if $I\subseteq J$ are two ideals of R, then $\Omega(I)\supseteq\Omega(J)$; and if $I\subseteq J\subseteq \mathrm{rad}(I)$, then $\Omega(I)=\Omega(J)$.*

(2) *If $\{I_\alpha\}$ is a family of ideals of R, then $\bigcap_\alpha \Omega(I_\alpha)=\Omega(\sum_\alpha I_\alpha)$.*

(3) *If I is an ideal of R and S is an overring of R with $R\subseteq S\subseteq\Omega_R(I)$, then $\Omega_S(IS)=\Omega_R(I)$.*

(4) *For each ideal I of R, $\Omega_R(I)=\bigcap_{P\in\mathrm{Spec}(R)}\Omega_{R_P}(IR_P)=\bigcap_{M\in\mathrm{Max}(R)}\Omega_{R_M}(IR_M)$.*

Proof. The first statement in part (1) is proved in [9, Lemma 1.6 and Theorem 1.7], and the second statement is an easy consequence of the first one.

Statement (2) follows easily from (1). (Also see [4, Lemma 3.1(f)].)

Statement (3) is [4, Lemma 3.1(l)].

Statement (4) is proved in [9, Proposition 1.9]. □

Recall that a *QQR-domain* is a domain R each of whose overrings is an intersection of localizations at prime ideals of R [5, p. 339] (see also [7,8]). According to Lemma 1.4(1), an Ω-domain is automatically a QQR-domain.

We shall say that an overring S of R is an *Ω-overring* (respectively, a T-overring) if $S=\Omega(I)$ (respectively, $S=T(I)$) for some ideal I of R.

We shall make frequent use of the following elementary result.

Proposition 1.5. *A domain R is an Ω-domain if and only if R is a QQR-domain and R_P is an Ω-overring for each prime ideal P of R.*

Proof. Suppose that R is a QQR-domain and that each R_P is an Ω-overring. Let S be an overring of R. Then $S=\bigcap_\alpha R_{P_\alpha}$ for some family $\{P_\alpha\}$ of prime ideals of R, and for each α, we have $R_{P_\alpha}=\Omega(I_\alpha)$ for some ideal I_α of R, whence by Lemma 1.4(2), we have $S=\Omega(\sum_\alpha I_\alpha)$. The converse follows easily from the definitions. $\square$

Since by [8, Corollary 1.7] a QQR-domain has Prüfer integral closure, we have the following:

Corollary 1.6. *If R is an Ω-domain, then its integral closure $\bar{R}$ is a Prüfer domain.*

Remark 1.7. It is shown in [8, Theorem 1.9] that the QQR-property is a local property. Unfortunately, despite Proposition 1.5, this is not the case for the Ω-property. For example, if R is any non-semilocal Dedekind domain, then, for each maximal ideal M of R, R_M is clearly an Ω-domain, but R_M is not an Ω-overring of R. Indeed, if I is a non-zero ideal of R, then there is a maximal ideal $N\neq M$ of R with $I\nsubseteq N$. By Lemma 1.4, this implies that $R_N\supseteq\Omega(I)$, whence $\Omega(I)\neq R_M$. We give a more interesting example of this failure in Example 2.12 below.

We set some notation for the remainder of the paper. For a prime ideal P of a domain R, the *pseudo-radical* of P is the ideal

$$P^*=\bigcap\{Q\mid Q \text{ is a prime ideal of } R \text{ with } Q\supsetneqq P\},$$

and we say that P is a *G(oldman)-ideal* if $P\neq P^*$ (see [6] and [12, Section 1.3]). (Here $P^*=R$ if P is maximal.) Note that maximal ideals are G-ideals. We also call a prime ideal P a *g(enerization)-ideal* if the set

$$P^{\downarrow}=\{Q\in\operatorname{Spec}(R)\mid Q\subseteq P\}$$

is an open subset of $\operatorname{Spec}(R)$ [15]. As usual, we denote by $\mathscr{V}(I)$ the closed subspace $\{Q\in\operatorname{Spec}(R)\mid Q\supseteq I\}$ and by $\mathscr{D}(I)$ the open subspace $\operatorname{Spec}(R)\setminus\mathscr{V}(I)$. Finally, we set

$$\mathscr{I}(P)=\bigcap\{Q\in\operatorname{Spec}(R)\mid Q\nsubseteq P\}=\bigcap\{Q\in\operatorname{Spec}(R)\mid R_Q\nsupseteq R_P\}.$$

Proposition 1.8. *Let P be a prime ideal of a domain R. Then the following conditions are equivalent*:

(a) *P is a g-ideal of R.*
(b) *$P^{\downarrow} = \mathscr{D}(I)$ for some finitely generated ideal I of R.*
(c) *$P^{\downarrow} = \mathscr{D}(f)$ for some element $f \in R \setminus P$.*
(d) *There is an element $f \in R \setminus P$ such that $R_P = R_f$.*
(e) *If $\{Q_\alpha\}$ is a family of prime ideals of R such that $\bigcap_\alpha Q_\alpha \subseteq P$, then $Q_\alpha \subseteq P$ for some α.*
(f) *$\mathscr{I}(P) \nsubseteq P$.*
(g) *$P^{\downarrow} = \mathscr{D}(\mathscr{I}(P))$.*

Proof. The equivalences (a)–(e) are stated in "collective" form in [15, Proposition 6]. A proof can be obtained by following the arguments given in [16, V, Propositions 1,2]. It is easy to see that (e) and (f) are equivalent, and it is clear that (g) implies (a). On the other hand, in general we have that $\overline{\mathrm{Spec}(R) \setminus P^{\downarrow}} = \mathscr{V}(\bigcap_{Q \in \mathrm{Spec}(R) \setminus P^{\downarrow}} Q) = \mathscr{V}(\bigcap_{Q \nsubseteq P} Q) = \mathscr{V}(\mathscr{I}(P))$. Hence if $P^{\downarrow}$ is open, then $\mathrm{Spec}(R) \setminus P^{\downarrow} = \mathscr{V}(\mathscr{I}(P))$, that is, $P^{\downarrow} = \mathscr{D}(\mathscr{I}(P))$. Thus (a) implies (g). $\square$

The following result is clear from Proposition 1.8(a) $\Leftrightarrow$ (e).

Corollary 1.9. *Every g-ideal is a G-ideal.*

The converse of Corollary 1.9 does not hold in general since, while a maximal ideal is necessarily a G-ideal, it need not be a g-ideal. For example, the maximal ideals in the ring $\mathbb{Z}$ of integers are not g-ideals, since it is easy to see that condition (e) of Proposition 1.8 does not hold. It is clear, however, that in a local ring the maximal ideal is a g-ideal. In a valuation domain, the notions are equivalent.

Proposition 1.10. *In a valuation domain, a prime ideal is a G-ideal if and only if it is a g-ideal.*

Proof. By Corollary 1.9 and the discussion above, it suffices to show that if P is a non-maximal prime G-ideal, then P is a g-ideal. However, it is easy to see that $V_P = V_f$ for any $f \in P^* \setminus P$, so this follows from Proposition 1.8(a) $\Leftrightarrow$ (d). $\square$

We shall call a domain R a *G-ideal domain* (respectively, a *g-ideal domain*) if every prime ideal of R is a G-ideal (respectively, a g-ideal).

Remark 1.11. Rings in which each prime ideal is a g-ideal were introduced and studied under the name "g-ring" by Picavet in [15,16]. Among other things, he proved that a g-ring is always semilocal [15, Proposition 7]. We note that a one-dimensional non-semilocal Prüfer domain with nonzero pseudoradical (i.e., $(0)^* \neq (0)$) is a G-ideal domain which is not a g-ideal domain. For example, we can take the integral closure of a one-dimensional valuation domain in a non-finite algebraic extension of its quotient field.

Proposition 1.12. *Let R be a semilocal Prüfer domain. Then the following statements are equivalent*:
(1) *R is a g-ideal domain.*
(2) *R is a G-ideal domain.*
(3) If *$\{P_n\}_{n \geq 0}$ is a strictly decreasing infinite sequence of prime ideals of R, and if P is any prime ideal of R, then $P \supseteq P_n$ for some $n \geq 0$.*
(4) *If $\{Q_\alpha\}$ is a family of prime ideals of R and if P is a prime ideal of R with $\bigcap_\alpha Q_\alpha \subseteq P$, then $Q_\beta \subseteq P$ for some β.*

Proof. Implication (1) implies (2) holds in general. Suppose that R is not a g-ideal domain. Then there is a prime non-g-ideal P in R. Since R is semilocal, there is a chain $\{Q_\alpha\}$ of prime ideals such that $Q_\alpha \nsubseteq P$ for each α and $Q = \bigcap_\alpha Q_\alpha \subseteq P$. It follows that the prime ideal Q is not a G-ideal. Hence (2) implies (1). The equivalence of (1) and (4) follows from Proposition 1.8(a) $\Leftrightarrow$ (e). It is clear that (4) implies (3). Finally, that (3) implies (1) is a consequence of [2, Theorem 1.5(e) $\Rightarrow$ (a)]. □

Corollary 1.13. *For a valuation domain V, the following statements are equivalent*:

(1) *V is a g-ideal domain.*
(2) *V is a G-ideal domain.*
(3) *For each nonmaximal ideal P of V, there is a prime ideal P^* right above P.*
(4) *For each prime ideal P of V, the descending chain condition on prime ideals holds in the ring V/P.*

2. Integrally closed Ω-domains

As a consequence of the results obtained in the preceding section, we show in Proposition 2.2 that questions (Q.1) and (Q.2) have positive answers for Ω-domains. We then undertake a study of Prüfer Ω-domains. Recall that by Corollary 1.6 an integrally closed Ω-domain is automatically a Prüfer domain.

Proposition 2.1. *If P is a prime g-ideal of a domain R, then R_P is an Ω-overring of R. It follows that a QQR-domain which is also a g-ideal domain is an Ω-domain.*

Proof. Let P be a prime g-ideal. By Proposition 1.8, we have $R_P = R_f$ for some element $f \in R \setminus P$. However, $R_f = \Omega(fR)$ by Lemma 1.4(1). The second statement follows from Proposition 1.5. □

Proposition 2.2. *If R is an Ω-domain, then each overring of R is an Ω-domain. In particular, the integral closure of an Ω-domain is an Ω-domain.*

Proof. Let S be an overring of R and T an overring of S. Since R is an Ω-domain, we have $T=\Omega_R(I)$ for some ideal I of R. It follows that $T=\Omega_S(IS)$ by Lemma 1.4(3). □

Lemma 2.3. *Let P be a nonmaximal prime ideal in a domain R. Then $\mathscr{I}(P)$ is contained in the Jacobson radical of R.*

Proof. Let M be a maximal ideal of R. Then $M\nsubseteq P$, whence $M\supseteq\mathscr{I}(P)$. □

Before stating our next result, we need some notation. For an R-submodule E of K, we set

$$\Omega^-(E):=\bigcap_{z\in E}\operatorname{rad}\,(R:_R zR).$$

By [4, Corollaries 3.15, 3.16], we have the following two facts:

(1) $\mathscr{I}(P)=\Omega^-(R_P)$ for each prime P and
(2) $\Omega(I)=\Omega(\Omega^-(\Omega(I)))$ for each ideal I.

Lemma 2.4. *Let R be a domain, let $P\in\operatorname{Spec}(R)$, and assume that $R_P=\Omega(I)$ for some ideal I of R. Then $R_P=\Omega(\mathscr{I}(P))$.*

Proof. By the facts mentioned above, we have

$$\Omega(\mathscr{I}(P))=\Omega(\Omega^-(R_P))=\Omega(\Omega^-(\Omega(I)))=\Omega(I)=R_P. \qquad \square$$

Recall that a prime ideal P of a domain R is said to be *unbranched* if P is the only P-primary ideal of R [5, p. 189]. In a Prüfer domain, this is equivalent to P being the union of the (chain of) primes contained in P [5, Theorem 23.3(e)].

Lemma 2.5. *Let (V,M) be a valuation domain, and suppose that $V=\Omega(I)$ for some proper ideal I of V. Then $I=M$, and M is unbranched.*

Proof. We have $V=\Omega(I)=\bigcap_{Q\nsupseteq I}V_Q=\bigcap_{Q\subsetneq I}V_Q$. It follows that $M=\bigcup_{Q\subsetneq I}Q$. Hence $I=M$, and M is unbranched. □

Lemma 2.6. *Let P be a nonzero prime ideal of a Prüfer domain R, and assume that $R_P=\Omega(I)$ for some ideal I of R with $I\subseteq P$. Then P is unbranched, and $IR_P=PR_P$. In particular, a nonzero prime non-g-ideal in a Prüfer Ω-domain is unbranched.*

Proof. By Lemma 1.4(1), $\Omega_{R_P}(IR_P)=R_P$. Hence by Lemma 2.5, either $IR_P=R_P$ (i.e., $I\nsubseteq P$) or $IR_P=PR_P$ and PR_P is unbranched in R_P. Since $I\subseteq P$, we are in the second case, and it follows that P is unbranched in R [5, Theorem 23.3(e)(6)].

The "in particular" statement now follows from Proposition 1.8(a) ⇔ (f) and Lemma 2.4. □

Lemma 2.7. *Let R be a Prüfer domain, and let P be a prime ideal of R.*

(1) *If P is contained in a prime g-ideal of R, then $P^{\downarrow} \setminus \{P\}$ is open.*
If, in addition, each nonzero prime non-g-ideal of R is unbranched, then:
(2) *if P is a non-g-ideal, then $P^{\downarrow} \setminus \{P\}$ is open, and*
(3) *if P is nonmaximal, then P is a G-ideal $\Leftrightarrow$ P is a g-ideal.*

Proof. (1) Let N be a prime g-ideal with $P \subseteq N$. Then $P^{\downarrow} \setminus \{P\} = N^{\downarrow} \cap \mathscr{D}(P)$, where $N^{\downarrow}$ is open (since N is a g-ideal).

(2) We may assume $P \neq 0$. It suffices to show that if $Q \in P^{\downarrow} \setminus \{P\}$, then there is an open subset U of Spec(R) with $Q \in U \subseteq P^{\downarrow} \setminus \{P\}$. If Q is a g-ideal, we may take $U = Q^{\downarrow}$. Otherwise, Q is unbranched, and $Q \subsetneq P$. Choose $x \in P \setminus Q$, and shrink P to a prime P_1 minimal over x. Since R is a Prüfer domain, we have $Q \subseteq P_1$. Since P_1 is branched, it is a g-ideal. Hence $U = P_1^{\downarrow} \setminus \{P_1\}$ is open by (1), and $Q \in U \subseteq P^{\downarrow} \setminus \{P\}$.

(3) Let P be a prime G-ideal, and let M be a maximal ideal with $P \subsetneq M$. As in the proof of (2), we may find a prime g-ideal P_0 with $P \subsetneq P_0 \subseteq M$. Since $P \subsetneq P^*$, it is then easy to see that $P^{\downarrow} = P_0^{\downarrow} \cap \mathscr{D}(P^*)$. Hence $P^{\downarrow}$ is open, and P is a g-ideal. The converse is true in general by Corollary 1.9. $\square$

Lemma 2.8. *Let R be a Prüfer domain in which every non-zero prime non-g-ideal is unbranched. Then a prime P of R is a non-g-ideal $\Leftrightarrow$ $P^{\downarrow} \setminus \{P\} = \mathscr{D}(\mathscr{I}(P))$.*

Proof. By Proposition 1.8(a) $\Rightarrow$ (g), it suffices to show that if P is not a g-ideal, then $P^{\downarrow} \setminus \{P\} = \mathscr{D}(\mathscr{I}(P))$. By Lemma 2.7, $P^{\downarrow} \setminus \{P\}$ is open. Since $P^{\downarrow}$ is not open, it must be the case that $P^{\downarrow} \setminus \{P\}$ is the interior of $P^{\downarrow}$. On the other hand, $\mathscr{V}(\mathscr{I}(P)) = \overline{\mathrm{Spec}(R) \setminus P^{\downarrow}}$, from which it follows that $\mathscr{D}(\mathscr{I}(P))$ is the interior of $P^{\downarrow}$. $\square$

Corollary 2.9. *Let R be a Prüfer Ω-domain. Then* Max(R) *is a closed subspace of* Spec(R). *Moreover, if $P \in$* Spec(R) *contains the Jacobson radical of R, then $P \in$* Max(R).

Proof. We may assume that R is not a field, in which case Max(R) is the complement in Spec(R) of $\bigcup_{M \in \mathrm{Max}(R)} (M^{\downarrow} \setminus \{M\})$. The first conclusion follows since Lemma 2.7 implies that each $M^{\downarrow} \setminus \{M\}$ is open. The second statement follows from the first and the fact that $\overline{\mathrm{Max}(R)} = \mathscr{V}(\mathscr{J})$, where $\mathscr{J}$ is the Jacobson radical of R. $\square$

Proposition 2.10. *Let R be a Prüfer domain, let $P \in$* Spec(R), *and let $\mathscr{J}$ denote the Jacobson radical of R. The following statements are equivalent:*

(1) $\mathscr{I}(P) = P^*$.
(2) $P \subseteq \mathscr{J}$.
(3) *P is comparable to all elements of* Spec(R).

Proof. (1) $\Leftrightarrow$ (3): We have $\mathscr{I}(P)=\bigcap_{Q\nsubseteq P} Q$. Since R is a Prüfer domain, it is easy to see that this intersection is equal to P^* precisely when P is comparable to every prime of R.

(3) $\Leftrightarrow$ (2): It is trivial that (3) $\Rightarrow$ (2). Suppose that Q is a prime of R, and let M be a maximal ideal with $Q\subseteq M$. Since R is a Prüfer domain, the primes within M are linearly ordered. Hence, since $P\subseteq \mathscr{J}\subseteq M$, we have that P and Q are comparable. □

Theorem 2.11. *The following are equivalent for a Prüfer domain R*:

(1) *R is an Ω-domain.*
(2) *Each nonzero prime non-g-ideal is unbranched.*
(3) *If P is a prime ideal of R, then either $\mathscr{D}(\mathscr{I}(P))=P^{\downarrow}$ or $\mathscr{D}(\mathscr{I}(P))=P^{\downarrow}\setminus\{P\}$ with P unbranched.*

Proof. (1) $\Rightarrow$ (2): This follows from Lemma 2.6.

(2) $\Rightarrow$ (3): This follows from Proposition 1.8(a) $\Rightarrow$ (g) and Lemma 2.8.

(3) $\Rightarrow$ (1): It suffices to show that R_P is an Ω-overring for each prime P of R. Let P be a prime; we shall show that $R_P=\Omega(\mathscr{I}(P))$. If $\mathscr{D}(\mathscr{I}(P))=P^{\downarrow}$, then $\Omega(\mathscr{I}(P))=\bigcap_{Q\nsupseteq \mathscr{I}(P)} R_Q=\bigcap_{Q\subseteq P} R_Q=R_P$. If, on the other hand, $\mathscr{D}(\mathscr{I}(P))=P^{\downarrow}\setminus\{P\}$ with P unbranched, then $\Omega(\mathscr{I}(P))=\bigcap_{Q\nsupseteq \mathscr{I}(P)} R_Q=\bigcap_{Q\subsetneq P} R_Q=R_P$ (since $P=\bigcup_{Q\subsetneq P} Q$).

As promised in Remark 1.7, we now give an example of a semilocal Prüfer domain R with the property that each localization is an Ω-domain but such that R is not an Ω-domain.

Example 2.12. Suppose that R is a Prüfer domain with spectrum as follows: R contains two maximal ideals M and N and a prime ideal $P\subseteq M\cap N$ such that $\operatorname{ht} M/P=1$, and two chains of primes $(0)=P_0\subsetneq P_1\subsetneq P_2\subsetneq\cdots$ and $N=Q_0\supsetneq Q_1\supsetneq Q_2\supsetneq\cdots$ with $\bigcup_{i=0}^{\infty} P_i=P=\bigcap_{j=0}^{\infty} Q_j$. (The existence of such a Prüfer domain follows from a construction of Lewis – see [14, Theorem 4.2].) It is easy to see that M is a non-g-ideal. Since M is branched, R is not an Ω-domain by Theorem 2.11. However, both R_M and R_N are Ω-domains, since every prime ideal of R_M is a g-ideal, and PR_N is the only non-zero prime non-g-ideal of R_N, which is unbranched by construction.

Recall that T-domains and FT-domains were characterized among valuation domains in [1]:

Proposition 2.13 (Brewer and Gilmer [1, Theorem 2.10]). *Let V be a valuation domain.*

(1) *V is an FT-domain if and only if V/P satisfies the descending chain condition for prime ideals for each nonzero prime ideal of V.*
(2) *V is a T-domain if and only if each prime ideal P of V such that $P=P^*$ is idempotent.*

Remark 2.14. By Corollary 1.13 and Proposition 2.13, a valuation domain V is an FT-domain if and only if each nonzero prime ideal of V is a g-ideal, and V is a T-domain if and only if each prime ideal of V which is not a g-ideal is idempotent.

Corollary 2.15. *If a valuation domain V is an Ω-domain, then V is also a T-domain.*

Proof. This follows from Theorem 2.11, Remark 2.14, and the fact that an unbranched prime in a valuation domain is idempotent [5, Theorem 17.3].

The converse of Corollary 2.15 is false, as the following example shows.

Example 2.16. Let W be a valuation domain with quotient field F such that $(0)=(0)^*$ and $Q \neq Q^*$ for each non-zero prime Q of W. Then W is an FT-domain (and hence also a T-domain and an Ω-domain). Now let (V_1, M_1) be a one-dimensional valuation domain with residue field F and satisfying $M_1 = M_1^2$. Clearly, V_1 is also an FT-domain. Let $\varphi : V_1 \to F$ denote the canonical projection. Finally, set $V = \varphi^{-1}(W)$, and let $P = \varphi^{-1}(0)$ denote the height one prime ideal of V. By construction, P is the only prime ideal of V which is not a g-ideal. Since $V_P = V_1$ and $PV_P = M_1$, we have $P = P^2$. Hence V is a T-domain by Proposition 2.13(2). However, since P is branched, V is not an Ω-domain by Theorem 2.11.

Proposition 2.17. *If R is a Prüfer Ω-domain and P is a prime ideal of R, then R/P is an Ω-domain.*

Proof. Let N/P be a nonzero prime non-g-ideal of R/P. We shall show that N/P is unbranched. We first observe that N is a prime non-g-ideal in R. Hence N is unbranched by Theorem 2.11. Thus N is the union of a chain $\{Q_\alpha\}$ of prime ideals of R. Since R is a Prüfer domain and $N \supsetneqq P$, it is clear that infinitely many of the Q_α must contain P. It is then easy to see that N/P is the union of these Q_α/P, and N/P is unbranched. Again by Theorem 2.11, R/P is an Ω-domain.

Lemma 2.18. *Let R be an Prüfer Ω-domain, let P and N be incomparable primes of R, and let $J = \bigcap_{Q \in N^\downarrow \setminus P^\downarrow} Q$. Then*

(1) *J is prime, and*
(2) *$J \nsubseteq P$.*

Proof. Since R is a Prüfer domain, J is the intersection of a chain of primes and is therefore itself prime. Since R is an Ω-domain, $R_P = \Omega(\mathscr{I}(P))$. For each $Q \in N^\downarrow \setminus P^\downarrow$, we have $Q \nsubseteq P$, so that $Q \supseteq \mathscr{I}(P)$. Hence $J \supseteq \mathscr{I}(P)$. Suppose that $J \subseteq P$. Then $\mathscr{I}(P) \subseteq P$, and P is not a g-ideal by Proposition 1.8(a) $\Leftrightarrow$ (f). Moreover, $\mathscr{I}(P)R_P = PR_P$ by Lemma 2.6. However, since J is prime and $\mathscr{I}(P) \subseteq J \subseteq P$, this implies that $J = P$. Since $J \subseteq N$, this is a contradiction. □

Proposition 2.19. *Let R be a semilocal Prüfer Ω-domain. Then every maximal ideal of R is a g-ideal.*

Proof. By Proposition 1.10, we may assume that R is not a valuation domain. Let M be a maximal ideal of R, and for each maximal ideal $N \neq M$, let $J_N = \bigcap_{Q \in N^\downarrow \setminus M^\downarrow} Q$. Then J_N is prime and $J_N \nsubseteq M$ by Lemma 2.18. However, it is clear that $\mathscr{I}(M) = \bigcap_{N \in \mathrm{Max}(R) \setminus \{M\}} J_N$, and, since R is semilocal, this implies that $\mathscr{I}(M) \nsubseteq M$. Thus M is a g-ideal.

Remark 2.20. Recall that, if R is any commutative ring with 1, then Spec(R) is quasi-compact. Thus if each maximal ideal is a g-ideal, then $\{M^\downarrow\}_{M \in \mathrm{Max}(R)}$ is an open cover of Spec(R), and R must be quasilocal. Thus Proposition 2.19 could be restated: If R is a Prüfer Ω-domain, then R is semilocal if and only if every maximal ideal of R is a g-ideal.

Lemma 2.21. *Let R be a semilocal Prüfer domain. Then R is a T-domain, if and only if, for each non-zero prime P of R we have $R_P = T(I)$ for some ideal I of R.*

Proof. One direction is obvious. Assume that each R_P has the form $T(I)$, and let S be an overring of R. Then S is semilocal, and $S = \bigcap_{i=1}^n R_{P_i}$ for some finite set of primes $\{P_i\}$ of R. If $R_{P_i} = T(I_i)$, then $S = \bigcap_{i=1}^n R_{P_i} = \bigcap_{i=1}^n T(I_i) = T(\sum_{i=1}^n I_i)$, the last equality following from [5, Exercise 9, p. 333].

Theorem 2.22. *Let R be a semilocal Prüfer Ω-domain. Then R is a T-domain.*

Proof. By the lemma, it suffices to show that R_P is a T-overring of R for each non-zero prime P of R. If P is a prime g-ideal, then by Proposition 1.8, we have $R_P = R_f = \Omega(fR) = T(fR)$ for some $f \in R \setminus P$. Let P be a prime non-g-ideal of R. Then $R_P = \Omega(\mathscr{I}(P))$ and $\mathscr{I}(P) \subseteq P$ by Proposition 1.8 and Lemma 2.4. For each maximal ideal N of R with $N \nsupseteq P$, let $J_N = \bigcap_{Q \in N^\downarrow \setminus P^\downarrow} Q$. We claim that for any $a \notin P$, we have $\mathrm{Rad}_{R_N}(a) \supseteq J_N R_N$. Otherwise, we have $a \in q \subsetneqq J_N$ for some prime q of R. But then we have $a \in q \subseteq P$ by construction of J_N.

By Lemma 2.18, we may pick $x_N \in J_N \setminus P$. Let $A = \prod_N x_N \cdot \mathscr{I}(P)$. We shall show that $R_P = T(A)$. By Theorem 2.11, $P^\downarrow \setminus \{P\} = \mathscr{D}(\mathscr{I}(P))$. We claim that $\mathscr{D}(\mathscr{I}(P)) = \mathscr{D}(A)$. The inclusion $\mathscr{D}(\mathscr{I}(P)) \supseteq \mathscr{D}(A)$ is clear. Let Q be prime with $\mathscr{I}(P) \nsubseteq Q$. Then since that $P^\downarrow \setminus \{P\} = \mathscr{D}(\mathscr{I}(P))$, we have $Q \subsetneqq P$, whence $\prod_N x_N \notin Q$. Thus $A \nsubseteq Q$, and we have $\mathscr{D}(\mathscr{I}(P)) = \mathscr{D}(A)$, as claimed. It follows that $T(A) \subseteq \Omega(A) = R_P$. Now pick $s \in R \setminus P$; we shall show that $s^{-1} \in T(A)$. We proceed locally. If M is maximal with $P \subseteq M$, then $\mathscr{I}(P)R_M \subseteq PR_M \subseteq sR_M$, whence $s^{-1}A \subseteq s^{-1}\mathscr{I}(P) \subseteq R_M$. Suppose that N is maximal with $P \nsubseteq N$. By the claim above, we have $J_N R_N \subseteq \mathrm{Rad}_{R_N}(x_N)$, whence $J_N R_N = \mathrm{Rad}_{R_N}(x_N)$ (since $x_N \in J_N$). Also by the claim, $\mathrm{Rad}_{R_N}(s) \supseteq J_N R_N$ since $s \notin P$. Hence sR_N contains a power of x_N. It follows that $s^{-1}A^k \subseteq R_N$ for some positive integer k. Since R is semilocal, we have $s^{-1} \in T(A)$, as desired. □

3. The general case

In this section, we attempt to characterize general Ω-domains, obtaining a satisfactory description in the semilocal case. Since by Proposition 1.5 an Ω-domain is necessarily a QQR-domain, it is convenient to begin with a characterization of local QQR-domains which are not integrally closed. This characterization, though cast somewhat differently, is essentially contained in [8].

Proposition 3.1. *Let (R,M) be a local domain which is not integrally closed. Set $k = R/M$.*

(1) *If R is a QQR-domain and $\bar{R}$ is local, then $\bar{R}$ is a valuation domain with maximal ideal M, M is unbranched, the extension $k \subseteq \bar{R}/M$ is a minimal extension of fields, and we have the following pullback diagram:*

$$\begin{array}{ccc} R & \longrightarrow & k \\ \downarrow & & \downarrow \\ \bar{R} & \longrightarrow & \bar{R}/M. \end{array}$$

Conversely, if $\bar{R}$ is a valuation domain with unbranched maximal ideal M such that $k \subseteq \bar{R}/M$ is a minimal extension of fields, then R is a QQR-domain.

(2) *If R is a QQR-domain and $\bar{R}$ is not local, then $\bar{R}$ is a Prüfer domain with exactly two maximal ideals N_1 and N_2, both unbranched, $M = N_1N_2$, $R/N_i = k$ for $i = 1,2$, and we have the following pullback diagram:*

$$\begin{array}{ccc} R & \longrightarrow & k \\ \downarrow & & \downarrow \\ \bar{R} & \longrightarrow & \bar{R}/N_1 \times \bar{R}/N_2 \cong k \times k. \end{array}$$

(The downward map on the right is the diagonal map.) Conversely, if $\bar{R}$ has two unbranched maximal ideals N_1 and N_2, such that $M = N_1N_2$ and $R/N_i = k$ for $i = 1,2$, then R is a QQR-domain.

Proof. (1) Suppose that R is a QQR-domain with $\bar{R}$ local. Then [8, Theorem 3.3] implies that $\bar{R}$ is a valuation domain with unbranched maximal ideal and that $\bar{R}$ is the unique minimal overring of R. To see that M is the maximal ideal of $\bar{R}$, first note that by [8, Lemma 2.3], M is the conductor of R in $\bar{R}$. Let Q be a nonmaximal prime ideal of $\bar{R}$. Since $\bar{R}$ is a valuation domain, either $Q \subseteq M$ or $M \subseteq Q$. However, since $R \subseteq \bar{R}$ is an integral extension, $Q \cap R$ is a nonmaximal ideal of R. Hence $Q \subseteq M$. Therefore, since M contains the union of the nonmaximal prime ideals of $\bar{R}$, it must be the case that M is the maximal ideal of $\bar{R}$. It follows that the diagram is a pullback. Since there are no rings between R and $\bar{R}$, $k \subseteq \bar{R}/M$ is a minimal extension of fields. The converse statement follows from [8, Theorem 3.3] and similar considerations.

(2) Now suppose that R is a QQR-domain with $\bar{R}$ not local. Then [8, Corollary 2.2 and Theorem 3.3] imply that $\bar{R}$ is a Prüfer domain with two unbranched maximal ideals (say) N_1 and N_2. Moreover, $M = N_1N_2$ by [8, Proposition 2.5]. Hence the following is a pullback diagram:

$$\begin{array}{ccc} R & \longrightarrow & k \\ \downarrow & & \downarrow \\ \bar{R} & \longrightarrow & \bar{R}/M \cong \bar{R}/N_1 \times \bar{R}/N_2. \end{array}$$

Since $\bar{R}$ is the unique minimal overring of R [8, Theorem 3.3], we must have $\bar{R}/N_i \cong k$ for $i = 1, 2$. For the converse, note that the given conditions imply that the diagram is a pullback, from which it follows that there are no domains properly between R and $\bar{R}$. Now apply [8, Theorem 3.3]. □

Now let R be a (not necessarily local) QQR-domain with integral closure $\bar{R}$. If M is a maximal ideal of R such that R_M is not integrally closed, then, according to Proposition 3.1, either $\overline{R_M}$ has a unique maximal ideal equal to MR_M or $\overline{R_M}$ has exactly two maximal ideals whose product is MR_M. It is convenient to distinguish these maximal ideals:

Definition 3.2. Let R be a QQR-domain. We say that a *maximal ideal* M *of* R *is of type* 0, 1, *or* 2, according as R_M is integrally closed, R_M is not integrally closed and $\bar{R}$ contains exactly one maximal ideal contracting to M, or R_M is not integrally closed and $\bar{R}$ contains two distinct maximal ideals which contract to M.

Remark 3.3. Let R be a QQR-domain. We examine $\mathrm{Max}(R)$ more closely. Let $M \in \mathrm{Max}(R)$. If M has type 0, then, since $\bar{R}$ is a Prüfer domain, and since R_M is integrally closed, we have that R_M is a valuation domain. Hence the primes contained in M form a chain. If M has type 1, then by Proposition 3.1(1), R_M is a pseudo-valuation domain and again the primes contained in M form a chain (see [11]). Now let M have type 2. We claim that there are two chains $\mathscr{C}_1$ and $\mathscr{C}_2$ of prime ideals such that M is the union of each chain and such that P_1 and P_2 are incomparable whenever $P_1 \in \mathscr{C}_1$ and $P_2 \in \mathscr{C}_2$. By Proposition 3.1, $\bar{R}$ contains exactly two maximal ideals N_1 and N_2 which contract to M, and both N_1 and N_2 are unbranched. Hence each N_i is the union of a chain of primes. In fact, if $x \in N_1 \setminus N_2$, then N_1 is the union of a chain of primes which contain x and are therefore not contained in N_2. Similarly, N_2 is the union of a chain of primes not contained in N_1. Contracting these chains to R verifies the claim.

Lemma 3.4. *Let R be a QQR-domain, let $P \in \mathrm{Spec}(R) \setminus \mathrm{Max}(R)$, and let $Q \in \mathrm{Spec}(\bar{R})$ satisfy $Q \cap R = P$. Then $R_P = \bar{R}_Q$. It follows that the contraction map from* $\mathrm{Spec}(\bar{R}) \setminus \mathrm{Max}(\bar{R})$ *to* $\mathrm{Spec}(R) \setminus \mathrm{Max}(R)$ *is a one-to-one correspondence.*

Proof. Let M be a maximal ideal of R with $P \subseteq M$. Then $Q\bar{R}_{R\setminus M} \cap R_M = PR_M$. If M has type 0, then we actually have $R_M = \bar{R}_{R\setminus M}$. Otherwise, note that MR_M is the conductor of R_M in $\bar{R}_M = \bar{R}_{R\setminus M}$, and since P does not contain the conductor, we have $(R_M)_{PR_M} = (\bar{R}_{R\setminus M})_{Q\bar{R}_{R\setminus M}}$. In any case, it follows that $R_P = \bar{R}_Q$. $\square$

Let R be a QQR-domain, and let $P \in \operatorname{Spec}(R)$. We claim that P is unbranched in R, if and only in, each of the (at most two) primes of $\bar{R}$ contracting to P is unbranched. To see this, first recall that "branchedness" is a local property. Hence the claim is true for non-maximal P by Lemma 3.4. For maximal P, this follows from Proposition 3.1 and [8, Lemmas 3.1, 3.2]. In particular, the maximal ideals of type 1 or 2 in a QQR-domain must be unbranched.

Our next result extends Proposition 3.1 to the semilocal case.

Proposition 3.5. *Let R be a semilocal QQR-domain. Then $\bar{R}$ is a semilocal Prüfer domain. Let $\{M_i\}_{i=1}^r$, $\{M_i\}_{i=r+1}^s$, and $\{M_i\}_{i=s+1}^t$ denote the sets of type 0, type 1, and type 2 maximal ideals, respectively. Set $k_i = R/M_i$ for $i = 1,\dots,t$. Finally, let N_i (N_{i1}, N_{i2}) contract to M_i for $i = 1,\dots,s$ (for $i = s+1,\dots,t$). Then $k_i \subseteq \bar{R}/N_i$ is a minimal extension of fields for $i = r+1,\dots,s$, $\bar{R}/N_{i1} \cong \bar{R}/N_{i2} \cong k_i$ for $i = s+1,\dots,t$, and we have the following pullback diagram*:

$$\begin{array}{ccc}
R & \longrightarrow & R\Big/\left(\prod_{i=r+1}^{t} M_i\right) \cong \prod_{i=r+1}^{t} k_i \\
\big\downarrow & & \big\downarrow \\
\bar{R} & \to \bar{R}\Big/\left(\prod_{i=r+1}^{s} N_i \times \prod_{i=s+1}^{t} N_{i1}N_{i2}\right) \cong & \prod_{i=r+1}^{s} \bar{R}/N_i \times \prod_{i=s+1}^{t} (k_i \times k_i).
\end{array}$$

(*The downward map on the right is inclusion in components $r+1$ to s, and diagonal in components $s+1$ to t.*)

Conversely, let $\bar{R}$ be a semilocal Prüfer domain with maximal ideals $\{N_i\}_{i=1}^r$, $\{N_i\}_{i=r+1}^s$, and $\{N_{i1}, N_{i2}\}_{i=s+1}^t$, and assume that each of the maximal ideals in the latter two sets in unbranched. Further assume that for each $i = s+1,\dots,t$ there is a field k_i with $\bar{R}/N_{i1} \cong \bar{R}/N_{i2} \cong k_i$. Finally, for each $i = r+1,\dots,s$, let $k_i \subseteq \bar{R}/N_i$ be a minimal extension of fields. Let R be the pullback of the following diagram:

$$\begin{array}{c}
\prod_{i=r+1}^{t} k_i \\
\big\downarrow \\
\bar{R} \to \prod_{i=r+1}^{s} \bar{R}/N_i \times \prod_{i=s+1}^{t} (k_i \times k_i).
\end{array}$$

Then R is a semilocal QQR-domain.

Proof. By [8, Theorem 1.9], the QQR-property a local property. Hence the result follows from Proposition 3.1 and the technique of localizing pullback diagrams. $\square$

Lemma 3.6. *Let R be an QQR-domain whose integral closure $\bar{R}$ is a (necessarily Prüfer) Ω-domain. Then R has a non-zero conductor $\mathscr{C}$ in $\bar{R}$, and we have*

$$\mathscr{C} = \bigcap \{M \in \mathrm{Max}(R) \mid M \textit{ has type } 1 \textit{ or type } 2\}$$

$$= \bigcap \{N \in \mathrm{Max}(\bar{R}) \mid N \cap R \textit{ has type } 1 \textit{ or type } 2\}.$$

Hence the following diagram is a pullback:

$$\begin{array}{ccc} R & \rightarrow & R/\mathscr{C} \\ \downarrow & & \downarrow \\ \bar{R} & \rightarrow & \bar{R}/\mathscr{C}. \end{array}$$

Proof. We may assume that R is not integrally closed, that is, that there is at least one maximal ideal of type 1 or 2. As observed above, such a maximal ideal must be unbranched. It follows that $\bar{R}$ contains a nonzero, nonmaximal ideal. Hence by Corollary 2.9, the Jacobson radical of $\bar{R}$ is nonzero. Let $x \in \bigcap\{N \in \mathrm{Max}\,(\bar{R}) \mid N \cap R$ has type 1 or type 2$\}$, and let M be a maximal ideal of R. If M has type 0 and $N \cap R = M$, then $x \in \bar{R} \subseteq \bar{R}_N = R_M$. If M has type 1 and $N \cap R = M$, then $x \in N \subseteq N\bar{R}_N = MR_M$. Finally, if M has type 2 and $N_1 \cap R = N_2 \cap R = M$, then $x \in N_1N_2 \subseteq N_1N_2\bar{R}_{\bar{R}\setminus(N_1 \cup N_2)} = MR_M$. Hence $x \in R$. It now suffices to show that if $y \in \mathscr{C}$, then $y \in \bigcap\{M \in \mathrm{Max}\,(R) \mid M$ has type 1 or type 2$\}$. However, for $y \in \mathscr{C}$ and M of type 1, we have $y\bar{R} \subseteq R$, so that $y\bar{R}_M \subseteq R_M$. Hence by Proposition 3.1, $y \in MR_M$, and it follows that $y \in M$. The argument for type 2 maximal ideals is similar. Thus $\mathscr{C}$ is non-zero and may be represented as indicated. $\square$

Remark 3.7. Note that by Proposition 3.5, the conclusion of Lemma 3.6 holds for an arbitrary semilocal QQR-domain (without assuming that $\bar{R}$ is an Ω-domain).

Proposition 3.8. *Let R be a QQR-domain whose integral closure $\bar{R}$ is an Ω-domain. Then the contraction map from* $\mathrm{Spec}(\bar{R}) \setminus \mathrm{Max}(\bar{R})$ *to* $\mathrm{Spec}(R) \setminus \mathrm{Max}(R)$ *is a homeomorphism.*

Proof. We may assume that $R \neq \bar{R}$, and we may as well assume that R (and $\bar{R}$) has non-maximal ideals. We have already observed in Lemma 3.4 that the map is a one-to-one correspondence. By Corollary 2.9, the Jacobson radical of $\bar{R}$ is non-zero. Hence by Lemma 3.6, R and $\bar{R}$ share the non-zero ideal $\mathscr{C}$. Moreover, it is not difficult to see that, for $Q \in \mathrm{Spec}(\bar{R}) \setminus \mathrm{Max}(\bar{R})$ and an ideal J of $\bar{R}$, we have $Q \not\supseteq J \Leftrightarrow Q \not\supseteq J\mathscr{C} \Leftrightarrow Q \cap R \not\supseteq J\mathscr{C}$. The result follows. $\square$

Lemma 3.9. *Let R be a QQR-domain whose integral closure $\bar{R}$ is an Ω-domain, let $P \in \operatorname{Spec}(R)$, and let $Q \in \operatorname{Spec}(\bar{R})$ satisfy $Q \cap R = P$.*

(1) *If P is non-maximal or P is a type 0 maximal ideal, then*
   (a) *$\mathscr{I}_R(P) = \mathscr{I}_{\bar{R}}(Q)$, and*
   (b) *P is a g-ideal $\Leftrightarrow$ Q is a g-ideal.*

(2) *If P is a type 1 or type 2 maximal ideal, and P is a g-ideal, then Q is a g-ideal.*

Proof. (1): We first claim that $\mathscr{I}_R(P) \subseteq \mathscr{I}_{\bar{R}}(Q)$. Let q be prime in $\bar{R}$ with $q \nsubseteq Q$. Then (by going up in the integral extension $R \subseteq \bar{R}$) we have $q \cap R \nsubseteq P$, and the claim follows easily. It is also easy to see that $\mathscr{I}_R(P) \supseteq \mathscr{I}_{\bar{R}}(Q) \cap R$. However, observe that $\mathscr{I}_{\bar{R}}(Q) \subseteq \mathscr{J} \subseteq \mathscr{C}$, where $\mathscr{C}$ is the conductor as described in Lemma 3.6. Statement (a) now follows, and (b) follows from (a).

(2): If P is a type 1 maximal ideal of R, then an argument similar to the one given above shows that $\mathscr{I}_R(P) \subseteq \mathscr{I}_{\bar{R}}(Q)$, and it follows easily that if $Q \supseteq \mathscr{I}_{\bar{R}}(Q)$, then $P \supseteq \mathscr{I}_R(P)$, that is, if Q is not a g-ideal, then P is not a g-ideal.

Finally, let P be type of 2, and assume that Q is not a g-ideal. Let Q' denote the other maximal ideal of $\bar{R}$ contracting to P. Let $J_{Q'} = \bigcap_{q \in (Q')^{\downarrow} \setminus Q^{\downarrow}} q$. Now $Q \supseteq \mathscr{I}_{\bar{R}}(Q) = J_{Q'} \cap (\bigcap \{q \in \operatorname{Spec}(\bar{R}) \mid q \nsubseteq Q \cup Q'\})$. By Lemma 2.18, $Q \nsupseteq J_{Q'}$. Hence $Q \supseteq \bigcap \{q \in \operatorname{Spec}(\bar{R}) \mid q \nsubseteq Q \cup Q'\}$, from which it follows that $P \supseteq \mathscr{I}(P)$. This completes the proof. □

Theorem 3.10. *Let R be a domain. Then the following statements are equivalent:*

(1) *R is an Ω-domain.*

(2) *R is a QQR-domain, each type 1 maximal and each type 2 maximal ideal of R is a g-ideal, and each prime non-g-ideal of R is unbranched.*

Proof. (1) $\Rightarrow$ (2): Of course, R is a QQR-domain. Let P be a prime non-g-ideal of R; we wish to show that P is unbranched. We may assume that P is a not a type 1 or type 2 maximal ideal of R. Let $Q \in \operatorname{Spec}(R)$ satisfy $Q \cap R = P$. By Lemma 3.9, Q is a non-g-ideal of $\bar{R}$. Since $\bar{R}$ is an Ω-domain, Q is unbranched, whence P is also unbranched. Now let M be a type 1 maximal ideal, and assume by way of contradiction that $\mathscr{I}_R(M) \subseteq M$. Let $N \in \operatorname{Max}(\bar{R})$ satisfy $N \cap R = M$. Also, let $p \in M^{\downarrow} \setminus \{M\}$, and let $q \in \operatorname{Spec}(\bar{R})$ satisfy $q \cap R = p$. By Theorem 2.11, $\mathscr{D}_{\bar{R}}(\mathscr{I}_{\bar{R}}(N)) = N^{\downarrow}$ or $N^{\downarrow} \setminus \{N\}$. In either case, we have $q \nsupseteq \mathscr{I}_{\bar{R}}(N)$, whence $q \nsupseteq \mathscr{C}\mathscr{I}_{\bar{R}}(N)$ and hence $p \nsupseteq \mathscr{I}_R(M)$. Thus $M^{\downarrow} \setminus \{M\} = \mathscr{D}_R(\mathscr{I}_R(M))$, and we have $\Omega_R(\mathscr{I}_R(M)) = \bigcap \{R_p \mid p \in \operatorname{Spec}(R), p \nsupseteq \mathscr{I}_R(M)\} = \bigcap \{R_p \mid p \in M^{\downarrow} \setminus \{M\}\} = \bigcap \{\bar{R}_q \mid q \in N^{\downarrow} \setminus \{N\} \supseteq \bar{R}_N \supsetneqq R_M$. This contradicts Lemma 2.4. Hence M must be a g-ideal. A similar (but slightly more complicated) argument shows that each type 2 maximal ideal is a g-ideal.

(2) $\Rightarrow$ (1): The hypothesis and Lemma 3.9 guarantee that each prime non-g-ideal of $\bar{R}$ is unbranched. Hence $\bar{R}$ is an Ω-domain. To show that R is an Ω-domain, we need only show that R_P is an Ω-overring of R for each prime non-g-ideal P of R. If P is a nonmaximal prime non-g-ideal of R and $Q \in \operatorname{Spec}(\bar{R})$ satisfies $Q \cap R = P$,

then making use of Lemma 3.9, we have $\Omega_R(\mathscr{I}_R(P)) = \bigcap\{R_p \mid p \in \operatorname{Spec}(R), p \not\supseteq \mathscr{I}(P)\} = \bigcap\{\bar{R}_q \mid q \in \operatorname{Spec}(\bar{R}), q \not\supseteq \mathscr{I}_{\bar{R}}(Q)\} = \Omega_{\bar{R}}(\mathscr{I}_{\bar{R}}(Q)) = \bar{R}_Q = R_P$. (The penultimate equality follows from the fact that $\bar{R}$ is a Ω-domain.) A similar argument works for type 0 maximal ideals. Localizations at type 1 or 2 maximal ideals are automatically Ω-overrings, since such maximal ideals are g-ideals by hypothesis. □

Recall that a prime ideal P of a domain R is said to be *divided* (in the sense of Dobbs [2]) if $P = PR_P$. It is well known that a divided prime of a domain R is comparable to every ideal of R. We show in Theorem 3.12 below that, if a domain R has a divided prime P, then the question as to whether R is an Ω-domain depends only on R/P, R_P, and, possibly, whether P is unbranched.

Lemma 3.11. *Let R be a domain with quotient field K, and let P be a divided prime of R. Set $k(P) = R_P/PR_P$ (which is canonically isomorphic to the quotient field of R/P). Now let S be an overring of R with $S \subsetneq R_P$. Then the following diagrams are pullbacks*:

$$\begin{array}{ccc} R & \to & R/P \\ \downarrow & & \downarrow \\ S & \to & S/P \\ \downarrow & & \downarrow \\ R_P & \xrightarrow{\varphi} & k(P). \end{array}$$

Moreover, if I is an ideal of R, then $S = \Omega_R(I)$ if and only if $P \subsetneq I$ and $S/P = \Omega_{R/P}(I/P)$.

Proof. Note that P is necessarily a prime ideal of S since P is divided. It is clear that the diagrams are pullbacks. Now suppose that $S = \Omega_R(I)$. We claim that $I \supsetneq P$. Otherwise, $I \subseteq P$, and hence $S = \Omega_R(I) = \bigcap_{Q \not\supseteq I} R_Q \supseteq R_P$ (since $Q \not\supseteq I$ implies that $Q \subseteq P$), a contradiction. Similarly, observe that if $I \supsetneq P$, then $\Omega(I) \subseteq R_P$. The conclusion now follows easily from the fact that for $x \in R_P$ and $a \in I$, we have $xa^n \in R \Leftrightarrow \varphi(x)\varphi(a)^n \in R/P$. □

Theorem 3.12. *Let P be a nonmaximal divided prime ideal of a domain R. Then R is an Ω-domain if and only if*

(1) *R_P is a valuation Ω-domain*,
(2) *R/P is an Ω-domain, and*
(3) *if P is a non-g-ideal, then P is unbranched.*

Proof. Assume that R is an Ω-domain. Then Proposition 2.2 implies that R_P is an Ω-domain, and, since P is non-maximal, R_P is a valuation domain by [8, Theorem 1.5]. This proves (1). Statement (2) follows from Lemma 3.11, and (3) follows from Theorem 3.10.

For the converse, let S be an overring of R; we wish to show that S is an Ω-overring. We first claim that S is comparable to R_P. To verify this, suppose that $S \nsubseteq R_P$, and pick $s \in S \setminus R_P$. Since R_P is a valuation domain and P is divided, we have $1/s \in PR_P = P$. It follows that if Q is a maximal ideal of S, then $1/s \notin Q \cap R$. Thus, again since P is divided, we have $Q \cap R \subsetneq P$, and hence $R_P \subseteq R_{Q \cap R} \subseteq S_Q$. Thus $R_P \subseteq S$, proving the claim. If $S \subsetneq R_P$, then S is an Ω-overring of R by Lemma 3.11. Suppose that $S = R_P$. If P is a g-ideal of R, then $S = R_P$ is an Ω-overring by Proposition 1.8. If P is not a g-ideal, then P is unbranched, and we have $S = R_P = \Omega(P)$. Finally, suppose that $S \supsetneq R_P$. Since R_P is a valuation domain, $S = R_Q$ for some prime $Q \subsetneq P$. If Q is a g-ideal of R, then S is an Ω-overring of R. If Q is not a g-ideal of R, then it is easy to see that QR_P is not a g-ideal of R_P; since R_P is a valuation Ω-domain, this implies that QR_P is unbranched in R_P, whence Q is unbranched in R. Since (as is easily shown) Q is divided, we have $R_Q = \Omega(Q)$. This completes the proof. □

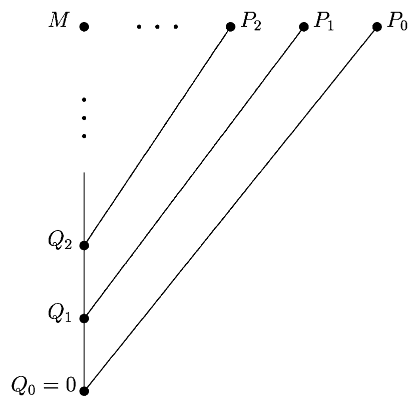
Theorem 3.13. *Let R be a semilocal domain. Then R is an Ω-domain $\Leftrightarrow$ R is a QQR-domain and $\bar{R}$ is an Ω-domain.*

Proof. If R is an Ω-domain, then R is a QQR-domain and $\bar{R}$ is a (Prüfer) Ω-domain (even without the semilocal hypothesis). Suppose that R is a QQR-domain and that $\bar{R}$ is an Ω-domain. By Theorem 3.10, we need only show that each non-maximal prime non-g-ideal of R is unbranched and that each maximal ideal (of type 1 or 2) is a g-ideal. The first statement follows easily from Lemma 3.9. Let M be a type 1 maximal ideal of R, and let N be the maximal ideal of $\bar{R}$ with $N \cap R = M$. It is not hard to show that $\mathscr{I}_R(M) = \mathscr{I}_{\bar{R}}(N) \cap R$. However, by Lemma 2.18, $\mathscr{I}_{\bar{R}}(N)$ is a finite intersection of primes, and, since N is a g-ideal by Proposition 2.19, none of these primes is contained in N. It follows that $\mathscr{I}_R(M)$ is also a finite intersection of primes, none of which is contained in M (by going up in the integral extension $R \subseteq \bar{R}$). Hence M is a g-ideal. If M has type 2, the argument is similar. Let N_1 and N_2 denote the maximal ideals of $\bar{R}$ which contract to M. For N' maximal in $\bar{R}$ with $N' \notin \{N_1, N_2\}$, set $J_{N'} = \bigcap \{Q \in \operatorname{Spec}(\bar{R}) \mid Q \subseteq N', Q \nsubseteq N_1 \cup N_2\}$. Then $J_{N'}$ is prime, and $J_{N'} \nsubseteq N_1 \cup N_2$ (since N_1 and N_2 are g-ideals). It follows that $J_{N'} \cap R \nsubseteq M$. Now since $\mathscr{I}_R(M) = \bigcap_{N' \notin \{N_1, N_2\}} (J_{N'} \cap R)$, we have $\mathscr{I}_R(M) \nsubseteq M$. Hence M is a g-ideal, as claimed. □

4. A non-semilocal example

In this section, we use a construction due to Fischer [3] to produce an example of a non-semilocal Prüfer Ω-domain. Note that in such an example, at least one maximal ideal must be a non-g-ideal (Remark 2.20). Hence a "simplest" example would have all but one of the maximal ideals being g-ideals. Of course, the maximal non-g-ideal must be unbranched.

Example 4.1. Let X denote the partially ordered set pictured below:



Now endow X with the *closure of points* topology: For each $x \in X$, let $x^{\uparrow} = \{y \in X \mid y \geq x\}$, and take the sets $x^{\uparrow}$ as a closed subbase for a topology on X. Thus the subbasic closed sets are the (sets containing the) points $M, P_0, P_1, P_2, \ldots$ and the sets $Q_i^{\uparrow} = \{Q_i, Q_{i+1}, \ldots\} \cup \{M\} \cup \{P_i, P_{i+1}, \ldots\}$. By [3, Lemma 2.7 and Theorem 2.1], there is a Bézout domain R whose spectrum is homeomorphic to the one just described. It is then easy to verify that the only prime non-g-ideal is M, and since M is unbranched by construction, R is a Prüfer Ω-domain by Theorem 2.11. Hence R is the desired example.

We close with a question: *Is an Ω-domain necessarily a T-domain*? Note that this question has a positive answer if we assume that the domain is semilocal and integrally closed [Theorem 2.22]. We have not been able to determine whether the example above is a T-domain.

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