NAGATA RINGS, KRONECKER FUNCTION RINGS AND RELATED SEMISTAR OPERATIONS: A SURVEY

presented by

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1. The Genesis

Toward the middle of the XIXth century, E.E. Kummer discovered that the ring of integers of a cyclotomic field does not have the unique factorization property.

Few years later, in 1847 Kummer introduced the concept of "ideal numbers" to re-establish some of the factorization theory for cyclotomic integers with prime exponents. (In 1856 he generalized his theory to the case of cyclotomic integers with arbitrary exponents.)

As R. Dedekind wrote in 1877 to his former student E. Selling, the *goal* of *a* general theory was immediately clear after Kummer's solution in the special case of cyclotomic integers: to extend Kummer's theory to the case of general algebraic integers.

Dedekind admitted to having struggled unsuccessfully for many years before he published the first version of his theory in 1871 (XI supplement to Dirichlet's "Vorlesungen über Zahlentheorie").

The theory of Dedekind domains, as it is known today, is based on original Dedekind's ideas and results: Dedekind's point of view is based on ideals ("ideal numbers") for generalizing the algebraic numbers; then he proved that, *in the ring of the integers of an algebraic number field, each proper ideal factors uniquely into a product of prime ideals.*

L. Kronecker has essentially achieved this goal in 1859, about 12 years after Kummer's pioneering work, but he published nothing until 1882 (the paper appeared in honor of the 50th anniversary of Kummer's doctorate).

Kronecker's theory holds in a larger context than that of ring of integers of algebraic numbers and solves a more general problem. The primary objective of his theory was to extend the set of elements and the concept of divisibility in such a way any finite set of elements has a GCD (greatest common divisor).

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Main references for the "classical" Kronecker function ring

L. Kronecker (1882), W. Krull (1936), H. Weyl (1940), H.M. Edwards (1990).

It is probably for this reason that the basic objects of Kronecker's theory –corresponding to Dedekind's "ideals"– are called "divisors".

Let D_0 be a PID with quotient field K_0 and let K be a finite field extension of K_0. The *Kronecker's divisors* are precisely all the possible GCD's of finite sets of elements of K that are algebraic over K_0; *a divisor* is *integral* if it is the GCD of a finite set of elements of the integral closure D of D_0 in K.

One of the key points of Kronecker's theory is that it is possible to give an explicit description of the "divisors". The divisors can be represented as equivalent classes of polynomials and a given polynomial in $D[X]$ represents the class of the integral divisor associated with the set of his coefficients.

More precisely, in the previous setting, with a modern terminology, *the Kronecker function ring of D* is given by:

$$
\begin{aligned}
\mathrm{Kr}(D) := & \left\{ \frac{f}{g} \mid f, g \in D[X] \ \textit{and} \ \boldsymbol{c}(f) \subseteq \boldsymbol{c}(g) \right\} \\
= & \left\{ \frac{f'}{g'} \mid f', g' \in D[X] \ \textit{and} \ \boldsymbol{c}(g') = D \right\},
\end{aligned}
$$

(where $\boldsymbol{c}(h)$ denotes *the content* of a polynomial $h \in D[X]$, i.e. the ideal of D generated by the coefficients of h).

Note that the previous equality holds since we are assuming that D is a Dedekind domain (being the integral closure of D_0, which is a PID, in a finite field extension K of the quotient field K_0 of D_0).
In this case, for each polynomial $g \in D[X]$, $\boldsymbol{c}(g)$ is an invertible ideal of D and, by choosing a polynomial $u \in K[X]$ such that $\boldsymbol{c}(u) = (\boldsymbol{c}(g))^{-1}$, then we have $f/g = uf/ug = f'/g'$, with $f' := uf$, $g' := ug \in D[X]$ and, obviously, $\boldsymbol{c}(g') = D$.

The fundamental properties of the Kronecker function ring are the following:

(1) $\mathrm{Kr}(D)$ *is a Bézout domain (i.e. each finite set of elements has a GCD and the GCD can be expressed as linear combination of these elements) and* $D[X] \subseteq \mathrm{Kr}(D) \subseteq K(X)$ *(in particular, the field of rational functions* $K(X)$ *is the quotient field of* $\mathrm{Kr}(D)$*).*

(2) *Let* $a_0, a_1, \ldots, a_n \in D$ *and set* $f := a_0 + a_1 X + \ldots + a_n X^n \in D[X]$*, then:*

$(a_0, a_1, \ldots, a_n)\mathrm{Kr}(D) = f\mathrm{Kr}(D)$ *(thus,* $\mathrm{GCD}_{\mathrm{Kr}(D)}(a_0, a_1, \ldots, a_n) = f$*),*
$f\mathrm{Kr}(D) \cap K = (a_0, a_1, \ldots, a_n)D = \boldsymbol{c}(f)D$ *(hence,* $\mathrm{Kr}(D) \cap K = D$*).*

Kronecker's classical theory leaded to two different major extensions:

- Beginning from 1936, W. Krull generalized the Kronecker function ring to the more general context of *integrally closed domains*, by introducing ideal systems associated to particular star operations: the e.a.b. (endlich arithmetisch brauchbar) star operations.

- Beginning from 1956, M. Nagata investigated, *for an arbitrary integral domain D*, the domain

$$D(X) := \left\{ \frac{f}{g} \mid f, g \in D[X] \text{ and } \boldsymbol{c}(g) = D \right\} .$$

It is now wellknown that $D(X)$ coincides with $\mathrm{Kr}(D)$ $(= \{f/g \mid f, g \in D[X] \text{ and } \boldsymbol{c}(f) \subseteq \boldsymbol{c}(g)\})$ if (and only if) D is a Prüfer domain.

2. Krull's generalization of the Kronecker function ring

One of the major difficulties for generalizing the Kronecker's theory is that Gauss Lemma for the content of polynomials holds for Dedekind domains (or, more generally, for Prüfer domains), but not in general: *let* $f, g \in D[X]$, *where* D *is a Prüfer domain, then:*

$$\boldsymbol{c}(fg) = \boldsymbol{c}(f)\boldsymbol{c}(g)\,.$$

In the general situation, we have the following result:

Dedekind-Mertens Lemma: *Let* D *be an integral domain and* $f, g \in D[X]$. *Let* $m := \deg(g)$, *then:*

$$\boldsymbol{c}(f)^m\boldsymbol{c}(fg) = \boldsymbol{c}(f)^{m+1}\boldsymbol{c}(g)\,.$$

In order to overcome this obstruction for the definition of a ring of "Kronecker functions type" in the general context, Krull introduced multiplicative ideal systems having a "nice" cancellation property, defined by what we call *e.a.b. star operations*, that is star operations $\star$ such that, for all nonzero finitely generated ideals $I,\ J,\ H$:

$$(IJ)^\star \subseteq (IH)^\star \ \Rightarrow\ J^\star \subseteq H^\star\,.$$

In this context, Krull recovers an useful identity for the contents of polynomials:

Let $\star$ be an e.a.b. star operation on an integral domain D (this condition implies that D is an integrally closed domain) and let $f,\ g \in D[X]$ then:

$$\boldsymbol{c}(fg)^\star = \boldsymbol{c}(f)^\star \boldsymbol{c}(g)^\star\,.$$

Remark 1 Let $\boldsymbol{F}(D)$ [respectively, $\boldsymbol{f}(D)$] be the set of all fractionary ideals [respectively, finitely generated fractionary ideals] of an integral domain D. A mapping $\star : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $I \mapsto I^\star$, is called *a star operation of D* if, for all $z \in K$, $z \neq 0$ and for all $I, J \in \boldsymbol{F}(D)$, the following properties hold:

$(\star\star_1)$ $(zD)^\star = zD$, $(zI)^\star = zI^\star$;

$(\star_2)$ $I \subseteq J \Rightarrow I^\star \subseteq J^\star$;

$(\star_3)$ $I \subseteq I^\star$ and $I^{\star\star} := (I^\star)^\star = I^\star$.

Krull introduced the concept of a star operation in his first *Beiträge paper* (1936). He used the notation " ′–Operation " ("Strich–Operation") for his generic operation. [In this paper you can find the terminology " ′–Operation " in footnote 13 and in the title of Section 6, among other places.]

Next page contains some of the key parts of Krull's Beiträge paper.

Beiträge zur Arithmetik kommutativer Integritätsbereiche. I. Multiplikationsringe, ausgezeichnete Idealsysteme und Kroneckersche Funktionalringe [43]

Math. Z. 41 (1936), 545–577
[JFM 62.1105.01; Zbl 015.00203]

In meinem Bericht über die neuere Entwicklung der kommutativen Idealtheorie[1]) habe ich gezeigt, daß die arithmetische Untersuchung und insbesondere die Entwicklung der Teilbarkeitslehre beliebiger ganz abgeschlossener Integritätsbereiche durch geeignete Verknüpfung bewertungs- und idealtheoretischer Methoden weitgehend gefördert und vielfach zu einem wenigstens vorläufigen Abschluß gebracht werden kann. Naturgemäß handelte es sich nur um eine Skizze, bei der allein die Umrisse im Großen deutlich herausgearbeitet wurden, während auf eine genauere Ausführung der Einzelheiten verzichtet werden mußte. Die folgenden Beiträge sind einerseits dazu bestimmt, die Lücken des Berichts auszufüllen, andererseits sollen sie nach Möglichkeit die Theorie über die bisherigen Grenzen hinaus entwickeln. Die Untersuchungen des ersten Beitrags beschäftigen sich mit allgemeinen, durch keinerlei Zusatzaxiom genauer gekennzeichneten ganz abgeschlossenen Integritätsbereichen. Die späteren Beiträge sollen vor allem Anwendungen auf mehr oder minder spezielle Ringklassen bringen. („Vollständig ganz" abgeschlossene Integritätsbereiche, endliche diskrete Hauptordnungen und ihre unendlichen algebraischen Erweiterungen, Potenzreihenringe usw.).

4. Wertideale und Funktionalring.

Es sei der Integritätsbereich $\mathfrak{J}$ fest gegeben, $\mathfrak{B}_1, \ldots, \mathfrak{B}_\tau, \ldots$ seien die sämtlichen $\mathfrak{J}$ umfassenden Bewertungsringe aus $\mathfrak{K}$, $w_\tau(a)$ bedeute den Wert des Körperelements a in der zu $\mathfrak{B}_\tau$ gehörigen Bewertung B_τ. — Ist $\mathfrak{a}$ ein beliebiges Ideal aus $\mathfrak{K}$, so setzen wir $\mathfrak{a}' = \mathfrak{a}_b = \underset{\tau}{\Delta}(\mathfrak{a} \cdot \mathfrak{B}_\tau)$, wobei die Durchschnittsbildung über alle $\mathfrak{B}_\tau$ zu erstrecken ist. $\mathfrak{a}_b$ besteht (außer der Null) aus allen und nur den Körperelementen a, zu denen es für jedes τ in $\mathfrak{a}$ ein der Ungleichung $w_\tau(a_\tau) \leqq w_\tau(a)$ genügendes Element a_τ gibt. — Die „*b-Operation*", die von $\mathfrak{a}$ zu $\mathfrak{a}' = \mathfrak{a}_b$ führt, genügt den folgenden Bedingungen[13]):

(1) $\mathfrak{J}' = \mathfrak{J}$. (2) $\mathfrak{a}' \supseteq \mathfrak{a}$; aus $\mathfrak{a} \supseteq \mathfrak{b}$ folgt $\mathfrak{a}' \supseteq \mathfrak{b}'$. (3) $(\mathfrak{a}')' = \mathfrak{a}'$.
(4) $(\mathfrak{a} + \mathfrak{b})' = (\mathfrak{a}' + \mathfrak{b}')'$. (5) $(\mathfrak{a} \cdot \mathfrak{b})' = (\mathfrak{a}' \cdot \mathfrak{b}')'$. (6) $(\mathfrak{a}' \frown \mathfrak{b}')' = \mathfrak{a}' \frown \mathfrak{b}'$.
(7) $(a)' = (a)$; $(a) \cdot \mathfrak{a}' = ((a) \cdot \mathfrak{a})'$.

[13]) Die folgenden Formeln werden uns in 6. zur Definition des Begriffes der allgemeinen '-Operation dienen. Deshalb schreiben wir schon hier bei ihnen überall $\mathfrak{a}'$ statt $\mathfrak{a}_b$. — Zu den Formeln (1) bis (7) vgl. Bericht 43., wo allerdings die Formeln (1) und (6) des Textes fehlen.

6. '-Operationen und *w*-Operationen.

Eine Rechenvorschrift, die jedem $\mathfrak{J}$-Ideal $\mathfrak{a}$ aus $\mathfrak{K}$ eindeutig ein Ideal $\mathfrak{a}'$ zuordnet, und zwar so, daß die in 4. für die b-Operation bewiesenen Formeln (1) bis (7) gelten, soll als „'-Operation" bezeichnet werden. Ein Ideal $\mathfrak{a}$, das der Gleichung $\mathfrak{a}' = \mathfrak{a}$ genügt, heißt „'-Ideal"; entsteht $\mathfrak{a}' = (a_1, \ldots, a_n)'$ aus einem endlichen Ideal durch den '-Prozeß, so ist $\mathfrak{a}'$ als „'-endlich" anzusehen. Gilt für eine '-Operation der Eindeutigkeitssatz 10 von 4., so wollen wir sie „*arithmetisch brauchbar*" nennen[17]).

[17]) Natürlich ist dabei (ebenso wie nachher beim Gaußschen Satz) $\mathfrak{a}'$ für $\mathfrak{a}_b$ zu setzen, und es hat das '-Produkt $\mathfrak{a}' \times \mathfrak{b}' = (\mathfrak{a}' \cdot \mathfrak{b}')'$ an die Stelle des b-Produktes $\mathfrak{a}_b \times \mathfrak{b}_b$ zu treten.

Satz 10. *Eindeutigkeitssatz: Aus* $\mathfrak{a}_b \times \mathfrak{b}_b \subseteq \mathfrak{a}_b \times \mathfrak{c}_b$ *folgt stets* $\mathfrak{b}_b = \mathfrak{c}_b$, *falls* $\mathfrak{a}_b$ *„b-endlich" ist, d. h. aus einem endlichen Ideal durch den b-Prozeß entsteht,* $\mathfrak{a}_b = (a_1, \ldots, a_r)_b$.

The notation " $*$–operation " ("star–operation") arises from Section 26 of the original version of Gilmer's "Multiplicative Ideal Theory" (Queen's, 1968).

Robert Gilmer explained to me that ≪ I believe the reason I switched from " $'$–Operation " to " $*$–operation " was because " $'$ " was not so generic at the time: I' was frequently used as the notation for the integral closure of an ideal I, just as D' was used to denote the integral closure of the domain D. (Such notation was used, for example, in both Nagata's Local Rings and in Zariski-Samuel's two volumes.) ≫

Moreover, Krull only considered the concept of an "arithmetisch brauchbar (a.b.) $'$–Operation", not an e.a.b. operation. [An *a.b.–operation* is a star operation $\star$ such that, if $I \in \boldsymbol{f}(D)$ and $J, K \in \boldsymbol{F}(D)$ and if $(IJ)^\star \subseteq (IH)^\star$ then $J^\star \subseteq H^\star$.] The e.a.b. concept stems from the original version of Gilmer's book (1968). The results of Section 26 show that this (presumably) weaker concept is all that one needs to develop a complete theory of Kronecker function rings.

Robert Gilmer explained to me that ≪ I believe I was influenced to recognize this because during the 1966 calendar year in our graduate algebra seminar (Bill Heinzer, Jimmy Arnold, and Jim Brewer, among others, were in that seminar) we had covered Bourbaki's Chapitres 5 and 7 of *Algèbre Commutative*, and the development in Chapter 7 on the v–operation indicated that e.a.b. would be sufficient.≫

I thank Robert Gilmer and Franz Halter-Koch for some information contained in this remark.

Using the star operations, in 1936 W. Krull defined a "well-behaved" Kronecker function ring in a more general setting than Kronecker's setting.

Let D be an integrally closed integral domain with quotient field K and let $\star$ be an e.a.b. star operation on D, then:

$$\mathsf{Kr}(D,\star) := \left\{ \frac{f}{g} \mid f, g \in D[X] \text{ and } \boldsymbol{c}(f)^\star \subseteq \boldsymbol{c}(g)^\star \right\}$$

is an integral domain with quotient field $K(X)$, called the $\star$–Kronecker function ring of D, *having the following properties:*

(1) $\mathsf{Kr}(D,\star)$ *is a Bézout domain and* $D[X] \subseteq \mathsf{Kr}(D,\star) \subseteq K(X)$.

(2) *Let* $a_0, a_1, \ldots, a_n \in D$ *and set* $f := a_0 + a_1X + \ldots + a_nX^n \in D[X]$, *then:*

$$(a_0, a_1, \ldots, a_n)\mathsf{Kr}(D,\star) = f\mathsf{Kr}(D,\star)\,,$$
$$(a_0, a_1, \ldots, a_n)\mathsf{Kr}(D,\star) \cap K = ((a_0, a_1, \ldots, a_n)D)^\star \;\; (\textit{i.e. } f\mathsf{Kr}(D,\star) \cap K = (\boldsymbol{c}(f))^\star)\;.$$

(In particular, $\mathsf{Kr}(D,\star) \cap K = D^\star = D\,$.*)*

In particular, the previous construction can be applied:

• when $\star$ is *the identity star operation* d on a Prüfer domain D; this case gives back the "classical" Kronecker function ring $\mathrm{Kr}(D)$, when D is a Dedekind domain (= Prüfer + Noetherian domain).

• when $\star$ is *the Artin's v–operation* on a Krull domain D (since in this case the v–operation is an e.a.b. star operation and is equivalent to the operation of "completion" with respect to the rank 1 discrete valuation overrings of D: $I \mapsto \cap\{ID_P \mid P \in \mathrm{Spec}^1(D)\}$).
This case gives rise to an effective extension: the construction of the Kronecker function ring for Krull domains.

[Note that:

Prüfer + Krull $\Leftrightarrow$ Dedekind domain.]

3. Nagata's "generalization" of the Kronecker function ring

Nagata's construction is possible for each integral domain D (even non integrally closed):

$$\mathrm{Na}(D) := D(X) := \left\{ \frac{f}{g} \mid f, g \in D[X] \text{ and } \boldsymbol{c}(g) = D \right\},$$

(cf. Krull (1943), Nagata's book (1962), Samuel (1964)),
but, in general Na(D) is not a Bézout domain.
It is not difficult to see that:
Na(D) *is a Bézout domain if (and only if)* D *is a Prüfer domain.*
Equivalently:
Na(D) *coincides with* Kr(D) *if (and only if)* D *is a Prüfer domain.*

The interest in the Nagata's ring $D(X)$ is due to the fact that this integral domain of rational functions has some "nice" properties that D itself need not have, mantaining in any case a strict relation with the ideal structure of D.

(a) *The map $P \mapsto PD(X)$ establishes a 1-1 correspondence between the maximal ideals of D and the maximal ideals of $D(X)$.*

(b) *For each ideal I of D,*

$$ID(X) \cap D = I\,, \qquad D(X)/ID(X) \cong (D/ID)(X)\,;$$

I is finitely generated if and only if $ID(X)$ is finitely generated.

Among the "new" properties acquired by $D(X)$ we mention the following:

(c) *the residue field at each maximal ideal of $D(X)$ is infinite;*

(d) *an ideal contained in a finite union of ideals is contained in one of them;*

(e) *each finitely generated locally principal ideal is principal (thus $\mathrm{Pic}(D(X)) = 0$).*

(cf. J. Arnold (1969), Gilmer-Mott (1970), Gilmer's book (1972), Quartararo-Butts (1975), D.D. Anderson (1977)).

4. Principal focuses of this talk

In a recent series of papers, in collaboration with K. Alan Loper, we investigated properties of *the Kronecker function rings* which arise from arbitrary semistar operations $\star$ on an integral domain D and we generalized Kang's notion of *a star Nagata ring* (1989) to the semistar setting.

The principal focuses of the present talk are the similarities between the ideal structure of the Nagata $\mathrm{Na}(D, \star)$ and Kronecker $\mathrm{Kr}(D, \star)$ *semistar* rings and between the natural semistar operations that these two types of function rings give rise to on D.

Moreover, I would like also to present some of the results of a work in progress with K.A. Loper in which these similarities lead naturally to study a "new" integral domain of rational functions
$\mathrm{Lo}(D, \star)$*, obtained as an intersection of local Nagata domains associated to a given semistar operation,* $\mathrm{Na}(D, \star) \subseteq \mathrm{Lo}(D, \star) \subseteq \mathrm{Kr}(D, \star)$.

$\mathrm{Lo}(D, \star)$ *generalizes at the same time* $\mathrm{Na}(D, \star)$ *and* $\mathrm{Kr}(D, \star)$ *and coincides with* $\mathrm{Na}(D, \star)$ *or* $\mathrm{Kr}(D, \star)$ *when the semistar operation* $\star$ *assumes "extreme values".*

5. Notation and basic facts on semistar operations

For the duration of this talk D will represent an integral domain with quotient field K.

Let $\overline{\boldsymbol{F}}(D)$ represent the set of all nonzero D–submodules of K.

Let $\boldsymbol{F}(D)$ represent the nonzero fractionary ideals of D (i.e. $E \in \overline{\boldsymbol{F}}(D)$ such that $dE \subseteq D$, for some nonzero element $d \in D$).

Finally, let $\boldsymbol{f}(D)$ represent the finitely generated D-submodules of K. Obviously:

$$\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D).$$

In 1994, Okabe and Matsuda introduced the more “flexible” notion of semistar operation $\star$ of an integral domain D, as a natural generalization of the notion of star operation, allowing $D \neq D^\star$.

More precisely, a mapping $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $z \in K$, $z \neq 0$ and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold:

$(\star_1)$ $(zE)^\star = zE^\star$;

$(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

$(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

When $D^\star = D$, we say that $\star$ is *a (semi)star operation of D*, since, restricted to $\boldsymbol{F}(D)$ it is *a star operation of D*

[i.e. $\star : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$ verifies the properties $(\star_2)$, $(\star_3)$ and
$(\star\star_1)$ $(zD)^\star = zD$, $(zE)^\star = zE^\star$].

For star operations, the notion of *$\star$–ideal* (that is, a nonzero ideal $I \subseteq D$, such that $I^\star = I$) leads to the definition of a canonically associated ideal system.

For semistar operations, we need a more general notion, that coincides with the notion of $\star$–ideal, when $\star$ is a (semi)star operation.
We say that a nonzero (integral) ideal I of D is a *quasi–$\star$–ideal* if $I^\star \cap D = I$.

For example, it is easy to see that, *if $I^\star \neq D^\star$, then $I^\star \cap D$ is a quasi–$\star$–ideal that contains I* (in particular, a $\star$–ideal is a quasi–$\star$–ideal).
Note that:
– when $D = D^\star$ the notions of quasi–$\star$–ideal and $\star$–ideal coincide;
– $I^\star \neq D^\star$ is equivalent to $I^\star \cap D \neq D$.

Similarly, we designate by *quasi–$\star$–prime* [respectively, *$\star$–prime*] of D a quasi–$\star$–ideal [respectively, an integral $\star$–ideal] of D which is also a prime ideal.

We designate by *quasi–$\star$–maximal* [respectively, *$\star$–maximal*] of D a maximal element in the set of all proper quasi–$\star$–ideals [respectively, integral $\star$–ideals] of D.

We denote by Spec$^\star(D)$ [respectively, Max$^\star(D)$, QSpec$^\star(D)$, QMax$^\star(D)$] the set of all $\star$–primes [respectively, $\star$–maximals, quasi–$\star$–primes, quasi–$\star$–maximals] of D.

As in the classical star-operation setting, we associate to a *semistar* operation $\star$ of D a new semistar operation $\star_f$ as follows. If $E \in \overline{\boldsymbol{F}}(D)$ we set:

$$E^{\star_f} := \cup\{F^\star \mid F \subseteq E,\, F \in \boldsymbol{f}(D)\}\,.$$

We call $\star_f$ *the semistar operation of finite type of D* associated to $\star$.

If $\star = \star_f$, we say that $\star$ is *a semistar operation of finite type of D*.
Note that $\star_f \leq \star$ and $(\star_f)_f = \star_f$, so $\star_f$ is a semistar operation of finite type of D.

Lemma 2 *Let $\star$ be a non-trivial semistar operation of finite type on D. Then*

(1) *Each proper quasi–$\star$–ideal is contained in a quasi–$\star$–maximal.*

(2) *Each quasi–$\star$–maximal is a quasi–$\star$–prime.*

(3) *Set*

$$\Pi^\star := \{P \in \mathrm{Spec}(D) \mid P \neq 0\,, P^\star \cap D \neq D\}\,.$$

Then $\mathrm{QSpec}^\star(D) \subseteq \Pi^\star$ and the set of maximal elements of $\Pi^\star$, denoted by $\Pi^\star_{\max}$, is nonempty and coincides with $\mathrm{QMax}^\star(D)$. □

For the sake of simplicity, when $\star = \star_f$, we will denote simply by $\mathcal{M}(\star)$, the nonempty set $\Pi^\star_{\max} = \mathrm{QMax}^\star(D)$.

6. Nagata semistar domain

A generalization of the "classical" Nagata ring construction was considered by Kang (1987, 1989).

We further generalize the previous construction so that, given *any* integral domain D and *any* semistar operation $\star$ on D, we define *the semistar Nagata ring* as follows:

$$\mathrm{Na}(D,\star) := \left\{ \frac{f}{g} \mid f, g \in D[X]\,,\ g \neq 0\,,\ \boldsymbol{c}(g)^{\star} = D^{\star} \right\}.$$

Note that, $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\star_f)$. Therefore, the assumption $\star = \star_f$ is not really restrictive when considering Nagata semistar rings.

If $\star = d$ is the identity (semi)star operation of D, then:

$$\mathrm{Na}(D,d) = D(X)\,.$$

Some results on *star* Nagata rings proved by Kang in 1989 are generalized to the semistar setting in the following:

Proposition 3 *Let* $\star$ *be a nontrivial semistar operation of an integral domain* D. *Set:*

$$N(\star) := N_D(\star) := \{h \in D[X] \mid \boldsymbol{c}(h)^\star = D^\star\} \ .$$

(1) $N(\star) = D[X] \setminus \cup\{Q[X] \mid Q \in \mathcal{M}(\star_f)\}$ *is a saturated multiplicatively closed subset of* $D[X]$ *and* $N(\star) = N(\star_f)$.

(2) $\mathrm{Max}(D[X]_{N(\star)}) = \{Q[X]_{N(\star)} \mid Q \in \mathcal{M}(\star_f)\}$.

(3) $\mathrm{Na}(D, \star) = D[X]_{N(\star)} = \cap\{D_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$.

(4) $\mathcal{M}(\star_f)$ *coincides with the canonical image in* $\mathrm{Spec}(D)$ *of the maximal spectrum of* $\mathrm{Na}(D, \star)$; *i.e.* $\mathcal{M}(\star_f) = \{M \cap D \mid M \in \mathrm{Max}(\mathrm{Na}(D, \star))\}$. □

Corollary 4 *Let* D *be an integral domain, then:*

Q *is a maximal* t*–ideal of* $D \Leftrightarrow Q = M \cap D$, *for some* $M \in \mathrm{Max}(\mathrm{Na}(D, v))$.

□

7. The semistar operation canonically associated to $\mathrm{Na}(D,\star)$

If Δ is a nonempty set of prime ideals of an integral domain D, then the semistar operation $\star_\Delta$ defined on D as follows

$$E^{\star_\Delta} := \cap\{ED_P \mid P \in \Delta\}, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D),$$

is called *the spectral semistar operation associated to* Δ.

Lemma 5 *Let D be an integral domain and let $\emptyset \neq \Delta \subseteq \mathrm{Spec}(D)$. Then:*

(1) $E^{\star_\Delta}D_P = ED_P$, *for each $E \in \overline{\boldsymbol{F}}(D)$ and for each $P \in \Delta$.*

(2) $(E \cap F)^{\star_\Delta} = E^{\star_\Delta} \cap F^{\star_\Delta}$, *for all $E, F \in \overline{\boldsymbol{F}}(D)$.*

(3) $P^{\star_\Delta} \cap D = P$, *for each $P \in \Delta$.*

(4) *If I is a nonzero integral ideal of D and $I^{\star_\Delta} \cap D \neq D$ then there exists $P \in \Delta$ such that $I \subseteq P$.* $\square$

- A semistar operation $\star$ of an integral domain D is called *a spectral semistar operation* if there exists $\emptyset \neq \Delta \subseteq \text{Spec}(D)$ such that $\star = \star_\Delta$.

- We say that $\star$ *possesses enough primes* or that $\star$ is *a quasi-spectral semistar operation of D* if, for each nonzero ideal I of D such that $I^\star \cap D \neq D$, there exists a quasi–$\star$–prime P of D such that $I \subseteq P$.

- Finally, we say that $\star$ is *a stable semistar operation on D* if

$$(E \cap F)^\star = E^\star \cap F^\star, \quad \text{for all } E, F \in \overline{\boldsymbol{F}}(D)\,.$$

Lemma 6 *Let $\star$ be a nontrivial semistar operation of an integral domain D. Then:*

(1) *$\star$ is spectral if and only if $\star$ is quasi-spectral and stable.*

(2) *Assume that $\star = \star_f$. Then $\star$ is quasi-spectral and $\mathcal{M}(\star) \neq \emptyset$.* $\square$

Theorem 7 *Let $\star$ be a nontrivial semistar operation and let $E \in \overline{\boldsymbol{F}}(D)$. Set*

$$\tilde{\star} := (\star_f)_{sp} := \star_{\mathcal{M}(\star_f)} \, .$$

[$\tilde{\star}$ is called the spectral semistar operation associated to $\star$.] Then:

(1) $E^{\tilde{\star}} = \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$ *[and $E^{\star_f} = \cap\{E^{\star_f}D_Q \mid Q \in \mathcal{M}(\star_f)\}$].*

(2) $\tilde{\star} \leq \star_f$ *.*

(3) $E\text{Na}(D,\star) = \cap\{ED_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$ *, thus:*
$E\text{Na}(D,\star) \cap K = \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$ *.*

(4) $E^{\tilde{\star}} = E\text{Na}(D,\star) \cap K$ *.* $\square$

Proposition 3(4) assures that, when a maximal ideal of $\text{Na}(D, \star)$ is contracted to D, the result is exactly a prime ideal in $\mathcal{M}(\star_f)$. This result can be reversed. Moreover, the semistar operation $\tilde{\star}$ generates the same Nagata ring as $\star$.

Corollary 8 *Let $\star$, $\star_1$, $\star_2$ be semistar operations of an integral domain D. Then:*

(1) $\text{Max}(\text{Na}(D, \star)) = \{QD_Q(X) \cap \text{Na}(D, \star) \mid Q \in \mathcal{M}(\star_f)\}$.

(2) $(\tilde{\star})_f = \tilde{\star} = \tilde{\tilde{\star}}$.

(3) $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$.

(4) $\text{Na}(D, \star) = \text{Na}(D, \tilde{\star})$.

(5) $\star_1 \leq \star_2 \;\Rightarrow\; \text{Na}(D, \star_1) \subseteq \text{Na}(D, \star_2) \;\Leftrightarrow\; \widetilde{\star_1} \leq \widetilde{\star_2}$. □

Remark 9 Note that, when $\star$ is the (semi)star v–operation, then the (semi)star operation $\tilde{v}$ coincides with *the (semi)star operation* w defined as follows:

$$E^w := \cup\{(E : H) \mid H \in \boldsymbol{f}(D) \text{ and } H^v = D\},$$

for each $E \in \overline{\boldsymbol{F}}(D)$. This (semi)star operation was considered by J. Hedstrom and E. Houston in 1980 under the name of F_∞–operation.

Later, starting in 1997, this operation was intensively studied by Wang Fanggui and R. McCasland under the name of w–operation. Note also that the notion of w–ideal coincides with the notion of semi-divisorial ideal considered by S. Glaz and W. Vasconcelos in 1977.

Finally, in 2000, for each (semi)star operation $\star$, D.D. Anderson and S.J. Cook considered the $\star_w$–operation which can be defined as follows:

$$E^{\star_w} := \cup\{(E : H) \mid H \in \boldsymbol{f}(D) \text{ and } H^\star = D\},$$

for each $E \in \overline{\boldsymbol{F}}(D)$. From their theory (and from the results by Hedstrom and Houston) it follows that:

$$\star_w = \tilde{\star}.$$

The relation between $\tilde{\star}$ and the localizing systems of ideals (in the sense of Gabriel and Popescu) was established by M. Fontana and J. Huckaba in 2000.

8. The Kronecker function ring in a general setting

The problem of the construction of a Kronecker function ring for general integral domains was considered indipendently by F. Halter-Koch (2003) and Fontana-Loper (2001, 2003).

Halter-Koch's approach is axiomatic and makes use of the theory of finitary ideal systems (star operations of finite type). He also establishes a connection with Krull's theory of Kronecker function rings and introduces the Kronecker function rings for integral domains with an ideal system which does not necessarily verify the cancellation property (e.a.b.).

Fontana-Loper's treatment is based on the theory of semistar operations.

Halter-Koch gives the following "abstract" definition:

Let K be a field, R a subring of $K(X)$ and $D := R \cap K$. If

(Kr.1) $X \in \boldsymbol{\mathcal{U}}(R)$;

(Kr.2) For each $f \in K[X]$, then $f(0) \in fR$;

then R is called *a K–function ring of D.*

Using only these two axioms, he proved that R behaves as a Kronecker function ring:

Theorem 10 *Let R be a K–function ring of $D = R \cap K$, then:*

(1) *R is a Bézout domain with quotient field $K(X)$.*

(2) *D is integrally closed in K .*

(3) *For each $f := a_0 + a_1X + \ldots + a_nX^n \in K[X]$, then $(a_0, a_1, \ldots, a_n)R = fR$.* □

One of the main goals for the classical theory of star operations has been to construct Kronecker function rings associated to a domain, in a more general context than the original one considered by L. Kronecker in 1882.
As we have already observed, in the Krull's setting, one begins with an integrally closed domain D and a star operation $\star$ on D with the cancellation property known as *e.a.b. (endlich arithmetisch brauchbar).*

Then *the star Kronecker function ring* is constructed as follows:

$$\mathrm{Kr}(D,\star) := \{f/g \mid f,g \in D[X],\ g \neq 0,\ \boldsymbol{c}(f)^\star \subseteq \boldsymbol{c}(g)^\star\}\,.$$

This domain turns out to be a Bézout overring of the polynomial ring $D[X]$ such that $\mathrm{Kr}(D,\star) \cap K = D$ (where K is the quotient field of D), cf. [Gilmer's book Section 32].

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Further related references
Arnold (1969), Arnold-Brewer (1971), Dobbs-Fontana (1986), D.F. Anderson-Dobbs-Fontana (1987), Okabe-Matsuda (1997).

If $\star$ is *any* semistar operation of *any* integral domain D, then we can introduce *the Kronecker function ring of D with respect to the semistar operation $\star$* in the following way:

$$\mathrm{Kr}(D,\star) := \{f/g \mid \ f,g \in D[X],\ g \neq 0, \ \text{and there exists} \\ h \in D[X]\setminus\{0\} \ \text{with } (\boldsymbol{c}(f)\boldsymbol{c}(h))^{\star} \subseteq (\boldsymbol{c}(g)\boldsymbol{c}(h))^{\star}\}.$$

At this point, we need some preliminaries in order

– to prove that this construction leads to a natural extension of the classical Kronecker function ring,

– to show the links between this general $\mathrm{Kr}(D,\star)$ and the "axiomatically defined" K-function ring, introduced by Halter-Koch and

– to show that $\mathrm{Kr}(D,\star)$ defines a "new" semistar operation on D, behaving with respect $\mathrm{Kr}(D,\star)$ in a "similar" way to $\tilde{\star}$ with respect to $\mathrm{Na}(D,\star)$.

It is possible to associate to any semistar operation $\star$ of D an e.a.b. semistar operation of finite type $\star_a$ of D, called *the e.a.b. semistar operation associated to* $\star$, defined as follows for each $F \in \boldsymbol{f}(D)$ and for each $E \in \overline{\boldsymbol{F}}(D)$:

$$F^{\star_a} := \cup\{((FH)^\star : H^\star) \mid H \in \boldsymbol{f}(D)\},$$
$$E^{\star_a} := \cup\{F^{\star_a} \mid F \subseteq E,\ F \in \boldsymbol{f}(D)\}.$$

The previous construction is essentially due to P. Jaffard (1960) and F. Halter-Koch (1997, 1998).
Obviously $(\star_f)_a = \star_a$. Note that:
– when $\star = \star_f$, then $\star$ is e.a.b. if and only if $\star = \star_a$.
– $D^{\star_a}$ *is integrally closed and contains the integral closure of* D *in* K.

When $\star = v$, then D^{v_a} coincides with *the pseudo-integral closure of* D introduced by D.F. Anderson, Houston and Zafrullah (1992).

Remark 11 In the "classical" context of *star* operations, $\star_a$ is a star operation and for this reason is defined on the "star closure" of D (cf. Okabe-Matsuda 1992, Halter-Koch (1997, 1998, 2003)).
More precisely (even if $\star$ is a semistar operation), we call *the $\star$–closure of D*:

$$D^{\mathrm{cl}^\star} := \cup\{(F^\star : F^\star) \mid F \in \boldsymbol{f}(D)\}.$$

It is easy to see that $D^{\mathrm{cl}^\star}$ *is an integrally closed overring of D*.

D is said *$\star$–closed* if $D = D^{\mathrm{cl}^\star}$.

We can now define a new (semi)star operation on D if $D = D^{\mathrm{cl}^\star}$ (or, in general, a semistar operation on D), $\mathrm{cl}^\star$ by setting for each $F \in \boldsymbol{f}(D)$, for each $E \in \overline{\boldsymbol{F}}(D)$:

$$\begin{aligned} F^{\mathrm{cl}^\star} &:= \cup\{((H^\star : H^\star)F)^\star \mid H \in \boldsymbol{f}(D)\}, \\ E^{\mathrm{cl}^\star} &:= \cup\{F^{\mathrm{cl}^\star} \mid F \subseteq E,\ F \in \boldsymbol{f}(D)\}. \end{aligned}$$

If we set $\bar{\star} := \mathrm{cl}^\star$, it is not difficult to see that $D^{\mathrm{cl}^{\bar{\star}}} = D^{\mathrm{cl}^\star}$ (and that it coincides with $D^{\star_a}$) and $D^{\mathrm{cl}^\star}$ contains the "classical" integral closure of D. Moreover (as semistar operations on D):

$$\star_{_f} \leq \mathrm{cl}^\star \leq \star_a, \quad (\star_{_f})_a = (\mathrm{cl}^\star)_a = (\star_a)_a = \star_a.$$

We now turn our attention to the valuation overrings. The notion that we recall next is due to P. Jaffard (1960) (cf. also Halter-Koch (1997)).

For a domain D and a semistar operation $\star$ on D, we say that a valuation overring V of D is *a $\star$–valuation overring of D* provided $F^\star \subseteq FV$, for each $F \in \boldsymbol{f}(D)$. Note that, by definition the $\star$–valuation overrings coincide with the $\star_f$–valuation overrings.

Proposition 12 *Let D be a domain and let $\star$ be a semistar operation on D.*

(1) *The $\star$–valuation overrings also coincide with the $\star_a$–valuation overrings.*

(2) $D^{\mathrm{cl}^\star} = \cap\{V \mid V$ *is a $\star$–valuation overring of $D\}$.*

(3) *A valuation overring V of D is a $\tilde{\star}$–valuation overring of D if and only if V is an overring of D_P, for some $P \in \mathcal{M}(\star_f)$.*

□

Theorem 13 *Let $\star$ be a semistar operation of an integral domain D with quotient field K. Then:*

(1) $\mathrm{Na}(D,\star) \subseteq \mathrm{Kr}(D,\star)$.

(2) *V is a $\star$–valuation overring of D if and only if $V(X)$ is a valuation overring of $\mathrm{Kr}(D,\star)$.*
The map $W \mapsto W \cap K$ establishes a bijection between the set of all valuation overrings of $\mathrm{Kr}(D,\star)$ and the set of all the $\star$–valuation overrings of D.

(3) $\mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\star_f) = \mathrm{Kr}(D,\star_a) = \cap\{V(X) \mid V \text{ is a } \star\text{–valuation overring of } D\}$ *is a Bézout domain with quotient field $K(X)$.*

(4) $E^{\star_a} = E\mathrm{Kr}(D,\star) \cap K = \cap\{EV \mid V \text{ is a } \star\text{–valuation overring of } D\}$*, for each $E \in \overline{\boldsymbol{F}}(D)$.*

(5) $R := \mathrm{Kr}(D,\star)$ *is a K–function ring of $R \cap K = D^{\star_a}$ (Halter-Koch's axiomatic definition).* □

9. Some relations between $\text{Na}(D,\star)$, $\text{Kr}(D,\star)$, $\widetilde{\star}$, and $\star_a$

An elementary first question to ask is whether the two semistar operations $\widetilde{\star}$ and $\star_a$ are actually the same - or usually the same - or rarely the same.

Proposition 12 indicates that for a semistar operation $\star$ on a domain D, the $\widetilde{\star}$–valuation overrings of D are all the valuation overrings of the localizations of D at the primes in $\mathcal{M}(\star_f)$.
On the other hand, it is known (cf. Halter-Koch (1997), Fontana-Loper (2001, 2003)) that the $\star_a$–valuation overrings (or, equivalently, the $\star$–valuation overrings) of D correspond exactly to the valuation overrings of the Kronecker function ring $\text{Kr}(D,\star)$.
In particular, each $\star_a$–valuation overring is also a $\widetilde{\star}$–valuation overring.

It is easy to imagine that these two collections of valuation domains can frequently be different. We only mention a couple of examples constructed explicitly in a joint work with K.A. Loper (2003).

Example 14 *There exists a (semi)star operation* $\star$ *of an integral domain* D *such that* $\tilde{\star} \neq \star_a$, *but the* $\tilde{\star}$*–valuation overrings coincide with the* $\star_a$*–valuation overrings (and so* $\mathrm{Kr}(D, \tilde{\star}) = \mathrm{Kr}(D, \star_a) = \mathrm{Kr}(D, \star)$ *).*

In Example 14 $\tilde{\star} \neq \star_a$, however $\tilde{\star} = \widetilde{(\star_a)}$. More generally,

Example 15 *There exists a (semi)star operation* $\star$ *of an integral domain* D *such that* $\tilde{\star} \neq \star_a$, *the* $\star_a$*–valuation overrings form a proper subset of the set of* $\tilde{\star}$*–valuation overrings and*

(a) $\tilde{\star} = \widetilde{(\star_a)}$, *or*

(b) $\tilde{\star} \lneq \widetilde{(\star_a)}$.

It is possible to prove "positive" statements about the relationship between $\widetilde{(\text{-})}$ and $(\text{-})_a$ under conditions made clear by the preceding examples.

However, we limit ourself to state here a result that generalizes the fundamental result that is at the basis of Krull's theory of Kronecker function rings:

$$\mathrm{Na}(D) = \mathrm{Na}(D,d) = \mathrm{Kr}(D,b) = \mathrm{Kr}(D) \quad \Leftrightarrow \quad D \text{ is a Prüfer domain.}$$

We recall that a *Prüfer $\star$–multiplication domain* (for short, a *P$\star$MD*) is an integral domain such that, for each $F \in \boldsymbol{f}(D)$, then: $(FF^{-1})^{\star_f} = D^{\star_f}$ $(= D^{\star})$ (i.e., each F is *$\star_f$–invertible*).

Some of the statements of the following theorem generalize some of the classical characterizations of the Prüfer v–multiplication domains (for short, PvMD) (cf. Griffin (1967), Arnold-Brewer (1971), Zafrullah (1984) and Kang (1989)).

Theorem 16 (Fontana-Jara-Santos, 2003). *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent:*

(i) *D is a P$\star$MD.*

(ii) $\mathrm{Na}(D, \star)$ *is a Prüfer domain.*

(iii) $\mathrm{Na}(D, \star) = \mathrm{Kr}(D, \star)$ *.*

(iv) $\tilde{\star} = \star_a$ *.*

(v) $\star_{_f}$ *is stable and e.a.b..*

In particular, D is a P$\star$MD if and only if it is a P$\tilde{\star}$MD. □

The following gives the converse of the implication PvMD $\Rightarrow$ PwMD proved by Wang Fanggui-McCasland (1999), cf. also D.D. Anderson-Cook (2000).

Corollary 17 *Let D be an integral domain. The following are equivalent:*

(i) *D is a PvMD.*

(ii) $\mathrm{Na}(D,t) = \mathrm{Kr}(D,t)$.

(iii) $w := \tilde{v} = v_a$.

(iv) *t is stable and e.a.b..*

In particular, D is a PvMD if and only if it is a PwMD. □

Corollary 18 *Let D be an integral domain and $\star$ a* star *operation on D.*

$$D \text{ is a P}\star\text{MD} \quad \Leftrightarrow \quad D \text{ is a P}v\text{MD and } t = \widetilde{\star} \ \ (\text{ or, equivalently, } t = \star_{_f} \ ).$$

Further relevant references on PvMDs and P$\star$MDs:
Mott-Zafrullah (1981), Houston-Malik-Mott (1984), Garcia-Jara-Santos (1999), Halter-Koch (2003).

10. Intersections of local Nagata domains, Na$(D, \star)$ **and** Kr$(D, \star)$

Given a semistar operation $\star$ on D, we have shown that the integral domains Na$(D, \star)$ and Kr$(D, \star)$ (and the related semistar operations $\tilde{\star}$ and $\star_a$) have for many aspects a similar behaviour.
This is the starting point of a work in progress in collaboration with K.A. Loper:
Is it possible to find a "new" integral domain of rational functions denoted by Lo$(D, \star)$ *(*obtained as an intersection of local Nagata domains associated to any semistar operation $\star$*) such that:*

- Na$(D, \star) \subseteq$ Lo$(D, \star) \subseteq$ Kr$(D, \star)$ *;*
- Lo$(D, \star)$ *"generalizes" at the same time* Na$(D, \star)$ *and* Kr$(D, \star)$ *and coincides with* Na$(D, \star) =$ Na$(D, \tilde{\star})$ *or* Kr$(D, \star) =$ Kr$(D, \star_a)$*, when the semistar operation (of finite type)* $\star$ *assumes the "extreme values" of the interval* $\tilde{\star} \leq \star \leq \star_a$ *?*

An overring T of D is called *a $\star$–overring of D* if, for each $F \in \boldsymbol{f}(D)$, then $F^\star \subseteq FT$ (or, equivalently, $F^\star T = FT$ thus, in particular, $T = T^{\star_f}$).
An overring T of D such that $T = T^{\star_f}$ is not necessarily a $\star$–overring of D.

If F is in $\boldsymbol{f}(D)$, we say that F is *$\star$–e.a.b.* if $(FG)^\star \subseteq (FH)^\star$, with $G,\ H \in \boldsymbol{f}(D)$, implies that $G^\star \subseteq H^\star$.
Note that *F is $\star$–e.a.b. if and only if $((FH)^\star : F^\star) = H^\star$, for each $H \in \boldsymbol{f}(D)$.*
The previous characterization gives "a posteriori" a motivation for the definition of $\star_a$ and shows that a semistar operation of finite type is e.a.b. if and only if $\star = \star_a$.

Lemma 19 *Let $\star$ be a semistar operation on an integral domain D, let $F \in \boldsymbol{f}(D)$ be $\star_{_f}$–invertible and let (L, N) be a local $\star$–overring of D. Then FL is a principal fractional ideal of L.*

Note that, in general, $\star$–(e.)a.b. does not imply $\star$–invertible, even for finite type semistar operations.
However, it is possible to show that, for finite type stable semistar operations $\star$ (i.e. when $\star = \tilde{\star}$), the notions of $\star$–e.a.b., $\star$–a.b. and $\star$–invertible coincide.

Next goal is to generalize Lemma 19 to the "e.a.b.–case".

Let $\star$ be a semistar operation on an integral domain D. *A $\star$–monolocality of D* is a local overring L of D such that:
– FL is a principal fractionary ideal of L, for each $\star$–e.a.b. $F \in \boldsymbol{f}(D)$;
– $L = L^{\star_f}$.

Obviously, each $\star$–valuation overring is a $\star$–monolocality.
It is not hard to prove that, for each $Q \in \mathcal{M}(\star_{_f})$, D_Q is a $\tilde{\star}$–monolocality.

Set:

$$\boldsymbol{\mathcal{L}}(\star) := \boldsymbol{\mathcal{L}}(D, \star) := \{L \mid L \text{ is a } \star\text{–monolocality of } D\},$$

$$\mathrm{Lo}(D, \star) := \cap\{L(X) \mid L \in \boldsymbol{\mathcal{L}}(D, \star)\}.$$

We are now in condition to state some of the results that we have already proved.

Theorem 20 *Let $\star$ be a semistar operation on an integral domain D with quotient field K.*

(1) $\mathrm{Na}(D,\star) \subseteq \mathrm{Lo}(D,\star) \subseteq \mathrm{Kr}(D,\star))$.

(2) $\mathrm{Lo}(D,\star) := \{f/g \in K(X) \mid \; f,g \in D[X],\ g \neq 0,\ \text{ such that } \boldsymbol{c}(f) \subseteq \boldsymbol{c}(g)^\star \; \boldsymbol{c}(g) \text{ is a } \star\text{–e.a.b. } \}$.

(3) *For each maximal ideal $\mathfrak{m}$ of $\mathrm{Lo}(D,\star)$, set $L(\mathfrak{m}) := \mathrm{Lo}(D,\star)_{\mathfrak{m}} \cap K$. Then:*
$L(\mathfrak{m})$ is a $\star$–monolocality of D (with maximal ideal $\mathfrak{M} := \mathfrak{m}\mathrm{Lo}(D,\star)_{\mathfrak{m}} \cap L(\mathfrak{m})$),
$\mathrm{Lo}(D,\star)_{\mathfrak{m}}$ coincides with the Nagata ring $L(\mathfrak{m})(X)$ and
$\mathfrak{m}$ coincides with $\mathfrak{M}(X) \cap \mathrm{Lo}(D,\star)$.

(4) *Every $\star$–monolocality of an integral domain D contains a minimal $\star$–monolocality of D. If we denote by $\boldsymbol{\mathcal{L}}(D,\star)_{min}$ the set of all the minimal $\star$–monolocalities of D, then*
$\boldsymbol{\mathcal{L}}(D,\star)_{min} = \{L(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(\mathrm{Lo}(D,\star))\}$ *and, obviously,*
$\mathrm{Lo}(D,\star) = \cap\{L(X) \mid L \in \boldsymbol{\mathcal{L}}(D,\star)_{min}\}$.

(5) *For each* $J := (a_0, a_1, \ldots, a_n)D \in \boldsymbol{f}(D)$, *with* $J \subseteq D$ *and* J $\star$*–e.a.b., let* $g := a_0 + a_1X + \ldots + a_nX^n \in D[X]$, *then:*

$$J\mathrm{Lo}(D,\star) = J^{\star}\mathrm{Lo}(D,\star) = g\mathrm{Lo}(D,\star)\,.$$

(6) *Let* $\overline{\star_{\ell}}$ *and* $\wedge_{\boldsymbol{\mathcal{L}}}$ *be the semistar operations of* D *defined as follows, for each* $E \in \overline{\boldsymbol{F}}(D)$,

$$E^{\star_\ell} := E\mathrm{Lo}(D,\star) \cap K\,,$$
$$E^{\wedge_{\mathcal{L}}} := \cap\{EL \mid L \in \boldsymbol{\mathcal{L}}(D,\star)\}\,.$$

Then:

$$\widetilde{\star} \leq \star_{\ell} = \wedge_{\boldsymbol{\mathcal{L}}} \leq \star_a\,.$$

(7) $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\widetilde{\star}) = \mathrm{Lo}(D,\widetilde{\star})\,.$

$\mathrm{Lo}(D,\star_a) = \mathrm{Kr}(D,\star_a) = \mathrm{Kr}(D,\star)\,.$