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Domains Satisfying the Trace Property

MARCO FONTANA,* JAMES A. HUCKABA,† AND IRA J. PAPICK†

Department of Mathematics, University of Missouri, Columbia, Missouri 65211

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1. INTRODUCTION

Let R be a commutative integral domain with identity and M a unital R-module. Recall that the trace of M, denoted $\gamma_R(M)$, is the ideal of R generated by the set $\{f(m)\colon f \in \operatorname{Hom}_R(M, R)\}$. When no confusion will result, we will suppress the R and write $\gamma(M)$.

In Section 2, we prove that if R is a valuation domain and M a unital R-module, then $\gamma(M)$ either equals R or a prime ideal of R. It is this result that motivates our work and gives rise to the following definition: A domain R is said to satisfy the *trace property* (for brevity, we will say R satisfies TP) provided that $\gamma(M)$ either equals R or a prime ideal of R for each R-module M. The goal of this paper is to characterize certain classes of domains satisfying TP.

Besides the result mentioned above (Proposition 2.1), Section 2 consists mainly of fundamental observations and examples which are very useful in Sections 3 and 4. In particular, we prove that R satisfies TP if and only if $\gamma(I)$ equals R or a prime ideal of R for each ideal I of R (Proposition 2.4). In connection with this result, we establish that $\gamma(I) = II^{-1}$ for any nonzero fractional ideal I of R (Lemma 2.2).

Section 3 is devoted to the study of Noetherian domains satisfying TP. We prove that a Noetherian domain R satisfies TP if and only if R is a Dedekind domain, or $\dim(R) = 1$ and R has a unique noninvertible maximal ideal M with $M^{-1} = \bar{R}$ (i.e., all other maximal ideals of R are

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invertible) (Theorem 3.5). We use this characterization to explicitly describe how Noetherian TP domains are constructed via a very specific pullback diagram (Theorem 3.6). Finally we use Theorems 3.5 and 3.6 to construct Noetherian TP domains with various properties (Examples 3.7).

Section 4 consists of an analysis of Prüfer domains satisfying TP. Recall that a domain R is said to be a $(\#)$-domain provided $\bigcap_{M \in \Delta_1} R_M \neq \bigcap_{M \in \Delta_2} R_M$, where Δ_1 and Δ_2 are distinct subsets of $\mathrm{Max}(R)$. R is called a $(\#\#)$-domain in case each overring of R is a $(\#)$-domain. We prove under the assumption of R being a Prüfer $(\#\#)$-domain, that R is a TP domain if and only if the noninvertible prime ideals of R are linearly ordered (Theorem 4.2). An example is provided (Example 4.3) to illustrate the invalidity of the Theorem, if the $(\#\#)$ condition is not included. Finally, we prove for a finite dimensional Prüfer domain R that R satisfies TP if and only if R is a $(\#\#)$-domain with its noninvertible prime ideals linearly ordered (Theorem 4.6).

2. Motivation and General Results

The origin of this paper arises from the following result which generalizes [1, Theorem 2.8].

Proposition 2.1. *If R is a valuation domain and M an R-module, then R satisfies TP.*

Proof. Assume $\gamma(M) \neq R$, $\gamma(M) \neq (0)$ and choose $P \in \mathrm{Spec}(R)$ with P minimal over $\gamma(M)$. We claim that $P = \gamma(M)$. Suppose not and let $a \in P \backslash \gamma(M)$. Note that $\gamma(M) \subset aR \subset R$, and so $\mathrm{Hom}_R(M, R) \subset a[\mathrm{Hom}_R(M, R)]$. Thus $\gamma(M) \subset (a)\,\gamma(M)$, and hence $\gamma(M) = (a^n)\,\gamma(M)$ for each positive integer n. Therefore, $\gamma(M) \subset \bigcap_{n=1}^{\infty} (a^n) = P_0$. However, P_0 is a prime ideal of R properly contained in P, which is a contradiction [7, Theorem 17.1].

We will show momentarily that the definition of the trace property can be restricted to integral ideals, but first we need a definition and an elementary lemma. Recall that an R-submodule I of $K = qf(R)$ is called a fractional ideal of R provided there exists $0 \neq r \in R$ such that $rI \subset R$.

Lemma 2.2. *If I is a fractional ideal of R, then $\gamma(I) = II^{-1}$.*

Proof. We may assume $I \neq (0)$. Let $f \in \mathrm{Hom}(I, R)$ and $0 \neq a \in I$. To show that $\gamma(I) \subset II^{-1}$, it suffices to prove that $f(a) \in II^{-1}$. We claim that $f(a)/a \in I^{-1}$. Let $b \in I$ and d a nonzero element of R with $dI \subset R$. Consider,

$$(f(a)/a)\,b = f(abd)/ad = (f(b)\,ad)/ad = f(b) \in R.$$

Thus $f(a)=a(f(a)/a)\in II^{-1}$. For the other inclusion let $0\neq a\in I$ and $u\in I^{-1}$. It is enough to show that $au\in\gamma(I)$. Define $f\colon I\to R$ by $f(t)=tu$. Clearly $f\in\operatorname{Hom}_R(I, R)$, and so $au=f(a)\in\gamma(I)$.

The following elementary remark, due to Bass [3, Proposition 7.2], will be very useful in the remainder of the paper.

Remark 2.3. If M is an R-module, then $\gamma(M)^{-1}=(\gamma(M)\colon\gamma(M))$.

PROPOSITION 2.4. *Let R be a domain. If $\gamma(I)$ either equals R or a prime ideal of R for each integral ideal I of R, then R has TP.*

Proof. Let M be an R-module and set $I=\gamma(M)$. We may assume $I\neq(0)$. Notice that $\gamma(I)=II^{-1}=I$ by Lemma 2.2 and Remark 2.3. Therefore $\gamma(M)$ either equals R or a prime ideal of R.

COROLLARY 2.5. *If R is a Dedekind domain, then R has TP.*

It is appropriate at this time to present an example of a local domain satisfying *TP* that is not integrally closed, and hence not a valuation domain. We will generalize this example in Section 3.

EXAMPLE 2.6. Let Q denote the rational numbers and let $V=Q(\sqrt{2})[[x]]=Q(\sqrt{2})+M$, where M is the unique maximal ideal of the discrete valuation domain V. Set $R=Q+M$. It follows that R is Noetherian, 1-dimensional, local with integral closure $V=M^{-1}$. Since V can be generated as an R-module by two elements, we have that each ideal of R is divisorial [13, Theorem 3.8]. Let I be a nonzero ideal of R such that $II^{-1}\neq R$. We claim that $M^{-1}=(II^{-1})^{-1}$. Clearly $M^{-1}\subset(II^{-1})^{-1}$, and since $(II^{-1})^{-1}=(II^{-1}:II^{-1})$ (Remark 2.3), we have by integrality that $R\subset(II^{-1})^{-1}\subset M^{-1}=V$. However, since each ideal of R is divisorial, we deduce that $M=II^{-1}$, the desired conclusion.

It is interesting to point out that even though domains satisfying *TP* need not be integrally closed, they do exhibit a property that is somewhat related to integral closure. Namely, if R is a domain satisfying *TP* and $a/b\in\bar{R}$ (=integral closure of R), then $a^2/b\in R$. To see this, let $T=R[a/b]=R+R(a/b)+\cdots+R(a^{n-1}/b^{n-1})$. Since $(R:T)\neq(0)$, we may apply Lemma 2.2 to obtain $\gamma_R(T)=(R:T)$. If $(R:T)=R$ we are done, so we may assume $(R:T)$ is a prime ideal of R. As $b^{n-1}\in(R:T)$, we get $b\in(R:T)$, and so $b(a^2/b^2)=a^2/b\in R$.

Our next sequence of results are pertinent to the general question of characterizing *TP* domains, and will be of some use in Sections 3 and 4.

PROPOSITION 2.7. *If R has TP, then* $\operatorname{grade}(R)\leqslant 1$; *i.e., R has no R-sequence of length greater than* 1.

Proof. Suppose grade$(R)>1$ and let a, b be an R-sequence of length two. Set $I=(a, b)$. Since $I^{-1}=R$ [12, Exercise 1, p. 102], $\gamma(I^2)=I^2I^{-2}=I^2$ is a prime ideal of R, and thus $I=I^2$. However, $I\neq I^2$ [12, Theorem 76] and this contradiction completes the proof.

In Section 3 we will give a complete characterization for Noetherian *TP* domains, and in Section 4 we handle the Prüfer *TP* case. However, at this moment we present two propositions which individually imply that an integrally closed Noetherian *TP* domain is Dedekind.

PROPOSITION 2.8. *If R is coherent, integrally closed and satisfies TP, then R is a Prüfer domain.*

Proof. Let $I=(a, b)$ where $a\neq 0$, $b\neq 0$. It suffices to show that I is invertible. Assume $II^{-1}\neq R$, and thus $II^{-1}=P$, a prime ideal of R. The coherence condition implies that I^{-1} is module-finite over R, and so P is a finitely generated ideal of R. Remark 2.3 gives us $P^{-1}=(P:P)$, and integrality forces $P^{-1}=R$. Hence, $\gamma(P^2)=P^2P^{-2}=P^2$ is a prime ideal. This is a contradiction in exactly the same way as in the proof of Proposition 2.7.

It is worthwhile to mention that the full power of coherence was not employed in the above proof, but merely the notion of 2-coherence, i.e., each 2-generated ideal of R is finitely presented. (This has been referred to as the finite conductor property in some of the literature.)

PROPOSITION 2.9. *If R is a Krull domain with TP, then R is a Dedekind domain.*

Proof. It is enough to show that $\dim(R)\leqslant 1$. Suppose $\dim(R)>1$ and let $P\in \operatorname{Spec}(R)$ such that $\operatorname{ht}(P)>1$. It follows that $PP^{-1}=P$, otherwise P is minimal over a principal ideal [12, Exercise 26, p. 43], and this cannot happen in a Krull domain. By integrality we have $P^{-1}=(P:P)=R$, and hence there exists a finitely generated ideal $J\subset P$ such that $J^{-1}=R$. As in the last two arguments this leads to a contradiction.

For an alternate proof of Proposition 2.9, use Proposition 2.7 in combination with [12, Exercises 2 and 4].

We will need another component for our characterization of Prüfer *TP* domains in Section 4, and our next proposition serves this purpose.

PROPOSITION 2.10. *If R is a TP domain and M a noninvertible maximal ideal of R, then each noninvertible ideal of R is contained in M.*

Proof. Let I be a noninvertible ideal of R and assume $I\not\subset M$. Consider, $\gamma(IM)=(IM)(IM)^{-1}=(IM)(I\cap M)^{-1}=(IM)(I^{-1}+M^{-1})$ [11, Lemma 3.7] $= II^{-1}M + IM = II^{-1}M \neq R$. Therefore $II^{-1}M$ is a prime ideal of R, which is clearly a contradiction, since $II^{-1} \not\subset M$.

COROLLARY 2.11. *If R is a domain satisfying TP and R has a noninvertible maximal ideal, then all other maximal ideals of R are invertible.*

We will end this section with a proposition which uncovers another interesting necessary condition for domains satisfying *TP*. In [1], we defined an ideal I of R to be L-stable provided $R^I = (I : I)$, where $R^I = \bigcup_{n=1}^{\infty}(I^n : I^n)$. It is worthwhile to mention that in a Prüfer domain each ideal is L-stable [1, Proposition 2.7].

PROPOSITION 2.12. *If R satisfies TP, then each prime ideal of R is L-stable.*

Proof. Let $(0) \neq Q \in \operatorname{Spec}(R)$. We may assume Q is not invertible, as invertible ideals are always L-stable [1, Lemma 2.1]. Thus for each n, Q^nQ^{-n} is a nonzero prime ideal of R. To complete the proof we will show that $(Q : Q) = (Q^n : Q^n)$ for each n. Clearly $(Q : Q) \subset (Q^n : Q^n)$, so let us concentrate on the other inclusion. Let $u \in (Q^n : Q^n)$ and set $P = Q^nQ^{-n}$. Since P is a prime ideal and $Q^n \subset Q^nQ^{-n} = P$, then $Q \subset P$. If $Q = P$ we are finished, as $uP \subset P$. If $Q \subsetneq P$, then $P^{-1} \subset (Q : Q)$ [6, Lemma 3.7] and our goal is accomplished.

3. NOETHERIAN *TP* DOMAINS

In this section we will give a complete characterization for Noetherian domains satisfying *TP*. We will need a bit of terminology before proceeding further. A *nonzero* ideal I of a domain R is called a *strong ideal* if $II^{-1} = I$, and I is called *strongly divisorial* if I is strong and divisorial. Let S denote the set of strongly divisorial ideals of R. Note that $R \in S$. We will now use this concept to describe $\bar{R}$, the integral closure of R.

LEMMA 3.1. *If R is a Noetherian domain satisfying TP, then $\bar{R} = \bigcup\{P^{-1} : P \in S \cap \operatorname{Spec}(R)$ or $P = R\}$.*

Proof. If we let R^* denote the complete integral closure of R, then we have $R^* = \bigcup\{I^{-1} : I \in S\}$ [2, Proposition 12]. However, the *TP* property implies that $R^* = \bigcup\{P^{-1} : P \in S \cap \operatorname{Spec}(R)$ or $P = R\}$, and $R^* = \bar{R}$ since R is Noetherian.

LEMMA 3.2. *Let R be a Noetherian TP domain. Then, R is integrally closed if and only if R is the only strongly divisorial ideal of R.*

Proof. ($\Leftarrow$) This direction is immediate from Lemma 3.1.

($\Rightarrow$) Assume $R=\bar{R}$. By Proposition 2.8 or 2.9, we deduce that R is a Dedekind domain. Hence each nonzero proper ideal of R is invertible, so R is the only strongly divisorial ideal of R.

Before attaining the promised characterization, we need some more structural information.

LEMMA 3.3. *If R is a Noetherian TP domain, then R_P is a Noetherian TP domain for each $P \in \operatorname{Spec}(R)$.*

Proof. Let J be an ideal of R_P. Write $J=IR_P$, where I is an ideal of R with $I \subset P$. If $II^{-1}=R$, then clearly $J(R_P:J)=R_P$. If $II^{-1}=Q$, $Q\in \operatorname{Spec}(R)$, then $J(R_P:J)=QR_P$. Hence, $J(R_P:J)$ either equals a prime ideal of R_P or R_P (depending on whether $Q\subset P$ or $Q \not\subset P$).

LEMMA 3.4. *If R is a Noetherian TP domain, then* $\dim(R)\leqslant 1$.

Proof. Suppose $\dim(R)>1$ and let M be a height two prime ideal of R. Since R_M is a Noetherian *TP* domain (Lemma 3.3), we may assume by changing notation that R is a local 2-dimensional *TP* domain with maximal ideal M, and that $R\neq\bar{R}$ (Proposition 2.9). Hence, there exists a strongly divisorial proper ideal P of R (Lemma 3.2), and it is prime by the *TP* property.

We shall consider three cases: (Throughout this proof we use the following convention: if I is a fractional ideal of R, then denote $(R:I)$ by I^{-1}.)

Case 1. There exists only one strongly divisorial prime ideal of R, namely P.

By applying Lemma 3.1, we see that $R \subsetneq \bar{R}=P^{-1}=(P:P)$, and so $(R:\bar{R})\neq(0)$. Let N be a height two maximal ideal of $\bar{R}$. Since $\bar{R}$ is Macaulay [12, Exercise 25, p. 104], we know $\operatorname{grade}(N)=2$ and hence $(\bar{R}:N^t)=\bar{R}$ for each positive integer t [12, Exercise 1, p. 102]. Thus, (*) $N^tN^{-t}=N^t(R:N^t)\subset N^t(\bar{R}:N^t)=N^t$ for each t. Note that N^t is a fractional ideal of R, as $(R:\bar{R})\neq(0)$, and so $\gamma(N^t)=N^tN^{-t}$ (Lemma 2.2) is either a *nonzero* prime ideal of R or equal to R. If $N^tN^{-t}=R$ for some t, then by (*), $1\in N^t$, a contradiction. Therefore we shall assume $N^tN^{-t}\in\operatorname{Spec}(R)$ for each t, and consider two subcases:

(i) $N^tN^{-t}=M$ for each t.

By (*) we see that $M\subset\bigcap_{t=1}^{\infty}N^t$, and this is a contradiction, since the Krull Intersection Theorem [12, Theorem 77] implies that $\bigcap_{t=1}^{\infty}N^t=(0)$.

(ii) $N^sN^{-s}=Q$, where s is a positive integer and Q is a height one prime ideal of R.

First, note the following inclusions:

$$\cdots \subset N^t N^{-t} \subset \cdots \subset N^2 N^{-2} \subset N N^{-1}.$$

Let s_0 be the least positive integer such that $N^{s_0} N^{-s_0} = Q$. Then $N^t N^{-t} = Q$ for $t \geqslant s_0$ and $N^t N^{-t} = M$ for $t < s_0$. Therefore, $Q \subset \bigcap_{t=1}^{\infty} N^t = (0)$, a contradiction.

Case 2. There exists more than one height one strongly divisorial prime ideal of R.

We will show that this case cannot happen. Suppose $P_1 \neq P_2$ are height one strongly divisorial prime ideals of R. Then,

$$(0) \neq P_1 P_2 \subset (P_1 P_2)(P_1 P_2)^{-1} \subset ((P_1 P_2)(P_1 P_2)^{-1})_v = Q.$$

Note that Q is a strongly divisorial ideal of R and $Q \subset P_1 \cap P_2$ [2, Lemma 15]. By the *TP* property, Q is a prime ideal of R and this contradicts the fact that $\mathrm{ht}(P_1) = \mathrm{ht}(P_2) = 1$.

Case 3. M is strongly divisorial and Q, a height one prime ideal of R, is strongly divisorial.

By Lemma 3.1, $\bar{R} = Q^{-1} \cup M^{-1} = Q^{-1} = (Q : Q)$, and the desired contradiction follows as in Case 1.

We are now ready for the main theorem of this section.

THEOREM 3.5. *Let R be a Noetherian domain. Then R satisfies TP if and only if*

(a) *R is a Dedekind domain, or*

(b) (i) $\dim(R) = 1$

(ii) *R has a unique noninvertible maximal ideal M with $M^{-1} = \bar{R}$ (i.e., all other maximal ideals of R are invertible.)*

Proof. ($\Rightarrow$) If $R = \bar{R}$, then Proposition 2.8 or 2.9 implies that R is a Dedekind domain. Hence we may assume $R \neq \bar{R}$ and $\dim(R) = 1$ (Lemma 3.4). By combining Lemma 3.2 with the proof of Case 2, Lemma 3.4, we see that there exists a unique strongly divisorial maximal ideal M of R. Clearly M is noninvertible and $\bar{R} = M^{-1}$ (Lemma 3.1). Finally, if N is any other maximal ideal of R, then Corollary 2.11 gives us that N is invertible.

($\Leftarrow$) Let I be a nonzero ideal of R. If R is a Dedekind domain, then $II^{-1} = R$ and we are done. Hence we may assume (b) and that $II^{-1} \neq R$. We claim that $II^{-1} \subset M$. Suppose not and assume $II^{-1} \subset N$, where N is a maximal ideal of R different from M, and hence invertible. Thus, $N^{-1} \subset$

$(II^{-1})^{-1}=(II^{-1}:II^{-1})\subset \bar{R}=M^{-1}$, and so $M\subset M_v\subset N_v=N$ [12, Theorem 94], which is a contradiction. Therefore $II^{-1}\subset M$, and moreover, $M^{-1}\subset (II^{-1})^{-1}=(II^{-1}:II^{-1})\subset \bar{R}=M^{-1}$. Hence $M^{-1}=(II^{-1})^{-1}=(II^{-1}:II^{-1})=\bar{R}$. Thus II^{-1} is an invertible ideal of $\bar{R}$ ($\bar{R}$ is Dedekind), and so $II^{-1}\cap R=II^{-1}$ is a divisorial ideal of R [6, Proposition 4.6]. Likewise, M is a divisorial ideal of R, and therefore $M=M_v=(II^{-1})_v=II^{-1}$.

The reader should note that with the aid of Theorem 3.5, it is trivial to see that Example 2.6 is a Noetherian *TP* domain. In fact, Example 2.6 is actually a Gorenstein *TP* domain [13, Theorem 3.8]. We will show momentarily that not all Noetherian *TP* domains are Gorenstein. Moreover, we will demonstrate how arbitrary Noetherian *TP* domains are constructed.

THEOREM 3.6. *R is a Noetherian TP domain if and only if there exists a Dedekind domain T containing R and an ideal I of T such that:*

(a) *T/I is a finitely generated k-module, where k is a subfield of the ring T/I:*

(b)

$$\begin{tikzcd} R \arrow[r] \arrow[d] & k \arrow[d, "u"] \\ T \arrow[r, "v"] & T/I \end{tikzcd}$$

is a pullback diagram, where u is the inclusion map and v is the canonical surjection.

Proof. ($\Rightarrow$) Let R be a Noetherian *TP* domain. If R is integrally closed, then (a) and (b) hold trivially for every maximal ideal $I=M$ of R with $k=R/M$. If R is not integrally closed, then take $T=\bar{R}$, the integral closure of R, and set $I=M$, the unique noninvertible maximal ideal of R. With the aid of Theorem 3.5, it is routine to verify that (a) and (b) hold.

($\Leftarrow$) Assume conditions (a) and (b), and without loss of generality we may assume $R\neq T$. Since R satisfies the pullback diagram in (b), we know that R is a 1-dimensional, Noetherian domain with integral closure $\bar{R}=T$ [4, Corollary 1.5]. Thus, by Theorem 3.5, it suffices to show that there exists a unique maximal ideal M of R such that $T=\bar{R}=M^{-1}$, and that all other maximal ideals of R are invertible.

Identify R with its canonical image inside $T=\bar{R}$, and observe that $\ker(R\twoheadrightarrow k)=I$ is a maximal ideal of R. If we take $I=M$, then we have $(R:\bar{R})=M$ [4, Theorem 1.4], and so $\bar{R}=(M:M)$. Since $R\neq\bar{R}$, then M is not invertible in R (otherwise $(M:M)=R$), and so $M^{-1}=(M:M)=\bar{R}$. If N is a maximal ideal of R and $N\neq M$ we will now show that N is invertible. Suppose that this is not the case and thus $N = NN^{-1}$. By integrality we have $(N : N) \subset \bar{R}$, which in turn gives us $N^{-1} \subset M^{-1}$. Moreover, as R is a 1-dimensional Noetherian domain, it follows that $M = M_v \subset N_v = N$, [12, Theorem 94] which is a contradiction. The proof is now complete.

EXAMPLES 3.7. (a) Our first example will be a Noetherian TP domain that is not Gorenstein. This is in contrast to Example 2.6. Let K be a field with subfield k such that $[K : k] > 2$. Let $V = K[[x]] = K + M$, where M is the unique maximal ideal of the discrete valuation domain V, and set $R = k + M$. As in Example 2.6, R is a 1-dimensional local Noetherian domain with integral closure $V = M^{-1}$. By Theorem 3.5 or 3.6 we see that R satisfies TP, yet R is not Gorenstein, since M^{-1} cannot be generated by two elements [13, Theorem 3.8].

(b) For our next example, we use Theorem 3.6 to construct a non-integrally closed Noetherian TP domain with infinitely many maximal ideals. In the notation of Theorem 3.6, let $T = k[x]$, k an algebraically closed field, and $I = \bigcap_{i=1}^{n} (x - a_i)$, where each $a_i \in k$. Let $u: k \to T/I \cong \prod_{i=1}^{n} k_i$ be the diagonal map, and let R be the pullback of the diagram in Theorem 3.6. Then R is the desired example.

4. PRÜFER TP DOMAINS

In this final section we study the structure of Prüfer domains satisfying TP. A crucial component of our work is the notion of a domain satisfying $(\#\#)$. Recall that R is said to be a *$(\#)$-domain* if Δ_1 and Δ_2 are distinct subsets of $\mathrm{Max}(R)$, then $\bigcap_{M \in \Delta_1} R_M \neq \bigcap_{M \in \Delta_2} R_M$, and R is called a *$(\#\#)$-domain* in case each overring of R is a $(\#)$-domain. In [9], Gilmer and Heinzer studied these domains and proved that R is a Prüfer $(\#\#)$-domain if and only if for each prime ideal P of R, there exists a finitely generated ideal $I \subset P$ such that each maximal ideal of R containing I contains P. We shall employ this result in our study of Prüfer TP domains.

Before stating our first characterization we need

LEMMA 4.1. *Let R be a Prüfer domain, $P \in \mathrm{Spec}(R)$ and Q a proper P-primary ideal of R. If $Q^{-1} = P^{-1}$, then $Q^{-n} = P^{-1}$ for each $n \geq 1$.*

Proof. First, assume that $P \neq P^2$. In this case $\{P^n\}_{n=1}^{\infty}$ is the set of P-primary ideals of R [7, Theorem 23.3]. It suffices to show that if $P^{-2} = P^{-1}$, then $P^{-n} = P^{-1}$. By induction we have,

$$P^{-n} = (R : P^n) = [(R : P^{n-1}) : P] = (P^{-1} : P) = (R : P^2) = P^{-2} = P^{-1}.$$

Next assume $P = P^2$ and that Q is a proper P-primary ideal of R such that $P^{-1} = Q^{-1}$. Again by induction we have,

$$\begin{aligned} Q^{-n} &= (R : Q^n) = [(R : Q^{n-1}) : Q] = (P^{-1} : Q) = (R : PQ) = [(R : Q) : P] \\ &= (Q^{-1} : P) = (P^{-1} : P) = (R : P^2) = (R : P) = P^{-1}. \end{aligned}$$

Recall that in Proposition 2.1 we proved that any valuation domain satisfies TP. Since any semilocal Prüfer domain satisfies $(\#\#)$ [9, Corollary 3], we point out that our next theorem significantly generalizes Proposition 2.1.

THEOREM 4.2. *Let R be a Prüfer domain satisfying $(\#\#)$. Then, R is a TP domain if and only if the noninvertible prime ideals of R are linearly ordered.*

Proof. $(\Rightarrow)$ First, assume that R has at least one noninvertible maximal ideal. By Proposition 2.10, we see in this case that R must have a unique noninvertible maximal ideal M_0, and that each nonmaximal prime ideal of R is contained in M_0 (nonmaximal prime ideals are noninvertible). Hence, the noninvertible prime ideals of R are linearly ordered.

Now suppose that each maximal ideal of R is invertible and let P, Q be two nonmaximal, hence noninvertible, incomparable prime ideals of R. Consider,

$$\begin{aligned} \gamma(PQ) = (PQ)(PQ)^{-1} &= (PQ)(P \cap Q)^{-1} \\ &= (PQ)(P^{-1} + Q^{-1}) && [11, \text{Lemma } 3.7] \\ &= PP^{-1}Q + QQ^{-1}P \\ &= PQ + PQ && [11, \text{Theorem } 3.8] \\ &= PQ. \end{aligned}$$

This is in contradiction with the TP property, since $\gamma(PQ)$ is not equal to R or a prime ideal of R.

$(\Leftarrow)$ Let $\{M_\alpha\}$ be the set of invertible maximal ideals of R, and let M_0 be the unique noninvertible maximal ideal of R (if it exists). Let I be a nonzero ideal of R and we may assume $II^{-1} \neq R$. We shall consider two cases.

Case 1. The minimal prime ideals of II^{-1} are maximal ideals.

First, we claim that no M_α can be minimal over II^{-1}. Suppose some M_{α_1} is minimal over II^{-1}. Then,

$$R \subsetneq M_{\alpha_1}^{-1} \subset (II^{-1})^{-1}. \qquad (*)$$

Write $\{M_\alpha\} \cup \{M_0\} = \{N_\beta\} \cup \{K_\gamma\}$, where the members of $\{N_\beta\}$ are maximal ideals of R containing II^{-1}, and the members of $\{K_\gamma\}$ are maximal ideals of R not containing II^{-1}. Hence, as $(II^{-1})^{-1}$ is a ring (Remark 2.3), and since each N_β is minimal over II^{-1}, we have by [11, Theorem 3.2],

$$(II^{-1})^{-1} = \left(\bigcap_\beta R_{N_\beta}\right) \cap \left(\bigcap_\gamma R_{K_\gamma}\right) = R. \qquad (**)$$

contradicting (*). Therefore, under the assumption of Case 1, no M_α is a minimal prime ideal of II^{-1}. Thus in this case, the only possibility is that M_0 is the unique minimal prime ideal of II^{-1}, and so $Q = II^{-1}$ is M_0-primary.

We now claim that $II^{-1} = M_0$. Assume otherwise, i.e., $II^{-1} = Q \subsetneq M_0$. Note that $P_0 = \bigcap_{n=1}^{\infty} Q^n$ is a prime ideal of R properly contained in M_0 [7, Theorem 23.3], and $T(Q) = R_{P_0} \cap (\bigcap_\alpha R_{M_\alpha})$ [7, Exercise 11, p. 331], where $T(Q)$ is the ideal transform of Q. However, by applying [9, Theorem 3, Corollary 2; 11, Corollary 3.4] we have,

$$\bigcup_{n=1}^{\infty} Q^{-n} = T(Q) \supsetneq R = M_0^{-1}.$$

From this, with the aid of Lemma 4.1, we deduce that $M_0^{-1} \subset Q^{-1}$, and thus

$$R = M_0^{-1} \subsetneq Q^{-1} = (II^{-1})^{-1}.$$

This is a contradiction, since by (**), $(II^{-1})^{-1} = R$. Therefore $II^{-1} = M_0$, and Case 1 is complete.

Case 2. There exists a nonmaximal prime ideal P that is minimal over II^{-1}.

Since the noninvertible prime ideals of R are linearly ordered by assumption, we note that M_0 (if it exists) is not minimal over II^{-1} (Proposition 2.10). Moreover, we claim that no M_α is minimal over II^{-1}. To see this, suppose that some M_{α_1} is minimal over II^{-1}. Then $M_{\alpha_1}^{-1} \subset (II^{-1})^{-1} \subset R_{M_{\alpha_1}}$ [11, Theorem 3.2], and so $M_{\alpha_1} M_{\alpha_1}^{-1} \subset M_{\alpha_1} R_{M_{\alpha_1}}$. This is a contradiction, as M_{α_1} is invertible. Therefore, by applying Proposition 2.10, we see that $P = \operatorname{rad}_R(II^{-1})$.

Now, we claim that II^{-1} is contained in no proper P-primary ideal of R. Suppose otherwise; i.e., $II^{-1} \subset Q \subsetneq P$, where Q is P-primary. Hence, $P^{-1} \subset Q^{-1} \subset (II^{-1})^{-1}$, and we may write

$$P^{-1} = R_p \cap \left(\bigcap_\rho R_{J_\rho}\right) \quad \text{and} \quad (II^{-1})^{-1} = R_P \cap \left(\bigcap R_{L_\mu}\right), \qquad (+)$$

where $\{J_\rho\}$, $\{L_\mu\}$ are, respectively, the sets of maximal ideals of R not containing P, and those not containing II^{-1} [11, Theorem 3.2]. Note that the maximal ideals in each of these sets must be invertible, and that $\{L_\mu\}\subset\{J_\rho\}$. Moreover, $\{J_\rho\}\subset\{L_\mu\}$. This inclusion follows, since if $II^{-1}\subset J_{\rho_1}$, then J_{ρ_1} is minimal over II^{-1}, which we have seen cannot happen. Therefore $P^{-1}=Q^{-1}=(II^{-1})^{-1}$.

To reach our desired contradiction we will prove that $P^{-1}\neq Q^{-1}$. Let $P_0=\bigcap_{n=1}^{\infty} Q^n$, and as before, P_0 is a prime ideal of R properly contained in P [7, Theorem 23.3]. It suffices, by Lemma 4.1, to show that $P^{-1}\subsetneq T(Q)$. By $(+)$, we know that $P^{-1}=R_P\cap(\bigcap_\rho R_{J_\rho})$, and it is straightforward to see that $T(Q)=R_{P_0}\cap(\bigcap_\rho R_{J_\rho})$ [7, Exercise 11, p. 331]. The $(\#\#)$ property gives us that $T(Q)\not\subset R_P$ [9, Theorem 3], [9, Corollary 1], and so $P^{-1}=R_P\cap(\bigcap_\rho R_{J_\rho})\subsetneq R_{P_0}\cap(\bigcap_\rho R_{J_\rho})=T(Q)$, the desired contradiction. Therefore, II^{-1} is contained in no proper P-primary ideal of R.

Now, we claim that P is a maximal ideal of the ring $P^{-1}=(P:P)$. Note that P is a prime ideal of P^{-1} [7, Theorem 26.1]. Suppose P is not a maximal ideal of P^{-1} and say $P\subsetneq Q$, where Q is a prime ideal of P^{-1}. Using the $(\#\#)$ condition in combination with [5, Proposition 11 and Lemma 10] it follows that $Q\cap R$ blows up in P^{-1}, a contradiction.

To complete Case 2, and hence the proof of this theorem, we will show that $II^{-1}=P$. Set $J=II^{-1}$, and recall that J is not contained in any proper P-primary ideal, $\operatorname{rad}_R(J)=P$, and $J^{-1}=(J:J)=(P:P)=P^{-1}$. With the aid of [7, Theorem 26.1], it is straightforward to prove that $\operatorname{rad}_{P^{-1}}(J)=P$. Hence, J is a P-primary ideal of P^{-1}, and so $J=J\cap R$ is a $P=P\cap R$-primary ideal of R. Therefore, $J=II^{-1}=P$, and the proof is complete.

Now, we would like to show, by way of an example, that Theorem 4.2 does not remain valid if the $(\#\#)$ property is deleted. (It is interesting to note that the $(\#\#)$ condition was not used in the necessity part of the proof.)

EXAMPLE 4.3. In [10, Sect. 6], Gilmer and Huckaba constructed an almost Dedekind domain R that is not Dedekind. It follows that R does not satisfy property $(\#)$, and hence does not satisfy property $(\#\#)$ [8, Theorem 3]. The set of maximal ideals of R is $\{M_i\}_{i=0}^{\infty}$, where M_0 is not finitely generated and M_i is principal for each $i\geqslant 1$. Hence, as $\dim(R)=1$, we have that the noninvertible prime ideals of R (namely, M_0) are linearly ordered. It remains to show that R does not satisfy TP. First, note that $M_0^{-1}=R$, [11, Corollary 3.4] and so $M_0^{-2}=R$. Second, we have $M_0\neq M_0^2$, as R_{M_0} is a Dedekind domain. Therefore, $\gamma(M_0^2)=M_0^2M_0^{-2}=M_0^2$, which is not a prime ideal of R.

Our next theorem (Theorem 4.6) characterizes the TP property in the setting of finite dimensional Prüfer domains. We need some additional information before proving this result.

In [11], Huckaba and Papick proved that if P is a nonmaximal prime ideal of a Prüfer domain R, then $P^{-1}=(P:P)$; i.e., P^{-1} is a ring. In general, we do not know that if I is an ideal of a Prüfer domain R, and I^{-1} is a ring, then $I^{-1}=(I:I)$. The next lemma provides a partial solution to this problem.

LEMMA 4.4. *Let R be a Prüfer domain and I a nonzero primary ideal of R. If I^{-1} is a ring, then $I^{-1}=(I:I)$.*

Proof. We know that $I^{-1}=(\bigcap_\alpha R_{P_\alpha})\cap(\bigcap_\beta R_{M_\beta})$ where $\{P_\alpha\}$ is the set of minimal prime ideals of I, and $\{M_\beta\}$ is the set of maximal ideals of R not containing I [11, Theorem 3.2]. Let $\{N_\gamma\}$ be the maximal ideals of R containing I, and let $u\in I^{-1}$ and $a\in I$. It suffices to show that $au\in IR_{N_\gamma}$ for each γ. Given any γ, there exists an α such that $I\subset P_\alpha\subset N_\gamma$. Write $u=r/s$, where $r\in R$ and $s\in R\backslash P_\alpha$. We claim that $a/s\in R_{N_\gamma}$, for if not, then $s/a=t\in R_{N_\gamma}$. Hence, $s=at\in P_\alpha R_{N_\gamma}\cap R=P_\alpha$, which is impossible. Whence, $a/s\in R_{N_\gamma}$. Therefore,

$$au=a(r/s)=r(a/s)\in IR_{P_\alpha}\cap R_{N_\gamma}=IR_{N_\gamma},$$

since I is a primary ideal of R. The proof of the lemma is now complete.

LEMMA 4.5. *Let R be a Prüfer TP domain. If P is a branched prime ideal of R, and if $\{M_\alpha\}$ is the set of maximal ideals of R that do not contain P, then there exists a finitely generated ideal I of R such that $I\subset P$ and $I\not\subset M_\alpha$ for each α.*

Proof. Let P be a branched prime ideal of R and assume that there does not exist an ideal I of R satisfying the conclusion of the lemma. By [9, Corollary 2] $\bigcap_\alpha R_{M_\alpha}\subset R_P$, and thus $P^{-1}=R_P\cap(\bigcap_\alpha R_{M_\alpha})=\bigcap_\alpha R_{M_\alpha}$ [11, Theorem 3.2]. Since P is branched, there exists a proper P-primary ideal Q. Let $P_0=\bigcap Q^n$ and recall that P_0 is a prime ideal of R with $P_0\subsetneq P$ [7, Theorem 23.3]. Thus,

$$\bigcap_\alpha R_{M_\alpha}=P^{-1}=R_P\cap\left(\bigcap_\alpha R_{M_\alpha}\right)\subset R_{P_0}\cap\left(\bigcap_\alpha R_{M_\alpha}\right)=T(Q)\subset\bigcap_\alpha R_{M_\alpha}.$$

Therefore, $P^{-1}=T(Q)$ and so $P^{-1}=Q^{-1}$. Hence Q^{-1} is a ring, and by Lemma 4.4 we know that $QQ^{-1}=Q$. The *TP* property implies that Q is a prime ideal, which is a contradiction.

We are now ready for the promised characterization.

THEOREM 4.6. *Let R be a finite dimensional Prüfer domain. Then, R satisfies TP if and only if R satisfies $(\#\#)$ and the noninvertible prime ideals of R are linearly ordered.*

Proof. $(\Leftarrow)$ This direction follows from Theorem 4.2 without the finite dimensional assumption.

$(\Rightarrow)$ The proof of Theorem 4.2 shows that the noninvertible prime ideals of R are linearly ordered, and Lemma 4.5 in combination with [9, Theorem 3] establishes that R has the $(\#\#)$ property.

REFERENCES

1. D. D. ANDERSON, J. HUCKABA, AND I. PAPICK, A note on stable domains, *Houston J. Math.*, in press.
2. V. BARUCCI, Strongly divisorial ideals and complete integral closure of an integral domain, *J. Algebra* **99** (1986), 132–142.
3. H. BASS, On the ubiquity of Gorenstein rings, *Math. Z.* **82** (1963), 8–28.
4. M. FONTANA, Topologically defined classes of commutative rings, *Ann. Mat. Pura Appl. (IV)* **CXXIII** (1980), 331–355.
5. M. FONTANA, J. HUCKABA, AND I. PAPICK, Some properties of divisorial prime ideals in Prüfer domains, *J. Pure Appl. Algebra*, in press.
6. R. FOSSUM, "The Divisor Class Group of a Krull Domain," Springer-Verlag, New York, 1973.
7. R. GILMER, "Multiplicative Ideal Theory," Dekker, New York, 1972.
8. R. GILMER, Overrings of Prüfer domains, *J. Algebra* **4** (1966), 331–340.
9. R. GILMER AND W. HEINZER, Overrings of Prüfer domains II, *J. Algebra* **7** (1967), 281–302.
10. R. GILMER AND J. HUCKABA, The transform formula for ideals, *J. Algebra* **21** (1972), 191–215.
11. J. HUCKABA AND I. PAPICK, When the dual of an ideal is a ring, *Manuscripta Math.* **37** (1982), 67–85.
12. I. KAPLANSKY, "Commutative Rings," Allyn & Bacon, Boston, 1970.
13. E. MATLIS, Reflexive domains, *J. Algebra* **8** (1968), 1–33.