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On the flat spectral topology

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SUNTO - *Si studia la topologia dello spettro primo di un anello (commutativo unitario) A avente come* chiusi *le immagini delle applicazioni* f^*: Spec $(B) \to$ Spec (A) *associate agli omomorfismi* $f: A \to B$ *piatti. Si dimostra che tale topologia è la meno fina per la quale gli insiemi* $D(g) = \{P \in \text{Spec}(A) : g \notin P\}$ *sono chiusi, per ogni* $g \in A$. *Inoltre, si prova che tale topologia coincide con la topologia dell'« ordine opposto » introdotta da Hochster. Si analizzano varie proprietà di tale topologia utilizzando omomorfismi di tipo (weak) going-down. Prendendo, invece, come* base *per gli* aperti *di* Spec (A) *le immagini delle applicazioni* f^*, *con* f *soddisfacente alla proprietà del going-down, si ottiene uno spazio topologico discreto di Alexandroff. Si studiano gli aspetti « duali » di tali topologie e se ne danno applicazioni allo studio dei g-anelli di Picavet.*

1. Introduction and summary.

Our starting point is the observation [2, p. 48] that if X denotes the set of prime ideals of a ring A (equipped with the Zariski topology X_Z), then the patch topology [13] on X coincides with the constructible topology [12] on X. It is well known [2, p. 48] that the latter topology has as its *closed* sets the images of the maps f^*: Spec $(B) \to$ Spec (A) arising from all ring homomorphisms $f: A \to B$. One

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might ask which coarser topologies on X arise if the maps f are subjected to suitable restrictions. By using the homological requirement that the available maps f make the corresponding B flat as A-module, we obtain a topology that we call the flat topology X_F on X. In more elementary terms (Theorem 2.2), X_F is the coarsest topology on X such that $X_g=\{P\in X \mid g\notin P\}$ is closed for each $g\in A$. Picavet earlier introduced X_F (called the « 0-topology » on X in [24]) by subjecting X_Z to the « opposite-order » topology of Hochster. Yet another view of X_F is given in Corollary 2.11, where it is shown that X_F arises, in the manner of [2], by making the ideal-theoretic requirement that the available maps f be « weak going-down » (defined in section 2). On the other hand, using those f which are « weak going-up » produces X_Z (Corollary 2.13).

Using (not necessarily weak) going-down maps f to form a topology on X whose basis of *open* sets are the images of f^* (contrast with the approach in [2]), we obtain a discrete Alexandroff topology X^{GD} on X. Again, there is a homological view: X^{GD} is the coarsest topology on X for which flat maps f cause $im\,(f^*)$ to be open.

Throughout, all rings are assumed commutative with 1; a subring must contain the 1 of the larger ring; and ring-homomorphisms are assumed unital, that is, send 1 to 1.

2. The flat spectral topology.

As in the introduction, let A be a ring and let $X=\mathrm{Spec}\,(A)$ be the set of prime ideals of A. The Zariski topology X_Z is the coarsest topology on X in which $X_g=\{P\in X \mid g\notin P\}$ is open for each $g\in A$. The typical closed set in X_Z is $V(I)=\{P\in X \mid I\subset P\}$ arising from an ideal I of A. As seen in [2], p. 48, the constructible topology X_C issuing from X_Z is the coarsest topology on X in which each X_g is both open and closed. The analysis in [2] also reveals that the collection of closed subsets of X_C is just $\{im\,(f^*)\subset X \mid f$ is a ring-homomorphism with domain $A\}$. That this collection of images forms the closed sets for *a* topology on X depends on the following two key assertions sketched in [2], Exercise 27, p. 48. Let $\{f_i\colon A\to B_i\}$ be a collection of ring-homomorphisms indexed by a set I, set $T=\bigotimes_{i\in I} B_i$ (tensoring over A) and $D=\prod_{i\in I} B_i$, and let $g\colon A\to T$ and $h\colon A\to D$ be the ring-homormorphisms induced by $\{f_i\}$. Then $im\,(g^*)=\bigcap_{i\in I} im\,(f_i^*)$; and, if I is finite, $im\,(h^*)=\bigcup_{i\in I} im\,(f_i^*)$. By invoking these same two assertions, we see similarly that one may introduce a topology X_F on X by decreeing that the typical closed set in X_F is of the form $im\,(f^*)$, where $f\colon A\to B$ is a ring-homomorphism inducing an A-flat module structure on B. (The point is that, in the above notation, T and (if I is finite) D are A-flat whenever each B_i is A-flat: cf. [3], Corollaire 1, p. 34 and Proposition 2, p. 28). We shall call X_F the *flat topology* (*on* X, or *for* A).

Before giving alternate descriptions of X_F, we need the following fragment of Lemma 2.5.

LEMMA 2.1. *Let L be a closed set in the flat topology X_F for a ring A. If $P, Q\in X$ (that is, P and Q are prime ideals of A) such that $P\subset Q$ and $Q\in L$, then $P\in L$.*

PROOF. By the construction of X_F, there is a ring-homomorphism $f\colon A\to B$ such that B is A-flat and $L=\mathrm{im}\,(f^*)$. In particular, $Q=f^*(Q_1)$ for some $Q_1\in Y=\mathrm{Spec}\,(B)$. However, by flatness, f satisfies the going-down property (GD, as in [14], p. 28): cf. [8]. Thus, there exists $P_1\in Y$ such that $P_1\subset Q_1$ and $f^*(P_1)=P$. In particular, $P\in im\,(f^*)$, as asserted.

THEOREM 2.2. *Let X_F be the flat topology (on X, for A). Then X_F is the coarsest topology on X in which X_g is closed for each $g\in A$. A basis for the open sets of X_F is given by $\{V(I) \mid I$ is a finitely generated ideal of $A\}$.*

PROOF. Let X' denote the coarsest topology on X in which X_g is closed for each $g\in A$. Observe for each $g\in A$, that the canonical map $f\colon A\to A_g$ satisfies $im\,(f^*)=X_g$. As A_g is A-flat [3], Théorème 1, p. 88, X_g is closed in X_F, so that $X'\subset X_F$, by the minimality of X'.

For the reverse inclusion, let L be a closed subset in X_F. We shall prove that $U=X\setminus L$ is open in X'. First, select a ring-homomorphism $h\colon A\to B$ such that B is A-flat (*via* h) and $im\,(h^*)=L$. We claim, for each $P\in X$, that $P\in U$ if and only if $h(P)B=B$. Indeed, the « if » half of the claim is apparent. On the other hand, if $P\in X$ and $h(P)B\neq B$, arrange $h(P)B\subset M$ for a suitable $M\in\mathrm{Spec}\,(B)$, and note that $P\subset h^*(M)$. As $h^*(M)\in L$, Lemma 2.1 gives $P\in L$, proving (the contrapositive of) the « only if » half. Now, to put the claim to use, note that for each $P\in U$, we can fix an equation

$$h(a_{1P})\, b_{1P} + h(a_{2P})\, b_{2P} + \dots + h(a_{n(P),P})\, b_{n(P),P} = 1$$

with $a_{ij} \in P$ and $b_{ij} \in B$. Evidently,

$$U = \bigcup_{P \in U} V(Aa_{1P} + \dots + Aa_{n(P),P}).$$

To complete the proof, it suffices to show that $V(c_1, \dots, c_m) = V(Ac_1 + \dots + Ac_m)$ is open in X' for any finite subset $\{c_1, \dots, c_m\}$ of A. This, in turn, follows since $V(c_1, \dots, c_m) = X \setminus (X_{c_1} \cup \dots \cup X_{c_m})$.

REMARK 2.3. (*a*). By the second assertion in Theorem 2.2, X_F coincides with the « 0 (rder)-topology » on X consider in [24], p. 88. Thus, by [13], Proposition 8, X_F is a spectral space. According to the characterization in [13], Proposition 4, X_F is a quasi-compact T_0-space, with a quasi-compact open basis closed under finite intersections, such that each nonempty irreducible closed subspace of X_F has a generic point. Direct verification of these properties of X_F is straightforward.

(*b*) In general, all that one may say is that X_F and X_Z are each coarser than X_C. Indeed, let A be a 1-dimensional quasilocal integral domain, with maximal ideal M. To see that $X_C \not\subseteq X_F$, note that $\{M\}$ is closed in X_C (consider f^* arising from the canonical epimorphism $f: A \to A/M$), although $\{M\}$ is not closed in X_F (by Lemma 2.1). Similarly, $X_Z \not\subseteq X_F$ since $\{M\}$ is closed in X_Z. Moreover, consideration of $\{M\}$ recovers the well-known fact that $X_C \not\subseteq X_Z$ (cf. [19]). Finally, we have $X_F \not\subseteq X_Z$, since $\{0\}$ is closed in X_F but not closed in X_Z.

(*c*) For any ring A, the minimal prime spectrum Y of A is quasi-compact in the subspace topology inherited from X_F. Indeed, suppose that $Y \subset \bigcup_{\alpha} V(a_{1\alpha}, a_{2\alpha}, \dots, a_{n(\alpha),\alpha})$ for suitable $a_{ij} \in A$. Since each nonempty closed subset of X_F contains a minimal prime (by Lemma 2.1 and [14], Theorem 10), $X \setminus \bigcup_{\alpha} V(a_{1\alpha}, \dots, a_{n(\alpha),\alpha}) = \phi$. As noted in part (*a*), X_F is quasi-compact, whence some union of finitely many of the $V(a_{1\alpha}, \dots, a_{n(\alpha),\alpha})$ suffices to cover X and, *a fortiori*, that union contains Y.

(*d*) In contrast with (*c*), the maximal spectrum W of A need not be quasi-compact in the subspace topology inherited from X_F. Indeed, such quasi-compactness for the case of a Noetherian ring A guarantees that A is semi-local. To see this, observe that $W \subset \bigcup_{M \in W} V(M)$, so that quasi-compactness of W leads to $W \subset V(M_1) \cup \dots \cup V(M_n)$

for finitely many $M_i \in W$. Then, by the prime avoidance lemma (cf. [14], Theorem 81), $W = \{M_1, \dots, M_n\}$, as asserted.

(*e*) Let $f: A \to B$ be a ring-homomorphism, inducing the function $f^*: Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$. Observe that f^*, viewed as a map from Y_F to X_F is continuous, since $f^{*-1}(V_A(a)) = V_B(f(a))$ for each $a \in A$. However, f^* need not be an open map for the flat topologies: for example, take f to be the inclusion map of the ring A from part (*b*) into its quotient field. Moreover, in contrast with the situation for the constructible topologies [2], Exercise 29, p. 49, f^* need not be a closed map for the flat topologies: for example, take f to be the canonical projection $A \to A/M$, where A is as in (*b*). However, it is easy to see in general, using [3], Corollaire 3, p. 35, that $f^*: Y_F \to X_F$ *is* closed in case f makes B a flat A-module.

(*f*) It is easy to show that a ring A is quasilocal if and only if X_F is irreducible. (*Proof*: For $a_i, b_j \in A$, note that $V(a_1, \dots, a_n) \cap V(b_1, \dots, b_m) \neq \phi$ if and only if the ideal $(a_1, \dots, a_n, b_1, \dots, b_m) \neq A$).
More generally, one may show, for any ring A, that there is an inclusion-preserving bijection from Spec(A) to the collection of closed irreducible subspaces of X_F, given by $P \mapsto \{Q \in X \mid Q \subset P\}$. Accordingly, the maximal spectrum of A is in one-to-one correspondence with the set of irreducible components of X_F.

(*g*) One way in which the flat topology behaves like the Zariski topology is the following. Let A be a ring and $B = A_{\mathrm{red}}$, the reduced ring associated to A. Let $f: A \to B$ be the canonical surjection, and set $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Then $f^*: Y_F \to X_F$ is a homeomorphism. For a *proof*, the first observation in (*e*) reduces the task to showing that f^* is an open map, and this follows in turn because $f^*(V_B(f(a))) = V_A(a)$ for each $a \in A$.

We next determine when X_F is a T_1-space. Another approach to the equivalence (1) $\Leftrightarrow$ (6) may be found in [19], Proposition 1.5. Further topological equivalences appear in [19], Théorème 1.3.

PROPOSITION 2.4. *Let A be a ring, and $X =$* Spec(A). *The following are equivalent:*

(1) *dim* $(A) = 0$;

(2) A_{red} *is von Neumann regular (« absolutely flat »);*

(3) *X_Z is a Hausdorff space;*

(4) *X_Z is a T_1-space;*

(5) $X_F = X_Z$;

(6) $X_Z = X_C$;

(7) $X_F = X_C$;

(8) *X_F is a Hausdorff space;*

(9) *X_F is a T_1-space.*

PROOF. It is well-known that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): cf. [2], Exercise 11, p. 44.

(3) $\Rightarrow$ (5): Assume (3). By standard techniques, X_a is (both open and) closed in X_Z, for each $a \in A$. Thus, the subbasic X_F-open sets $V(a)$ are open in X_Z, whence $X_F \subset X_Z$. For the reverse inclusion, Remark 2.3 (*g*) reduces consideration to the case in which A is a reduced ring. In particular, A is von Neumann regular. Now, it suffices to show that each basic X_Z-open set X_a is X_F-open. If $f\colon A \to A/Aa$ is the canonical surjection, note that $im(f^*) = V(a) = X \setminus X_a$ is closed in X_F since von Neumann regularity forces A/Aa to be A-flat [3], Exercise 17, p. 64.

(5) $\Rightarrow$ (6): Since $X_Z \subset X_C$ in general, (6) holds precisely in case X_a is closed in X_Z for each $a \in A$. This, in turn, follows from (5) because each X_a is closed in X_F.

(6) $\Rightarrow$ (7): As $X_F \subset X_C$ in general, (7) holds precisely in case X_a is open in X_F for each $a \in A$. Since X_C is Hausdorff in general [2], Exercise 28, p. 48, (6) guarantees that A_{red} is von Neumann regular, so that (7) now follows as in the earlier proof that (3) $\Rightarrow$ (5).

Finally, (7) $\Rightarrow$ (8) since X_C is Hausdorff; (8) $\Rightarrow$ (9) trivially; and (9) $\Rightarrow$ (1) by Lemma 2.1. The proof is complete.

The next result adapts a most useful conceptual insight from [13]. First, some terminology: if A is a ring and L is a subset of $X = \operatorname{Spec}(A)$, then L is said to be *stable under generalization* (resp., *stable under specialization*) if, whenever $Q \subset P$ (resp., $P \subset Q$) for $P, Q \in X$ and $P \in L$, one must then have $Q \in L$.

LEMMA 2.5. *Let A be a ring, and L a subset of $X =$ Spec (A). Then:*

(*a*) *L is closed in X_F if and only if L is closed in X_C and L is stable under generalization.*

(*b*) *L is closed in X_Z if and only if L is closed in X_C and L is stable under specialization.*

PROOF. (*a*) One may verify directly that the constructible topology (in the sense of [12], (7.2.11), p. 337) issuing from X_F is just X_C. (This was also observed in [24], p. 88). Accordingly, [13], first corollary of Theorem 1 (cf. also [12], Corollaire 7.3.2, p. 339) yields the following. L is closed in X_F if and only if L is closed in X_C and $L = \bigcup_{P \in L} \overline{P}$, where $\overline{P}$ denotes the closure in X_F of the singleton set $\{P\}$. It now suffices to establish that $\overline{P} = \{Q \in X \mid Q \subset P\}$. To this end, note first that $\{Q \in X \mid Q \subset P\}$ is X_F-closed, since it is the image of the spectral map induced by the canonical homomorphism $A \to A_P$; and then use Lemma 2.1.

(*b*) View X_C as the constructible topology (in the sense of [12]) issuing from X_Z and argue as above, using [13], first corollary of Theorem 1. To effect the translation, observe that the closure in X_Z of a singleton $\{P\}$ is just $\{Q \in X \mid P \subset Q\}$.

REMARK 2.6. As an application of Lemma 2.5, we sketch a proof that X_F is a connected space if and only if the only idempotent elements of A are 0 and 1 (and hence, by [3], Corollaire 2, p. 132, if and only if X_Z is a connected space). For the « only if » half, observe that the existence of a nontrivial element $e = e^2 \in A$ results in X being the union of the nonempty, mutually disjoint X_F-open sets $V(e)$ and $V(1-e)$. As for the « if » half, suppose that $X = L_1 \cup L_2$, where L_1 and L_2 are mutually disjoint, nonempty X_F-closed sets. It suffices to prove that X_Z is not connected (for a nontrivial idempotent is thereby guaranteed). To this end, we need only prove that each L_i is X_Z-closed, that is, according to Lemma 2.5 (*b*), that L_i is stable under specialization. This in turn follows from Lemma 2.5 (*a*), since the X_F-closed set $X \setminus L_i$ is stable under generalization.

We next turn to a more pertinent application of Lemma 2.5. As usual (cf. [2], p. 68), *GD* (resp., *GU*) will denote the going-down (resp., going-up) property satisfied by certain ring-homomorphisms.

Note that assertion (*b*) in the next result is well-known in case Y_Z is a Noetherian space [2], Exercise 11, p. 79.

PROPOSITION 2.7 *Let* $f: A \to B$ *be a ring-homomorphism, with* $f^*: Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$ *the induced function. Then:*

(*a*) *f satisfies GD if and only if* $f^*: Y_F \to X_F$ *is a closed map.*

(*b*) *f satisfies GU if and only if* $f^*: Y_Z \to X_Z$ *is a closed map.*

PROOF. (*a*) Assume that f satisfies GD. To prove that f^* is F-closed, let L be a closed subset of Y_F. By the going-down assumption, $f^*(L)$ is stable under generalization. Moreover, L is closed in Y_C, so that $f^*(L)$ is closed in X_C (cf. [2], Exercise 29, p. 49). Thus, by Lemma 2.5 (*a*), $f^*(L)$ is closed in X_F, as desired.

Conversely, suppose that f^* is F-closed. To prove that f satisfies GD, consider P_1, $P \in X$ and $Q \in Y$ such that $P_1 \subset P$ and $f^*(Q) = P$. Our task is to find $Q_1 \in Y$ such that $Q_1 \subset Q$ and $f^*(Q_1) = P_1$. To this end, first notice that $L = \{W \in Y: W \subset Q\}$ is closed in Y_F, and so by hypothesis, $f^*(L)$ is closed in X_F. As $P \in f^*(L)$, Lemma 2.1 guarantees that $P_1 \in f^*(L)$, as required.

(*b*) The « only if » half follows as in (*a*), *mutatis mutandis.* The « if » half is well-known (cf. [2], Exercise 10, p. 68).

The next result collects some variations on the above theme.

PROPOSITION 2.8. *Let* $f: A \to B$ *be a ring-homomorphism, with induced function* $f^*: Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$. *Then:*

(*a*) *If f satisfies GD and* f^* *is C-open, then* f^* *is Z-open.*

(*b*) *If f satisfies GU and* f^* *is C-open, then* f^* *is F-open.*

(*c*) *If* Y_Z *is a Noetherian space and* f^* *is F-open, then f satisfies GU.*

(*d*) *If* X_Z *is a Noetherian space and f satisfies GU, then* f^* *is F-open.*

PROOF. By virtue of Lemma 2.5 (*b*), assertion (*a*) is a consequence of the remark that for each Y_Z-open subset U, one has that $f^*(U)$ is X_C-open and stable under generalization. Similarly, (*b*) follows by use of Lemma 2.5 (*a*).

(*c*): To show that f satisfies GU, consider P, $P_1 \in X$ and $Q \in Y$ such that $P \subset P_1$ and $f^*(Q) = P$. We must find $Q_1 \in Y$ such that $Q \subset Q_1$ and $f^*(Q_1) = P_1$. By the Noetherian hypothesis, [21], Proposition 2.1 shows that Q is the radical of a finitely generated ideal $B b_1 + \dots + B b_n$ of B. Then $V(Q) = V(b_1, \dots, b_n)$ is Y_F-open, and so by hypothesis, $U = f^*(V(Q))$ is X_F-open. Since $P \in U$, Lemma 2.1 guarantees that $P_1 \in U$, as desired.

(*d*): By Proposition 2.7 (*b*), the going-up hypothesis guarantees that f^* sends each basic Y_F-open subset $V(b_1, \dots, b_n)$ to a suitable X_Z-closed subset, that is, to $V(I)$ for a suitable ideal I of A. However, as in the proof of (*c*), the Noetherian hypothesis arranges finitely many $a_i \in A$ such that $V(I) = V(a_1, \dots, a_m)$, an X_F-open subset, completing the proof.

Recall that this section began with a homologically-motivated construction of X_F in the spirit of the construction in [2] of X_C. (Notice the importance of *which* homomorphisms are used. For example, if one constructs the closed sets by using *faithfully* flat maps in the role played by flat maps earlier, the result is the indiscrete topology). In seeking an analogous construction of X_F with an ideal-theoretic flavor, one might consider using going-down homomorphisms to replace the flat maps in the earlier construction. A difficulty arises, however, if one attempts to adapt the machinery from [2] in this way, for we do not know that going-down homomorphisms $f: A \to B$ and $g: A \to C$ must necessarily induce a going-down homomorphism $h: A \to B \otimes_A C$. (Of course, there is no problem if, for example, B is A-flat *via* f). According to Proposition 2.7 (*b*), the question is whether h^* must be F-closed. In general, we can show only that $im(h^*)$ is F-closed. Accordingly, we temporarily abandon considerations involving GD, and introduce another, more adaptable ideal-theoretic notion as follows.

Let $f: A \to B$ be a ring-homomorphism, with induced function $f^*: Y = \mathrm{Spec}(B) \to X = \mathrm{Spec}(A)$. We shall say that f satisfies *weak going-down,* denoted WGD (resp., *weak going-up,* denoted WGU) if, whenever P, $P_1 \in X$ and $Q \in Y$ are such that $P_1 \subset P$ (resp., $P \subset P_1$) and $f^*(Q) = P$, there must exist $Q_1 \in Y$ such that $f^*(Q_1) = P_1$; that is, if and only if $im(f^*)$ is stable under generalization (resp., specialization). Evidently, a homomorphism which satisfies GD (resp., GU) must also satisfy WGD (resp., WGU). As the next remark shows, the converse fails.

REMARK 2.9. Consider, as in Remark 2.3 (*b*), a quasilocal 1-dimensional integral domain A, with maximal ideal M, quotient field K, and residue class field $k=A/M$. Let $f\colon A \to K\times k$ be the canonical homomorphism. Then f satisfies *WGD* and *WGU*, but f satisfies neither *GD* nor *GU*. The point is that the two prime ideals of $K\times k$ are not comparable although $im\,(f^*)=\mathrm{Spec}\,(A)$.

The next result contains the key to an ideal-theoretic construction of X_F.

LEMMA 2.10. *Let A be a ring, and L a subset of $X=\mathrm{Spec}\,(A)$. The following are equivalent:*

(1) *L is a closed subset of X_F;*

(2) *$L=im\,(f^*)$ for some ring-homomorphism $f\colon A\to B$ satisfying GD;*

(3) *$L=im\,(f^*)$ for some ring-homomorphism $f\colon A\to B$ satisfying WGD.*

PROOF. Since each flat homomorphism satisfies *GD*, our definition of X_F guarantees that (1) $\Rightarrow$ (2). Of course, (2) $\Rightarrow$ (3) trivially. Finally, (3) $\Rightarrow$ (1) by Lemma 2.5 (*a*).

COROLLARY 2.11. *The collection of closed subsets in the flat topology X_F (on X, for A) may be described in each of the following ways:*

(i) $\{im\,(f^*) \mid f\colon A\to B$ *is a ring-homomorphism satisfying GD*$\}$;

(ii) $\{im\,(f^*) \mid f\colon A\to B$ *is a ring-homomorphism satisfying WGD*$\}$.

The point of introducing the *WGD* concept is that, by *strict* analogy with the situations for X_C and X_F described in the first paragraph of this section, one may verify *directly* (what is now known by other means) that the collection in (ii) above forms the class of closed sets for *a* topology on X. Precisely, we have the following result.

PROPOSITION 2.12. *Let $\{f_i\colon A\to B_i\}$ be a collection of ring-homomorphisms indexed by a set I, such that f_i satisfies WGD for each $i\in I$. Set $T=\otimes B_i$ (tensoring over A) and $D=\Pi B_i$, and let $g\colon A\to T$ and $h\colon A\to D$ be the homomorphisms induced by $\{f_i\}$. Then g satisfies WGD; and, if I is finite, h satisfies WGD.*

PROOF. Note that $im\,(f_i^*)$ is stable under generalization for each $i\in I$, since f_i satisfies *WGD*. Thus, $im\,(g^*)=\cap\, im\,(f_i^*)$ is also stable under generalization, whence g satisfies *WGD*. The final assertion follows similarly by recalling that $im\,(h^*)=\cup\, im\,(f_i^*)$ in case I is finite.

Another benefit derived from Proposition 2.12 is the next result. Its proof results by replacing « generalization » by « specialization » throughout the above proof.

PROPOSITION 2.12 (bis). *The assertion obtained by letting « WGU » replace all occurrences of « WGD » in the statement of Proposition 2.12 is valid.*

Now, by *strict* analogy with the ideal-theoretic approaches to X_C and X_F, we are ready to recover X_Z.

COROLLARY 2.13. *The collection of closed sets in the Zariski topology X_Z (on X, for A) is $\{im\,(f^*) \mid f\colon A\to B$ is a ring-homomorphism satisfying WGU$\}$.*

PROOF. Each closed set in X_Z does arise in the indicated way, for if I is an ideal of A and $f\colon A\to A/I$ is the canonical surjection, it is evident that f^* satisfies *WGU* and $im\,(f^*)=V\,(I)$. Conversely, we claim that if $L=im\,(g^*)$ for a ring-homomorphism $g\colon A\to B$ satisfying *WGU*, then L is X_Z-closed. Indeed, note first that L is X_C-closed since g^* is a C-closed map. Moreover, the *WGU* condition assures that L is closed under specialization, so that an application of Lemma 2.5 (*b*) completes the proof.

Despite the comments following the proof of Proposition 2.8, the going-down property's behavior does lead to an interesting topology (see Theorem 2.16 below). A related result is given next.

LEMMA 2.14. *Let $(I, \leq)$ be a direct set, (B_i, g_{ij}) a direct system of rings indexed by I, and $\{h_i\}$ a set of ring-homomorphisms, such that each $h_i\colon A\to B_i$ satisfies GD and $g_{ij}\,h_i=h_j$ whenever $i\leq j$ in I. Set $B=\varinjlim B_i$ and $h=\varinjlim h_i$. Then $h\colon A\to B$ also satisfies GD.*

PROOF. We must show that if $Q\in\mathrm{Spec}\,(B)$ and $P=h^*\,(Q)\in\mathrm{Spec}\,(A)$, then the canonical function $F\colon \mathrm{Spec}\,(B_Q)\to\mathrm{Spec}\,(A_P)$ is surjective. To this end, let $g_i\colon B_i\to B$ be the structure map for each $i\in I$, set satisfies GD, we see that F_j: Spec $((B_j)_{Q_j}) \to$ Spec (A_P) is surjective for each j. As $\varinjlim (B_j)_{Q_j}$ and B_Q are isomorphic A_P-algebras [12], Proposition 6.1.6 (ii), p. 130, it follows that $im(F) = im$ (Spec $(\varinjlim (B_j)_{Q_j}) \to$ $\to$ Spec $(A_P)) = \cap\, im(F_j) =$ Spec (A_P), as required.

REMARK 2.15. The analogue of Lemma 2.14 in which « GD » is replaced throughout by « GU » is valid. [Sketch of *proof*: The task is to show that Spec $(B/Q) \to$ Spec (A/P) is surjective whenever $h^*(Q)=P$. Argue as above, noting that $\varinjlim (B_j/Q_j)$ and B/Q are isomorphic A/P-algebras]. Of course, the analogue using the lying-over property (LO [14], p. 28) is also valid.

We pause to recall (cf. [1], p. 28) that a *discrete Alexandroff space* is a T_0-topological space X such that, for each $Y \subset X$, the closure of Y is the union of the closures of the singleton subsets of Y; equivalently, a T_0-space in which every intersection of (arbitrarily many) open subsets is open.

THEOREM 2.16. *Let A be a ring, with $X=$ Spec (A). Then there exists a discrete Alexandroff topological structure X^{GD} on X such that each of the following three collections forms a basis for the open sets in X^{GD}:*

(i) $\{im(f^*) \mid f: A \to B$ *is a ring-homomorphism satisfying* $GD\}$;

(ii) $\{im(f^*) \mid f: A \to B$ *is a ring-homomorphism making B a flat A-module*$\}$;

(iii) $\{\{Q \in X \mid Q \subset P\} \mid P \in X\}$.

Moreover, X^{GD} is the coarsest topology on X in which im (Spec $(A_P) \to$ $\to$ Spec (A)) *is open for each $P \in X$.*

PROOF. Observe that im (Spec $(A_P) \to$ Spec $(A)) = \{Q \in X \mid Q \subset P\}$. In particular, X is the union of the sets in the collection described in (i). Moreover, that collection is stable under arbitrary intersections, by virtue of Corollary 2.11. (If one could answer affirmatively the question raised after Proposition 2.8, then another proof would follow from Lemma 2.14 and [2], Exercise 26, p. 48). Thus, the collection in (i) forms an open basis for a discrete Alexandroff topology on X. Let X^{GD} denote this topological structure. Since each flat homomorphism satisfies going-down, it only remains to prove the following assertion. Each open subset U of X^{GD} is expressible as a union of some of the sets which are members of the collection described in (iii). For the proof, write $U = im(f^*)$, where $f: A \to B$ is a suitable homomorphism satisfying GD. Now, for each $Q \in Y =$ Spec (B), let h_Q be the composite $A \to B \to B_Q$; then $im(f^*) = \bigcup_{Q \in Y} im(h_Q^*)$. If $P = f^*(Q) \in X$, observe that h_Q is the composite of the canonical maps $g_Q: A \to A_P$ and $f_Q: A_P \to B_Q$, so that $h_Q^* = g_Q^* \circ f_Q^*$. However, f_Q^* is surjective since f satisfies GD, and so $im(h_Q^*) = im(g_Q^*)$, which is a member of the collection described in (iii). This completes the proof.

It was shown in [24], Proposition 1, section 5 and [9], Proposition 1, section 3 that a spectral space X is discrete Alexandroff if and only if X is homeomorphic to $(\text{Spec}\,A)_Z$ for some g-ring A. Recall that a ring A is said to be a *g-ring* if, for each $P \in$ Spec (A), some element $f \in A \setminus P$ satisfies $A_P \cong A_f$; equivalently, if $\{Q \in$ Spec $(A) \mid Q \subset P\}$ is Spec $(A)_Z$-open for each $P \in$ Spec (A) (cf. [24], Proposition 1, p. 87). Every g-ring is a Goldman ring in the sense of [9]. In the case of integral domains, g-rings have been studied in [25] under the name of « locally pqr-domains » and are closely related to QR-domains ([11], [6], [23]) and open domains [22].

The next result characterizes g-rings in terms of some of the earlier notions. The equivalence of (a) and (b) was first noted in [24], Proposition 2, section 5.

THEOREM 2.17. *For a ring A, the following are equivalent:*

(a) *A is a g-ring;*

(b) *If $f: A \to B$ is a ring-homomorphism making B a flat A-module, then f^*:* $(\text{Spec}\,B)_Z \to (\text{Spec}\,A)_Z$ *is an open map;*

(c) *If $f: A \to B$ is a ring-homomorphism, then f satisfies GD (if and) only if f^* is a Z-open map.*

PROOF. It is well-known (cf. [8]) that a ring-homomorphism h satisfies GD whenever h^* is Z-open, thus explaining the parenthetical part of (c). Evidently, (c) $\Rightarrow$ (b) $\Rightarrow$ (a), by the above remarks. Finally, to show (a) $\Rightarrow$ (c), let $f: A \to B$ satisfy GD, and consider the image under f^* of a basic Z-open set Y_b, for some $b \in B$. As in the proof of Theorem 2.16, for $Q \in Y_b$ and $P = f^*(Q) \in$ Spec (A), let h_Q be the composite map $A \to A_P \to B_Q$ and let $g_Q: A \to A_P$ be the canonical map. Then, since each Spec $(B_Q) \to$ Spec (A_P) is surjective, we have

$$f^*(Y_b) = \bigcup_{Q \in Y_b} im\,(h_Q^*) = \bigcup_{Q \in Y_b} im\,(g_Q^*).$$

Now, since (*a*) assumes that A is a *g*-ring, each $Q \in Y_b$ yields an element $d \in A \setminus P$ such that $A_P \cong A_d$. Then $im\,(g_Q^*)$ is the basic *Z*-open set X_d of Spec (A), whence $f^*(Y_b)$ is open in (Spec $A)_Z$, completing the proof.

REMARK 2.18. (*a*) The converse of condition (*b*) in Theorem 2.17 is false, even if A is a *g*-ring. It suffices to consider, for example, a quasilocal 1-dimensional integral domain A which is not integrally closed and to let $f: A \to A'$ be the inclusion of A into its integral closure. One sees immediately in this case that A is a *g*-ring, f satisfies both *GD* and *GU*, and (by (*c*) of Theorem 2.17 and Proposition 2.7 (*b*)) f^* is both *Z*-open and *Z*-closed, although f is not flat [26], Proposition 2.

(*b*) Theorem 2.17 may be combined with [9], Corollary of Proposition 1, section 3 and Proposition 5, section 1 to produce easy proofs of the characterizations of open domains in [22], Proposition 3.2 and Theorem 3.16.

The next part of this section studies the Noetherian property for the spectral spaces considered above. Picavet [24], Proposition 4, p. 89 has shown that a ring A is a *g*-ring if and only if (Spec $A)_F$ is a Noetherian topological space. A condition characterizing Noetherianness of X_Z is given in [24], Proposition 4, p. 81. We proceed to a more tractable property which also serves to characterize Noetherianness of X_Z. For motivation, the reader may wish to review the proof of Proposition 2.8 (*c*). First, observe that the « duality » between the flat topology and the Zariski topology is enhanced by the following consequence of [13], Proposition 8. For each ring A, there exist a ring B and inclusion-reversing homeomorphisms (Spec $A)_F \to$ (Spec $B)_Z$ and (Spec $A)_Z \to$ (Spec $B)_F$.

THEOREM 2.19. *Let A be a ring, with $X=$ Spec (A). Consider the following statements:*

(i) *X_Z is a Noetherian space;*

(ii) *$V(I)$ is open in X_F for each ideal I of A;*

(iii) *$V(P)$ is open in X_F for each $P \in X$;*

(iv) *A satisfies the ascending chain condition for prime ideals. Then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

PROOF. (i) $\Rightarrow$ (ii): As X_Z is Noetherian, [21], Proposition 2.1 gives, for each ideal I of A, a finitely generated ideal J of A such that I and J have the same radical. Then $V(I) = V(J)$, which is open in X_F.

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (ii): It suffices to observe that if $\{P_i\}$ is the set of prime ideals of A which contain a given ideal I of A, then $V(I) = \cup V(P_i)$.

(ii) $\Rightarrow$ (i): If $\{I_i\}$ is any collection of ideals of A, then $\cap V(I_i) = V(\Sigma I_i)$. Thus, by Theorem 2.2, it follows from (ii) that X_F has a basis which is stable under arbitrary intersections, so that X_F is a discrete Alexandroff space. Now if B is taken as in the sentence preceding the statement of the theorem, (Spec $B)_Z$ is discrete Alexandroff, so that B is a g-ring, whence the above comments assure that $X \cong$ (Spec $B)_F$ is a Noetherian space, as desired.

(i) $\Rightarrow$ (iv): Trivial, since (i) is equivalent to the ascending chain condition for radical ideals of A.

REMARK 2.20. (*a*) One should note that condition (iv) in Theorem 2.19 is not universally satisfied. It suffices to consider a valuation domain A whose maximal ideal is the union of the nonmaximal prime ideals of A.

(*b*) In [10], Example 1, Gilmer constructs a 1-dimensional Bézout Goldman domain A with infinitely many prime ideals. We claim that X_Z is not Noetherian, indeed that $V(P)$ fails to be X_F-open for some maximal $P \in$ Spec (A). (Thus, as a consequence, (iv) $\not\Rightarrow$ (iii), in the notation of Theorem 2.19, even for coherent A). For a *proof*, first note that $\{0\}$ is open in X_Z, and hence open in X_C, since A is a Goldman domain. Now, if the assertion fails, the compact space X_C is covered by the union of the *C*-open singleton sets $\{Q\}$, as Q ranges over Spec (A). The existence of a finite subcover guarantees that Spec (A) is finite, the desired contradiction.

(*c*) As a prelude to a later paper (*), we turn next to applica-

(*) D. E. DOBBS - M. FONTANA - I. J. PAPICK, *On certain distinguished spectral sets.* Annali Mat. Pura Appl. (to appear).

map. Then, since each $\mathrm{Spec}(B_Q) \to \mathrm{Spec}(A_P)$ is surjective, we have

$$f^*(Y_b) = \bigcup_{Q \in Y_b} im(h_Q^*) = \bigcup_{Q \in Y_b} im(g_Q^*).$$

Now, since (*a*) assumes that A is a *g*-ring, each $Q \in Y_b$ yields an element $d \in A \setminus P$ such that $A_P \cong A_d$. Then $im(g_Q^*)$ is the basic *Z*-open set X_d of $\mathrm{Spec}(A)$, whence $f^*(Y_b)$ is open in $(\mathrm{Spec}\, A)_Z$, completing the proof.

REMARK 2.18. (*a*) The converse of condition (*b*) in Theorem 2.17 is false, even if A is a *g*-ring. It suffices to consider, for example, a quasilocal 1-dimensional integral domain A which is not integrally closed and to let $f: A \to A'$ be the inclusion of A into its integral closure. One sees immediately in this case that A is a *g*-ring, f satisfies both *GD* and *GU*, and (by (*c*) of Theorem 2.17 and Proposition 2.7 (*b*)) f^* is both *Z*-open and *Z*-closed, although f is not flat [26], Proposition 2.

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The next part of this section studies the Noetherian property for the spectral spaces considered above. Picavet [24], Proposition 4, p. 89 has shown that a ring A is a *g*-ring if and only if $(\mathrm{Spec}\, A)_F$ is a Noetherian topological space. A condition characterizing Noetherianness of X_Z is given in [24], Proposition 4, p. 81. We proceed to a more tractable property which also serves to characterize Noetherianness of X_Z. For motivation, the reader may wish to review the proof of Proposition 2.8 (*c*). First, observe that the « duality » between the flat topology and the Zariski topology is enhanced by the following consequence of [13], Proposition 8. For each ring A, there exist a ring B and inclusion-reversing homeomorphisms $(\mathrm{Spec}\, A)_F \to (\mathrm{Spec}\, B)_Z$ and $(\mathrm{Spec}\, A)_Z \to (\mathrm{Spec}\, B)_F$.

THEOREM 2.19. *Let A be a ring, with $X = \mathrm{Spec}(A)$. Consider the following statements*:

(i) *X_Z is a Noetherian space;*

(ii) *$V(I)$ is open in X_F for each ideal I of A;*

(iii) *$V(P)$ is open in X_F for each $P \in X$;*

(iv) *A satisfies the ascending chain condition for prime ideals. Then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

PROOF. (i) $\Rightarrow$ (ii): As X_Z is Noetherian, [21], Proposition 2.1 gives, for each ideal I of A, a finitely generated ideal J of A such that I and J have the same radical. Then $V(I) = V(J)$, which is open in X_F.

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (ii): It suffices to observe that if $\{P_i\}$ is the set of prime ideals of A which contain a given ideal I of A, then $V(I) = \bigcup V(P_i)$.

(ii) $\Rightarrow$ (i): If $\{I_i\}$ is any collection of ideals of A, then $\bigcap V(I_i) = V(\Sigma I_i)$. Thus, by Theorem 2.2, it follows from (ii) that X_F has a basis which is stable under arbitrary intersections, so that X_F is a discrete Alexandroff space. Now if B is taken as in the sentence preceding the statement of the theorem, $(\mathrm{Spec}\, B)_Z$ is discrete Alexandroff, so that B is a g-ring, whence the above comments assure that $X \cong (\mathrm{Spec}\, B)_F$ is a Noetherian space, as desired.

(i) $\Rightarrow$ (iv): Trivial, since (i) is equivalent to the ascending chain condition for radical ideals of A.

REMARK 2.20. (*a*) One should note that condition (iv) in Theorem 2.19 is not universally satisfied. It suffices to consider a valuation domain A whose maximal ideal is the union of the nonmaximal prime ideals of A.

(*b*) In [10], Example 1, Gilmer constructs a 1-dimensional Bézout Goldman domain A with infinitely many prime ideals. We claim that X_Z is not Noetherian, indeed that $V(P)$ fails to be X_F-open for some maximal $P \in \mathrm{Spec}(A)$. (Thus, as a consequence, (iv) $\not\Rightarrow$ (iii), in the notation of Theorem 2.19, even for coherent A). For a *proof*, first note that $\{0\}$ is open in X_Z, and hence open in X_C, since A is a Goldman domain. Now, if the assertion fails, the compact space X_C is covered by the union of the *C*-open singleton sets $\{Q\}$, as Q ranges over $\mathrm{Spec}(A)$. The existence of a finite subcover guarantees that $\mathrm{Spec}(A)$ is finite, the desired contradiction.

(*c*) As a prelude to a later paper (*), we turn next to applica-

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tions of the inclusion-reversing aspect of the homeomorphisms noted prior to the statement of Theorem 2.19. For the remaining parts of this remark, we *fix notation* $X=\mathrm{Spec}(A)$, with B as in those inclusion-reversing homeomorphisms.

Our first application asserts that if X_F is a Noetherian space, then A satisfies the descending chain condition for prime ideals. For a *proof*, note that the hypothesis amounts to $(\mathrm{Spec}\,B)_Z$ being Noetherian, so that Theorem 2.19 assures that B satisfies the ascending chain condition for prime ideals, from which the desired assertion concerning A follows immediately.

(*d*) We next provide a companion for Remark 2.3 (*c*). If each prime ideal of A contains a unique minimal prime ideal, then the minimal prime spectrum Y of A is Hausdorff in the subspace topology inherited from X_F. For a *proof*, note that the hypothesis asserts that each prime ideal of B is contained in a unique maximal ideal of B, *i. e.* that B is a *pm*-ring in the sense of [7]. Thus, by [7], Theorem 1.2, the maximal spectrum of B is Hausdorff in the subspace topology inherited from $(\mathrm{Spec}\,B)_Z$. However, Y is homeomorphic with that maximal spectrum, completing the proof.

(*e*) A companion for Remark 2.3 (*d*) is given next. Let W be the maximal spectrum of A, with the subspace topology inherited from X_F. If A is a *pm*-ring and if the function $X_F \to W$ (sending each prime of A to the unique maximal containing it) is continuous, then W is compact. For a *proof*, note that W is a retract of X_F. Thus, the minimal prime spectrum U of B, with the subspace topology inherited from $(\mathrm{Spec}\,B)_Z$, is a retract of $(\mathrm{Spec}\,B)_Z$. Replacing B by its associated reduced ring, we infer from [15], Theorem 2 that B is a Baer ring. By [15], Theorem 1 (cf. also [27], Proposition 3.4), U is compact. As W and U are homeomorphic, the proof is complete.

(*f*) It is evident from Remarks 2.3 (*f*) and 2.6 that irreducible components and connected components behave quite differently in the passage from X_Z to X_F. In this vein, we next establish a bijection between the set of connected components of X_Z and the set of connected components of X_F.

For the *proof*, first note that the function $a \mapsto X_a$ sets up a bijection between the set of idempotents of A and the set of clopen subsets of X_Z (cf. [3], p. 131). However, it follows readily from Lemma 2.5 that X_Z and X_F have the same clopen sets. Thus, if one identifies X (as a set only) with $Y=\mathrm{Spec}(B)$, a bijection between

the set of idempotents of A and the set of idempotents of B results from the recipe: $a \mapsto b$ if $X_a = Y_b$. The key fact to which we now appeal is a result of Cox-Pendleton [5], Proposition 4.7: the set of connected components of X_Z (resp., of $X_F \cong Y_Z$) is in one-to-one correspondence with the collection of those sets of idempotents of A (resp., B) which are maximal with respect to generating a proper ideal of A (resp., B). To conclude the proof, it is enough to verify that if $\{a_i\}$ is a collection of idempotents of A, with $\{b_i\}$ the corresponding collection of idempotents of B, then $\Sigma A a_i = A$ if and only if $\Sigma B b_i = B$. This, in turn, follows since:

$$\Sigma Aa_i = A \Leftrightarrow \bigcup X_{a_i} = X \Leftrightarrow \bigcup Y_{b_i} = Y \Leftrightarrow \Sigma Bb_i = B.$$

As a corollary, we see that X_F has only finitely many connected components if A has either only finitely many minimal primes or only finitely many maximal ideals. Indeed, in the first case, it follows that X_Z has only finitely many irreducible components and, *a fortiori*, only finitely many connected components, so an application of the preceding result establishes this case. For the second case, one reasons similarly, after noticing that B has only finitely many minimal primes.

In [20], Proposition 3.10 (as translated via [22]), it is shown that if A is a strong G-domain (in other words, an open Prüfer domain), then X_Z is a Noetherian space if and only if $\mathrm{Spec}(A)$ is finite. Note in this case that X_F is automatically Noetherian, since open domains must be g-rings. These observations motivate the final result of this paper.

THEOREM 2.21. *For a ring A, with $X=\mathrm{Spec}(A)$, the following are equivalent*:

(1) *X_C is a discrete space;*

(2) *X is a finite set;*

(3) *Both X_F and X_Z are Noetherian spaces;*

(4) *A is a g-ring, and for each $P \in X$, there exists a finitely-generated ideal I of A such that P is minimal among primes of A containing I.*

PROOF. (1) $\Rightarrow$ (2): The underlying set of any discrete compact topological space must be finite.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (4): A is a g-ring since X_F is a Noetherian space. Moreover, for each $P \in X$, the assumption that X_Z is Noetherian provides a finitely-generated ideal J of A such that (the radical ideal) P is the radical of J. It follows that P is minimal among primes over J (cf. [14], Theorem 26).

(4) $\Rightarrow$ (1): It is enough to show that the singleton set $\{P\}$ is open in X_C for each $P \in X$. As A is a g-ring, $A_P \cong A_f$ for some $f \in A \setminus P$, so that $\{Q \in X \mid Q \subset P\} = X_f$. By hypothesis, P is minimal over a finitely generated ideal I of A. It is now straightforward to check that $\{P\} = X_f \cap V(I)$, a basic open set in X_C, to complete the proof.

To close the paper, we present a companion for Corollaries 2.11 and 2.13 and Theorem 2.16. The general question concerns introducing topologies on $X = \text{Spec}(A)$ by using a basis for *open* sets consisting of certain $im(f^*)$, as f ranges over a collection of ring-homomorphisms with domain A. Notice that if the maps f are not restricted in any additional way, one thereby obtains the discrete topology on X. One way to see this is to observe that each singleton subset of X is open in the generated topology, since such subsets are closed in X_C.

PROPOSITION 2.22. *Let A be a ring, with $X = \text{Spec}(A)$. Then there exists a topological structure* T *on X such that each of the following four collections forms a basis for the open sets in* (X, T):

(i) $\{V(P) \mid P \in X\}$;

(ii) $\{im(f^*) \mid f: A \to B$ *is a ring-homomorphism satisfying* $GU\}$;

(iii) $\{im(f^*) \mid f: A \to B$ *is a ring-homomorphism satisfying* $WGU\}$;

(iv) $\{im(f^*) \mid f: A \to B$ *is a surjective ring-homomorphism*$\}$.

PROOF. It is evident that the collection described in (i) is a basis for some topology, T, on X. To see that (iv) also describes a basis for T, rewrite (iv) as $\{V(I) \mid I$ is an ideal of $A\}$, and note that $V(I) = \cup \{V(P) \mid P \in V(I)\}$. Now, since the canonical projection $A \to A/I$ satisfies GU (and WGU) for each I, it only remains to show that $im(f^*) \in \text{T}$ whenever $f: A \to B$ satisfies $(W)\,GU$. This, however, follows immediately since $im(f^*)$ is stable under specialization.

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