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Star operations and pullbacks. (English. English summary)

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In this paper the authors investigate the behaviour of the star operations in a general pullback setting and with respect to surjective homomorphisms of integral domains. Thus consider the following situation (P): T represents a domain, M an ideal of T, k the factor ring T/M, $\varphi: T \to k$ the canonical projection, D a domain subring of k whose field of quotients L is contained in k, $R = \varphi^{-1}(D)$ the pullback of D, K the field of quotients of R, and $S = \varphi^{-1}(L)$.

Denote by $\mathrm{Star}(A)$ the set of all star operations on a domain A. The authors define in a natural way a map

$$\Phi: \mathrm{Star}(D) \times \mathrm{Star}(T) \to \mathrm{Star}(R),\ (\star_D, \star_T) \mapsto \diamond.$$

They also characterize the star operation $\diamond$ by introducing new techniques for projecting and lifting star operations under surjective homomorphisms of domains. Let A be a domain, and f be a surjective homomorphism of A to a domain B. Then they define in natural ways maps $(-1)^f: \mathrm{Star}(B) \to \mathrm{Star}(A)$, $\star \mapsto \star^f$ and $(-1)_f: \mathrm{Star}(A) \to \mathrm{Star}(B)$, $* \mapsto *_f$.

In the situation (P) the authors show the following: (1) $(-1)_\varphi: \mathrm{Star}(R) \to \mathrm{Star}(D)$ is order-preserving and surjective. (2) $(-1)^\varphi: \mathrm{Star}(D) \to \mathrm{Star}(R)$ is order-preserving and injective. (3) $\diamond = (\star_D)^\varphi \wedge (\star_T)^\iota$, where $(\star_T)^\iota$ is a semistar operation on R induced by $\star_T$ canonically. (4) If $D \subsetneq k$, then $* \le (*_\varphi)^\varphi$ for each $* \in \mathrm{Star}(R)$. (5) Assume that $M \ne (0)$ and $D \subsetneq k$. Then they show that $\diamond = (\diamond_\varphi)^\varphi \wedge (\diamond_\iota)^\iota$, and investigate the family $\{\star_T \in \mathrm{Star}(T) \mid \star_T \le (v_R)_\iota\}$.

Furthermore, the authors study the transfer in a pullback or with respect to a surjective homomorphism of some relevant properties or classes of star operations. For example, they show in the situation (P) that $(t_R)_\varphi = t_D$, $(w_R)_\varphi = w_D$ and $\tilde{*}_\varphi = \widetilde{(*_\varphi)}$ for each $* \in \mathrm{Star}(R)$. They apply part of their theory to give a complete positive answer to a problem posed by D. F. Anderson in 1992 concerning the star operations on the "$D+M$" construction. *Ryūki Matsuda* (Ibaraki)

[References]

1. T. Akiba, A note on AV-domains, Bull. Kyoto Univ. Ed. Ser. B 31 (1967) 1–3. MR0218339 (36 #1426)
2. D.D. Anderson, Star operations induced by overrings, Comm. Algebra 16 (1988) 2535–2553. MR0955324 (89f:13032)

3. D.D. Anderson, D.F. Anderson, Some remarks on star operations and the class group, J. Pure Appl. Algebra 51 (1988) 27–33. MR0941887 (89f:13024)
4. D.D. Anderson, D.F. Anderson, Examples of star operations, Comm. Algebra 18 (1990) 1621–1643. MR1059752 (91d:13001)
5. D.D. Anderson, D.F. Anderson, M. Zafrullah, Splitting the t-class group, J. Pure Appl. Algebra 74 (1991) 17–37. MR1129127 (93d:13023)
6. D.D. Anderson, S.J. Cook, Two star-operations and their induced lattices, Comm. Algebra 28 (2000) 2461–2475. MR1757473 (2001c:13033)
7. D.D. Anderson, E. Houston, M. Zafrullah, t-linked extensions, the t-class group and Nagata's Theorem, J. Pure Appl. Algebra 86 (1993) 109–124. MR1215640 (94e:13036)
8. D.F. Anderson, A general theory of class groups, Comm. Algebra 16 (1988) 805–847. MR0932636 (89f:13023)
9. D.F. Anderson, Star operations and the $D+M$ construction, Rend. Circ. Mat. Palermo (2) Suppl. 41 (1992) 221–230. MR1196616 (93k:13002)
10. D.F. Anderson, The class group and the local class group of an integral domain, in: S. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, in: Math. Appl., vol. 520, Kluwer Academic, Dordrecht, 2000, pp. 33–55. MR1858156 (2003b:13015)
11. D.F. Anderson, S. El Baghdadi, S.-E. Kabbaj, On the class group of $A+XB[X]$ domains, in: D. Dobbs, M. Fontana, S.-E. Kabbaj (Eds.), Advances in Commutative Ring Theory, in: Lecture Notes in Pure and Appl. Math., vol. 205, Dekker, New York, 1999, pp. 73–85. MR1767451 (2001e:13015)
12. E. Bastida, R. Gilmer, Overrings and divisorial ideals of the form $D+M$, Michigan Math. J. 20 (1973) 79–95. MR0323782 (48 #2138)
13. M. Boisen, P. Sheldon, CPI-extensions: overrings of integral domains with special prime spectrum, Canad. J. Math. 24 (1977) 722–737. MR0447205 (56 #5520)
14. N. Bourbaki, Algèbre Commutative, Hermann, Paris, 1961. MR0171800 (30 #2027)
15. J. Brewer, E. Rutter, $D+M$ constructions with general overrings, Michigan Math. J. 23 (1976) 33–41. MR0401744 (53 #5571)
16. D. Dobbs, Divided rings and going-down, Pacific J. Math. 67 (1976) 353–363. MR0424795 (54 #12753)
17. D. Dobbs, E. Houston, T. Lucas, M. Zafrullah, t-linked overrings and Prüfer v-multiplication domains, Comm. Algebra 17 (1989)

2835–2852. MR1025612 (90j:13016)
18. W. Fanggui, R.L. McCasland, On w-modules over strong Mori domains, Comm. Algebra 25 (1997) 1285–1306. MR1437672 (98g:13025)
19. M. Fontana, Topologically defined classes of commutative rings, Ann. Mat. Pura Appl. (4) 123 (1980) 331–355. MR0581935 (81j:13001)
20. M. Fontana, S. Gabelli, On the class group and the local class group of a pullback, J. Algebra 181 (1996) 803–835. MR1386580 (97h:13011)
21. M. Fontana, J. Huckaba, Localizing systems and semistar operations, in: S. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, Kluwer Academic, Dordrecht, 2000, pp. 169–187, Chapter 8. MR1858162 (2002k:13001)
22. M. Fontana, K.A. Loper, Kronecker function rings: a general approach, in: D.D. Anderson, I.J. Papick (Eds.), Ideal Theoretic Methods in Commutative Algebra, in: Lecture Notes in Pure and Appl. Math., vol. 220, Dekker, 2001, pp. 189–205. MR1836601 (2002h:13029)
23. M. Fontana, K.A. Loper, A Krull-type theorem for the semistar integral closure of an integral domain, Commutative Algebra, Arab. J. Sci. Eng. Sect. C Theme Issues 26 (2001) 89–95. MR1843459 (2002e:13019)
24. S. Gabelli, E. Houston, Coherentlike conditions in pullbacks, Michigan Math. J. 44 (1997) 99–123. MR1439671 (98d:13019)
25. S. Gabelli, E. Houston, Ideal theory in pullbacks, in: S. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, in: Math. Appl., vol. 520, Kluwer Academic, Dordrecht, 2000, pp. 199–227. MR1858163 (2003a:13001)
26. R. Gilmer, Multiplicative Ideal Theory, Dekker, New York, 1972. MR0427289 (55 #323)
27. B.V. Greenberg, Coherence in polynomial rings, J. Algebra 50 (1978) 12–25. MR0472794 (57 #12484)
28. F. Halter-Koch, Ideal Systems: An Introduction to Multiplicative Ideal Theory, Dekker, New York, 1998. MR1828371 (2001m:13005)
29. J.R. Hedstrom, E.G. Houston, Pseudo-valuation domains, Pacific J. Math. 75 (1978) 137–147. MR0485811 (58 #5615a)
30. J.R. Hedstrom, E.G. Houston, Some remarks on star-operations, J. Pure Appl. Algebra 18 (1980) 37–44. MR0578564 (81m:13008)
31. E.G. Houston, M. Zafrullah, Integral domains in which each t-ideal is divisorial, Michigan Math. J. 35 (2) (1988) 291–300.

MR0959276 (89i:13027)
32. P. Jaffard, Les Systèmes d'Idéaux, Dunod, Paris, 1960. MR0114810 (22 #5628)
33. B.G. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$, J. Algebra 123 (1989) 151–170. MR1000481 (90e:13017)
34. M.D. Larsen, P.J. McCarthy, Multiplicative Ideal Theory, Academic Press, New York, 1971. MR0414528 (54 #2629)
35. T. Lucas, Examples built with $D+M$, $A+XB[X]$ and other pullback constructions, in: S. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, in: Math. Appl., vol. 520, Kluwer Academic, Dordrecht, 2000, pp. 341–368. MR1858170 (2002g:13007)
36. R. Matsuda, I. Sato, Note on star operations and semistar operations, Bull. Fac. Sci. Ibaraki Univ. Ser. A 28 (1996) 5–22. MR1408283 (97f:13003)
37. R. Matsuda, T. Sugatani, Semistar operations on integral domains, II, Math. J. Toyama Univ. 18 (1995) 155–161. MR1369703 (97b:13002)
38. A. Mimouni, TW-domains and strong Mori domains, J. Pure Appl. Algebra 177 (2003) 79–93. MR1948840 (2003j:13032)
39. M. Nagata, Local Rings, Interscience, New York, 1972. MR0155856 (27 #5790)
40. A. Okabe, R. Matsuda, Star operations and generalized integral closures, Bull. Fac. Sci. Ibaraki Univ. Ser. A 24 (1992) 7–13. MR1177280 (93h:13006)
41. A. Okabe, R. Matsuda, Semistar operations on integral domains, Math. J. Toyama Univ. 17 (1994) 1–21. MR1311837 (95k:13027)
42. M.H. Park, Group rings and semigroup rings over strong Mori domains, J. Pure Appl. Algebra 163 (2001) 301–318. MR1852122 (2002g:13038)
43. A. Seidenberg, A note on dimension theory of rings, Pacific J. Math. 3 (1953) 505–512; Part II, Pacific J. Math. 4 (1954) 603–614. MR0054571 (14,941c)
44. C. Traverso, Seminormality and Picard group, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) 24 (1970) 585–595. MR0277542 (43 #3275)
45. W. Vasconcelos, The local rings of global dimension two, Proc. Amer. Math. Soc. 35 (1972) 381–386. MR0308115 (46 #7230)
46. M. Zafrullah, Various facets of rings between $D[X]$ and $K[X]$, Comm. Algebra 31 (5) (2003) 2497–2540. MR1976289 (2004d:13029)
47. O. Zariski, P. Samuel, Commutative Algebra, vol. II, Van Nostrand, Princeton, 1960. MR0120249 (22 #11006)