arXiv:0809.2947v1 [math.AC] 17 Sep 2008

ON v–DOMAINS AND STAR OPERATIONS

D.D. ANDERSON, DAVID F. ANDERSON,
MARCO FONTANA AND MUHAMMAD ZAFRULLAH

ABSTRACT. Let $*$ be a star operation on an integral domain D. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated fractional ideals of D. Call D a $*$–Prüfer (respectively, $(*, v)$–Prüfer) domain if $(FF^{-1})^* = D$ (respectively, $(F^vF^{-1})^* = D$) for all $F \in \boldsymbol{f}(D)$. We establish that $*$–Prüfer domains (and $(*, v)$–Prüfer domains) for various star operations $*$ span a major portion of the known generalizations of Prüfer domains inside the class of v–domains. We also use Theorem 6.6 of the Larsen and McCarthy book [Multiplicative Theory of Ideals, Academic Press, New York–London, 1971], which gives several equivalent conditions for an integral domain to be a Prüfer domain, as a model, and we show which statements of that theorem on Prüfer domains can be generalized in a natural way and proved for $*$–Prüfer domains, and which cannot be. We also show that in a $*$–Prüfer domain, each pair of $*$-invertible $*$-ideals admits a GCD in the set of $*$-invertible $*$-ideals, obtaining a remarkable generalization of a property holding for the "classical" class of Prüfer v–multiplication domains. We also link D being $*$–Prüfer (or $(*, v)$–Prüfer) with the group $\mathrm{Inv}^*(D)$ of $*$-invertible $*$-ideals (under $*$-multiplication) being lattice-ordered.

The so called v–domains (i.e., the integral domains such that every nonzero finitely generated fractional ideal is v–invertible) include several distinguished classes of Prüfer-like domains, but not much seems to be known about them. (For a brief history of v-domains and an annotated list of references on this important class of domains, see [45]). The aim of this article is to prove new properties of v–domains in their most general form, using star operations, and to give a unifying pattern in this body of results. As a consequence, after specializing the star operation to some relevant cases, we also obtain several properties already known for various classes of Prüfer-like domains, providing a clear indication how these properties and classes of domains are related to one another.

Let D be an integral domain with quotient field K, and let $\boldsymbol{F}(D)$ denote the set of nonzero fractional ideals of D. Also, let $\boldsymbol{f}(D) := \{A \in \boldsymbol{F}(D) \mid A$ is finitely

Date: September 17, 2008.

2000 *Mathematics Subject Classification.* 13A15, 13F05, 13G05.

Key words and phrases. Prüfer domain, GCD domain, star operation, v–domain, $*$–Prüfer domain, completely integrally closed domain, $*$–invertible ideal.

Acknowledgments. We would like to thank Franz Halter-Koch for several helpful comments on a previous version of the present paper and for suggesting lines for further investigation and extension to the general setting of ideal systems on monoids.

During the preparation of this paper, the third named author was partially supported by a research grant PRIN-MiUR.

generated} and $\boldsymbol{F}^v(D) := \{A \in \boldsymbol{F}(D) \mid A = A^v\}$ the set of fractional divisorial ideals, where $A^v := \left(A^{-1}\right)^{-1}$. Let $*$ be a star operation on D. (For a review of star operations, the reader may consult Gilmer [21, Sections 32 and 34] or Halter-Koch [23] for a general approach in the language of ideal systems on monoids.) Call $A \in \boldsymbol{F}(D)$ $*$*-invertible* if $(AA^{-1})^* = D$. We say that D is a *completely integrally closed domain* (for short, CICD) if $D = \widetilde{D} := \{x \in K \mid$ there exists $0 \neq d \in D$ such that $dx^n \in D$ for all integers $n \geq 1\}$. It is well known that D is a completely integrally closed domain if and only if $D = (AA^{-1})^v$ $(= (A^vA^{-1})^v)$ for all $A \in \boldsymbol{F}(D)$ [21, Theorem 34.3]; in particular, a CICD is a v–domain.

These days it is customary to take a concept defined or characterized using the standard v-operation and to ask for domains that are characterized or defined by replacing the v-operation by a general star operation. It appears that in the case of CICD's there are at least two star operation analogues. Let $*$ be a general star operation on D. Call D a $*$*–completely integrally closed domain* (for short, $*$–CICD) if $(AA^{-1})^* = D$ for all $A \in \boldsymbol{F}(D)$, and call D a $(*, v)$*–completely integrally closed domain* (for short, $(*, v)$–CICD) if $(A^vA^{-1})^* = D$ for all $A \in \boldsymbol{F}(D)$. Clearly a $*$–CICD, or a $(*, v)$–CICD, is a v–domain, since $(A^*)^v = A^v = (A^v)^*$ for all $A \in \boldsymbol{F}(D)$. Moreover, a $*$–CICD is a $(*, v)$–CICD, but not conversely.

Let $*_1, *_2$ be two star operations on D. Recall that $*_1 \leq *_2$ if $A^{*_1} \subseteq A^{*_2}$ for all $A \in \boldsymbol{F}(D)$. For instance, we have $w \leq t \leq v$, where the t*–operation* (respectively, w*–operation*) is defined by setting $A^t := \bigcup\{F^v \mid F \subseteq A,\ F \in \boldsymbol{f}(D)\}$ (respectively, $A^w := \bigcap\{AD_P \mid P$ is a maximal t–ideal of $D\}$) for all $A \in \boldsymbol{F}(D)$; cf. for instance [21, Theorem 34.1], [40], [5, Section 2], and [15, Sections 3 and 4].

Clearly, if $*_1, *_2$ are two star operations on D with $*_1 \leq *_2$, then a $*_1$–CICD (respectively, $(*_1, v)$–CICD) D is a $*_2$–CICD (respectively, $(*_2, v)$–CICD). Recall that $* \leq v$ for every star operation $*$ on D [21, Theorem 34.1], and hence $(AA^{-1})^* = D$ implies $(AA^{-1})^v = D$ for $A \in \boldsymbol{F}(D)$, i.e., a $*$-invertible ideal is v-invertible. Therefore a $(*, v)$–CICD (and, in particular, a $*$–CICD) is a v–CICD (= CICD). However, since $A \in \boldsymbol{F}(D)$ being $*$-invertible implies $A^* = A^v$ [44, page 433], a distinction between $*$–CICD's and $(*, v)$–CICD's appears highly unlikely. But, as we shall see, there is a marked distinction between them in several cases.

In a preliminary part of this paper, we discuss the motivations and the advantages for studying star operation analogues of CICD's, we give some characterizations of $(*, v)$–CICD's, we give interpretations of $(*, v)$–CICD's for different star operations $*$, we compare them with $*$–CICD's, and we review results known for both.

Having dealt with this topic of immediate interest in Section 1, we investigate in Section 2 the main theme of this paper studying a "star operation version" of v–domains. We call D a $*$*–Prüfer domain* if every nonzero finitely generated ideal of D is $*$-invertible (i.e., $(FF^{-1})^* = D$ for all $F \in \boldsymbol{f}(D)$), and we call D a $(*, v)$*–Prüfer domain* if F^v is $*$-invertible (i.e., $(F^vF^{-1})^* = D$) for all $F \in \boldsymbol{f}(D)$. Clearly, if $*_1, *_2$ are two star operations on D with $*_1 \leq *_2$, then a $*_1$–Prüfer domain (respectively, $(*_1, v)$–Prüfer domain) D is a $*_2$–Prüfer domain (respectively, $(*_2, v)$–Prüfer domain). Clearly a $*$–Prüfer domain, or a $(*, v)$–Prüfer domain, is a v–domain. Moreover, a $*$–Prüfer domain is a $(*, v)$–Prüfer domain, but not conversely.

These domains have been partially studied in [7] as special cases of rather general results [7, Theorem 4.1 and Corollary 4.3]. Since the proofs provided in [7] were sort of dismissive, we provide here direct proofs of some of the relevant results stated in [7] and we prove some more. We establish in this section that $*$–Prüfer domains (and $(*, v)$–Prüfer domains) for various star operations $*$ span a major portion of the known generalizations of Prüfer domains inside the class of v–domains. For example, for $* = d$ (i.e., the identity star operation), we get a d–Prüfer domain which is precisely a Prüfer domain; for $* = t$, we get a t–Prüfer domain which is precisely a Prüfer v–multiplication domain (or a PvMD); and of course for $* = v$, we get the usual v–domain. In this section, we also use Theorem 6.6 of Larsen and McCarthy [30], which gives several equivalent conditions for an integral domain to be a Prüfer domain, as a model, and we show which statements of that theorem on Prüfer domains can be generalized in a natural way and proved for $*$–Prüfer domains and which cannot be. In particular, we show that D is a $*$–Prüfer domain if and only if $((A \cap B)(A + B))^* = (AB)^*$ for all $A, B \in \boldsymbol{F}(D)$. This type of result is known for Prüfer domains [21, Theorem 25.2] and for PvMD's [22, Theorem 5], but is definitely not known for v-domains.

The last part of the paper deals with a general form of GCD for $*$–Prüfer domains (in particular, for v–domains) and connections with lattice-ordered abelian groups. The key fact is that an integral domain D is a $*$–Prüfer domain if and only if $A+B$ is $*$-invertible for all $*$-invertible $A, B \in \boldsymbol{F}(D)$. Recall that D is a *GCD domain* if for all $x, y \in D^\times := D\backslash\{0\}$, we have $\mathrm{GCD}(x, y) \in D$. Now a Bézout domain D (e.g., a PID) is slightly more than a GCD domain in that for all nonzero ideals aD and bD, we have a unique ideal dD with $aD + bD = dD$, where d is a GCD of a and b. Moreover, note that aD and bD are invertible ideals and that in a Prüfer domain nonzero finitely generated ideals are invertible. If we regard, for every pair of integral invertible ideals A and B of a Prüfer domain, the invertible ideal $C := A + B$ as the GCD of A and B, then we find that $A, B \subseteq C$. Hence, $A_1 := AC^{-1} \subseteq D$ and $B_1 := BC^{-1} \subseteq D$, and so $A = A_1C$ and $B = B_1C$, where $A_1 + B_1 = D$. Thus, in a Prüfer domain, each pair of integral invertible ideals has a GCD of sorts. In [12, Section 1], the above observations were used to show that in t–Prüfer domains (= Prüfer v–multiplication domains), each pair of integral t-invertible t-ideals has a GCD of sorts, generalizing to this setting some aspects of the GCD theory of Bézout domains. In the general context of $*$–Prüfer domains, we show that each pair of integral $*$-invertible $*$-ideals has a GCD of sorts. This result is a slightly bigger jump than the t–Prüfer domain case in that, in a $*$–Prüfer domain, a $*$-invertible $*$-ideal may not be of finite type.

1. Star Completely Integrally Closed Domains

Recall that a ring R is a *multiplication ring* if for all ideals A and B of R with $A \subseteq B$, there exists an ideal C of R such that $A = BC$ (cf. for instance, [30, Definition 9.12, page 209]). Clearly, a Dedekind domain is a multiplication ring, and more precisely, for an integral domain the notions of Dedekind domain and multiplication ring coincide [30, Theorem 9.13].

Given a star operation $*$ on an integral domain D, it is natural to call D a $*$–*multiplication domain* (respectively, $(*, v)$–*multiplication domain*) if for all $A, B \in \boldsymbol{F}(D)$ with $A^* \subseteq B^*$ (respectively, $A^* \subseteq B^v$), there exists $C \in \boldsymbol{F}(D)$ such that $A^* = (BC)^*$ (respectively, $A^* = (B^vC)^*$). Note that star multiplication domains, and in particular, divisorial multiplication domains were recently investigated in relation to Gabriel topologies by J. Escoriza and B. Torrecillas [13].

As usual, for all $A, B \in \boldsymbol{F}(D)$, we denote the fractional ideal $(A :_K B) := \{x \in K \mid xB \subseteq A\}$ by $(A : B)$ and the ideal $(A : B) \cap D$ by $(A :_D B)$.

Proposition 1.1. *Let $*$ be a star operation on an integral domain D. Then*

(a) *D is a $*$–CICD if and only if D is a $*$–multiplication domain, and*
(b) *D is a $(*, v)$–CICD if and only if D is a $(*, v)$–multiplication domain.*

Proof. (a) If D is a $*$–CICD and $A^* \subseteq B^*$, then $C := B^{-1}A \in \boldsymbol{F}(D)$ satisfies $(BC)^* = (BB^{-1}A)^* = ((BB^{-1})^*A)^* = (DA)^* = A^*$.

Conversely, for each $A \in \boldsymbol{F}(D)$, let $0 \neq a \in A$, and so $aD \subseteq A^*$. By assumption, there exists $C \in \boldsymbol{F}(D)$ such that $(AC)^* = aD$, i.e., $(Aa^{-1}C)^* = D$. Note that $B := a^{-1}C \in \boldsymbol{F}(D)$ and $a^{-1}C \subseteq (D : A)$. Therefore we conclude that $D = (Aa^{-1}C)^* \subseteq (AA^{-1})^* \subseteq D$, i.e., $(AA^{-1})^* = D$.

(b) If D is a $(*, v)$–CICD and $A^* \subseteq B^v$, then $C := B^{-1}A \in \boldsymbol{F}(D)$ satisfies $(B^vC)^* = (B^vB^{-1}A)^* = ((B^vB^{-1})^*A)^* = (DA)^* = A^*$.

Conversely, for each $A \in \boldsymbol{F}(D)$, let $0 \neq a \in A$, and so $aD \subseteq A^v$. By assumption, there exists $C \in \boldsymbol{F}(D)$ such that $(A^vC)^* = aD$, i.e., $(A^va^{-1}C)^* = D$. Note that $B := a^{-1}C \in \boldsymbol{F}(D)$ and $a^{-1}C \subseteq (D : A^v) = (D : A)$. Therefore we conclude that $D = (A^va^{-1}C)^* \subseteq (A^vA^{-1})^* \subseteq D$, i.e., $(A^vA^{-1})^* = D$. □

Proposition 1.2. *Let $*$ be a star operation on an integral domain D. Then D is a $(*, v)$–CICD if and only if $(AB)^{-1} = (A^{-1}B^{-1})^*$ for all $A, B \in \boldsymbol{F}(D)$.*

Proof. Suppose that D is a $(*, v)$–CICD and consider $A, B \in \boldsymbol{F}(D)$. Then $D = ((AB)^v(AB)^{-1})^*$. Multiplying both sides of the above equation by $A^{-1}B^{-1}$ and applying $*$, we get:

$$\begin{aligned}(A^{-1}B^{-1})^* &= (A^{-1}B^{-1}(AB)^v(AB)^{-1})^* \supseteq (A^{-1}B^{-1}A^vB^v(AB)^{-1})^* \\ &= ((A^{-1}A^v)(B^{-1}B^v)(AB)^{-1})^* \supseteq (A^{-1}A^v)^*(B^{-1}B^v)^*((AB)^{-1})^* \\ &= ((AB)^{-1})^* = (AB)^{-1}\,.\end{aligned}$$

For the reverse inclusion, note that $A^{-1}B^{-1} \subseteq (AB)^{-1}$, and so $(A^{-1}B^{-1})^* \subseteq ((AB)^{-1})^* = (AB)^{-1}$.

Conversely, if $(AB)^{-1} = (A^{-1}B^{-1})^*$ for all $A, B \in \boldsymbol{F}(D)$, then in particular $(A^{-1}A^v)^{-1} = (A^vA^{-1})^*$ for all $A \in \boldsymbol{F}(D)$. Now, as $A^vA^{-1} = A^{-1}A^v \subseteq D$, we have $D \subseteq (A^{-1}A^v)^{-1} = (A^vA^{-1})^* \subseteq D$. Thus $(A^vA^{-1})^* = D$ for all $A \in \boldsymbol{F}(D)$; so D is a $(*, v)$–CICD. □

Proposition 1.3. *Let $*$ be a star operation on an integral domain D. Then*

(a) *If D is a $*$–CICD, then D is a $(*,v)$–CICD.*
(b) *If D is a $(*,v)$–CICD, then D is a completely integrally closed domain.*

Proof. From the definition and from the fact that $(A^v)^{-1} = A^{-1}$ for all $A \in \boldsymbol{F}(D)$, it follows immediately that a $*$–CICD is a $(*,v)$–CICD. Furthermore, if we have $(A^vA^{-1})^* = D$ for all $A \in \boldsymbol{F}(D)$, then $(A^vA^{-1})^v = D$, and so $(AA^{-1})^v = (A^vA^{-1})^v = D$, i.e., D is a completely integrally closed domain. □

These results are simple and straightforward, but their value is in the interpretation of the $(*,v)$–CICD for different star operations $*$. We shall give examples of $(*,v)$–CICD's that are not $*$–CICD's for the same $*$. Most of our examples come from [7], which provides a lot of quotient-based characterizations of $*$–CICD's and of $(*,v)$–CICD's. Since the method of proof in [7] was somewhat involved, we include direct proofs of these characterizations here.

Proposition 1.4. [7, Corollary 3.4] *Let $*$ be a star operation on an integral domain D. Then the following conditions are equivalent.*

(i) $(A : B)^* = (AB^{-1})^*$ *for all* $A, B \in \boldsymbol{F}(D)$.
(ii) $(A : B^{-1})^* = (AB)^*$ *for all* $A, B \in \boldsymbol{F}(D)$.
(iii) $(A^* : B) = (AB^{-1})^*$ *for all* $A, B \in \boldsymbol{F}(D)$.
(iv) $(A^* : B^{-1}) = (AB)^*$ *for all* $A, B \in \boldsymbol{F}(D)$.
(v) *D is a $*$–CICD.*
(vi) *D is a CICD and $A^* = A^v$ for all $A \in \boldsymbol{F}(D)$.*
(vii) $(A^v : B^{-1}) = (A^vB)^*$ *for all* $A, B \in \boldsymbol{F}(D)$.

Proof. Let us note that (v)⇔(vi) is well known and it is the only part of the proof directly given in [7, Corollary 3.4 and Proposition 3.2]. For the rest, we use the following plan: (i)⇒(iii)⇒(v)⇒(i) and (ii)⇒(iv)⇒(vii)⇒(v)⇒(ii).

(i)⇒(iii). Replace A by A^* in (i) to get $(A^* : B)^* = (A^*B^{-1})^*$, and note that $(A^* : B)$ is a $*$-ideal and that $(A^*B^{-1})^* = (AB^{-1})^*$. So $(A^* : B) = (AB^{-1})^*$ for all $A, B \in \boldsymbol{F}(D)$.

(iii)⇒(v). Set $B = A^*$ in (iii) to get $(A^* : A^*) = (A(A^*)^{-1})^* = (AA^{-1})^*$. Noting that $D \subseteq (A^* : A^*) = (AA^{-1})^* \subseteq D$, we have $(AA^{-1})^* = D$ for all $A \in \boldsymbol{F}(D)$.

(v)⇒(i). Note that $AB^{-1} \subseteq (A : B)$; so $(AB^{-1})^* \subseteq (A : B)^*$. For the reverse inclusion, let $x \in (A : B)^*$. Then $xB \subseteq (A : B)^*B$, and so $xB^* \subseteq ((A : B)^*B)^* = ((A : B)B)^* \subseteq A^*$. This gives $xB^* \subseteq A^*$. Multiplying by B^{-1} on both sides and applying $*$, we have $(xB^*B^{-1})^* \subseteq (A^*B^{-1})^* = (AB^{-1})^*$. Invoking (v), we get $x \in (AB^{-1})^*$.

(ii)⇒(iv). Same as (i)⇒(iii).

(iv)⇒(vii). Replace A by A^v in (iv) to get $((A^v)^* : B^{-1}) = (A^vB)^*$, and note that $(A^v)^* = A^v$.

(vii)⇒(v). Set $A = B^{-1}$ in (vii) to get $(B^{-1} : B^{-1}) = (B^{-1}B)^*$, and proceed as in the proof of (iii)⇒(v) in order to get $(B^{-1}B)^* = D$.

(v)⇒(ii). The proof is more or less similar to the proof of (v)⇒(i). More precisely, $AB \subseteq (A : B^{-1})$, and so $(AB)^* \subseteq (A : B^{-1})^*$. Conversely, let $x \in (A : B^{-1})^*$. Then $xB^{-1} \subseteq (A : B^{-1})^*B^{-1}$, and so $x(B^{-1})^* \subseteq ((A : B^{-1})^*B^{-1})^* =$

$((A : B^{-1})B^{-1})^* \subseteq A^*$. This gives $x(B^{-1})^* \subseteq A^*$. Multiplying by B on both sides and applying $*$, we have $(x(B^{-1})^*B)^* \subseteq (A^*B)^* = (AB)^*$. Invoking (v), we get $x \in (AB)^*$. □

For the $(*, v)$–CICD case, we have the following set of quotient-based characterizations.

Proposition 1.5. [7, Corollary 3.5] *Let $*$ be a star operation on an integral domain D. Then the following conditions are equivalent.*

(i) *D is a $(*, v)$–CICD.*
(ii) $(A^v : B) = (A^vB^{-1})^*$ *for all* $A, B \in \boldsymbol{F}(D)$.
(iii) $(A^v : B^{-1}) = (A^vB^v)^*$ *for all* $A, B \in \boldsymbol{F}(D)$.
(iv) $(A : B)^v = (A^vB^{-1})^*$ *for all* $A, B \in \boldsymbol{F}(D)$.
(v) $(A : B^{-1})^* = (AB^v)^*$ *for all* $A, B \in \boldsymbol{F}(D)$.
(vi) $(A^* : B^{-1}) = (AB^v)^*$ *for all* $A, B \in \boldsymbol{F}(D)$.

Proof. (i)⇒(ii). Obviously $A^vB^{-1} \subseteq (A^v : B)$, and so $(A^vB^{-1})^* \subseteq (A^v : B)^* = (A^v : B)$. For the reverse inclusion, let $x \in (A^v : B)$. Then $xB \subseteq A^v$, and so $xB^v \subseteq A^v$. Multiplying the last equation by B^{-1} and applying $*$, we get $(xB^vB^{-1})^* \subseteq (A^vB^{-1})^*$. Invoking (i), we have $x \in (A^vB^{-1})^*$, and from this follows $(A^v : B) \subseteq (A^vB^{-1})^*$.

(ii)⇒(iii). Replace B by B^{-1} in $(A^v : B) = (A^vB^{-1})^*$ for all $A, B \in \boldsymbol{F}(D)$.

(iii)⇒(i). Set $B = A^{-1}$ in the equality $(A^v : B^{-1}) = (A^vB^v)^*$ to get $(A^v : A^v) = (A^vA^{-1})^*$. But since $D \subseteq (A^v : A^v) = (A^vA^{-1})^* \subseteq D$, we conclude that $(A^vA^{-1})^* = D$ for all $A \in \boldsymbol{F}(D)$, and so D is a $(*, v)$–CICD.

(i)⇒(iv). Note that $AB^{-1} \subseteq (A : B)$. So $(AB^{-1})^v \subseteq (A : B)^v$. This gives $A^vB^{-1} \subseteq (AB^{-1})^v \subseteq (A : B)^v$, and from this we conclude that $(A^vB^{-1})^* \subseteq (A : B)^v$. For the reverse inclusion, let $x \in (A : B)^v$. Then $xB \subseteq (A : B)^vB$, and so $xB^v \subseteq ((A : B)^vB)^v = ((A : B)B)^v \subseteq A^v$. This gives $xB^v \subseteq A^v$, and on multiplying by B^{-1} on both sides, we get $xB^vB^{-1} \subseteq A^vB^{-1}$. Applying $*$ on both sides and invoking (i), we conclude that $x \in (A^vB^{-1})^*$. This establishes the reverse inclusion.

(iv)⇒(i). Set $B = A$ in $(A : B)^v = (A^vB^{-1})^*$ to get $(A : A)^v = (A^vA^{-1})^*$. But then $D \subseteq (A : A)^v = (A^vA^{-1})^* \subseteq D$ for all $A \in \boldsymbol{F}(D)$. That is, $(A^vA^{-1})^* = D$ for all $A \in \boldsymbol{F}(D)$, and this is (i).

(v)⇒(vi). This is obvious once we replace A by A^* and note that $(A^* : B^{-1})^* = (A^* : B^{-1})$.

(vi)⇒(i). Set $A = B^{-1}$ in (vi) to get $(B^{-1} : B^{-1}) = (B^{-1}B^v)^*$, which can be used to conclude that $(B^{-1}B^v)^* = D$ for all $B \in \boldsymbol{F}(D)$.

(i)⇒(v). Clearly $AB^v \subseteq (A : B^{-1})$, and so $(AB^v)^* \subseteq (A : B^{-1})^*$. For the reverse inclusion, let $x \in (A : B^{-1})^*$. Then $xB^{-1} \subseteq (A : B^{-1})^*B^{-1}$. So $xB^{-1} \subseteq ((A : B^{-1})^*B^{-1})^* = ((A : B^{-1})B^{-1})^* \subseteq A^*$. Multiplying both sides of $xB^{-1} \subseteq A^*$ by B^v, we have $xB^{-1}B^v \subseteq A^*B^v$. Applying $*$ and invoking (i), we conclude that $x \in (A^*B^v)^* = (AB^v)^*$. □

Remark 1.6. It is easy to verify that statement (ii) of Proposition 1.5 can be equivalently stated as in [7, Corollary 3.5 (2)]:

(ii′) $(A^v : B^v) = (A^v B^{-1})^*$ *for all* $A, B \in \boldsymbol{F}(D)$.

The next result provides a useful characterization of $*$-invertible fractional ideals and sheds new light on Proposition 1.4.

Proposition 1.7. *Let* $*$ *be a star operation on an integral domain* D*, and let* $H \in \boldsymbol{F}(D)$*. Then* H *is* $*$*-invertible if and only if* $(A : H)^* = (A^* : H)^* = (AH^{-1})^*$ *for all* $A \in \boldsymbol{F}(D)$.

Proof. Note that, in general, we have $(A : H)^* \subseteq (A^* : H)^* = (A^* : H)$ for all $A, H \in \boldsymbol{F}(D)$ [21, page 406, Exercise 1].

Assume that H is $*$-invertible, and let $x \in (A^* : H)$. Therefore $xH \subseteq A^*$. Multiplying both sides by H^{-1} and applying $*$, we get $x \in (A^*H^{-1})^* = (AH^{-1})^*$. This gives $(A^* : H) \subseteq (AH^{-1})^*$. Next, let $y \in AH^{-1}$. Multiplying both sides by H, we get $yH \subseteq AH^{-1}H \subseteq A$, and thus $y \in (A : H)$. So $AH^{-1} \subseteq (A : H)$, and consequently $(AH^{-1})^* \subseteq (A : H)^*$. Putting it all together, we get $(A : H)^* \subseteq (A^* : H) \subseteq (AH^{-1})^* \subseteq (A : H)^*$, which establishes the equalities.

Conversely, assume that $(A : H)^* = (AH^{-1})^*$ for all $A \in \boldsymbol{F}(D)$. In particular, for $A = H$, we have $D \subseteq (H : H)^* = (HH^{-1})^* \subseteq D$, and so H is $*$-invertible. □

Remark 1.8. Note that Proposition 1.7 can be also deduced from [23, Corollary 12.1]. We thank Halter-Koch for pointing out this fact and for informing us that, using the ideal systems approach on commutative monoids, he has proved a general result on invertibility [25] that implies the previous Propositions 1.1, 1.4, 1.5, and 1.7.

We next give some examples of $*$–CICD's.

Example 1.9. *Let* $*$ *be a star operation on an integral domain* D.

Case: $* = v$.

The following properties are equivalent.

(i) *D is a v–CICD.*
(ii) *D is a (v, v)–CICD.*
(iii) *D is a CICD.*
(iv) $(AB)^{-1} = (A^{-1}B^{-1})^v$ *for all* $A, B \in \boldsymbol{F}(D)$.
(v) *D is a v–multiplication domain.*

The previous statement is an immediate consequence of Propositions 1.1, 1.2, 1.3, and the fact that, from the definition of a $*$–CICD, the notions of v–CICD and CICD coincide. Note that the equivalence (iii)⇔(v) gives back [13, Theorem 3.7 ((1)⇔(2))].

The case of a star operation of finite character is particularly interesting. Let $*$ be a star operation on an integral domain D. The operation defined by $A^{*_f} := \bigcup\{F^* \mid F \subseteq A\,,\ F \in \boldsymbol{f}(D)\}$ for all $A \in \boldsymbol{F}(D)$ is a star operation on D, called the *star operation of finite character associated to* $*$. When $* = *_f$, $*$ is called a *star* *operation of finite character.* As usual, we denote by t the star operation of finite character associated to the v–operation, i.e., $t := v_f$. We have $*_f \leq *$ for each star operation $*$, and hence, as we have already observed in the introduction, a $*_f$–CICD is a $*$–CICD. Note that a $*_f$–CICD is a special case of a $*_f$–Prüfer domain. It is obvious from the definitions that the notion of $*_f$–Prüfer domain coincides with that of *Prüfer $*$–multiplication domain* (for short, P$*$MD), i.e., an integral domain such that $(FF^{-1})^{*_f} = D$ for all $F \in \boldsymbol{f}(D)$ [27], [14], and [24].

In order to give better interpretations of $*_f$–CICD's and $(*_f, v)$–CICD's, we start by recalling that an integral domain D is a *Dedekind domain* (respectively, *Krull domain*) if and only if every $A \in \boldsymbol{F}(D)$ is invertible (respectively, t-invertible) (see e.g. [21, Theorem 37.1] and [28, Theorem 3.2]). Let d be the identity star operation. Since $d \leq *$ (respectively, $*_f \leq t$ [21, Theorem 34.1 (4)]) for all star operations $*$ on D, if $AA^{-1} = D$ (respectively, $(AA^{-1})^{*_f} = D$), then also $(AA^{-1})^* = D$ (respectively, $(AA^{-1})^t = D$). Therefore a Dedekind domain is a $*$–CICD for all star operations $*$ on D and a $*_f$–CICD is not just a CICD, but more precisely, it is a Krull domain such that $A^{*_f} = A^t$ $(= A^v)$ for all $A \in \boldsymbol{F}(D)$ (Proposition 1.4 ((v)$\Rightarrow$(vi))).

The previous remarks provide a motivation for the following terminology. Let us call a Krull domain such that $A^{*_f} = A^t$ for all $A \in \boldsymbol{F}(D)$ a *$*$–Dedekind domain.* Clearly, a $*_f$–CICD coincides with a $*$–Dedekind domain (which is identical by definition to a $*_f$–Dedekind domain), a v–Dedekind domain is just a Krull domain, and a $*$–Dedekind domain is a particular P$*$MD. Next, call a $(*_f, v)$–CICD a *$(*, v)$–Dedekind domain*; in other words, a *$(*, v)$–Dedekind domain* is an integral domain D such that A^v is $*_f$-invertible for all $A \in \boldsymbol{F}(D)$. Obviously the notions of $(*, v)$–Dedekind domain and $(*_f, v)$–Dedekind coincide.

Example 1.10. *Let $*$ be a star operation on an integral domain D.*

Case: $* = d$ (where d is the identity star operation).

The following properties are equivalent.

(i) *D is a d–Dedekind domain (= d–CICD).*
(ii) *D is a Dedekind domain.*
(iii) *$\boldsymbol{F}^v(D) = \boldsymbol{F}(D)$ and $(AB)^{-1} = A^{-1}B^{-1}$ for all $A, B \in \boldsymbol{F}(D)$.*

As a matter of fact, a Dedekind domain is an integral domain such that every nonzero fractional ideal is invertible (cf. for instance [21, Theorem 37.1]). The equivalence of (ii) and (iii) is in [42, Corollary 1.3] or [6, Theorem 2.8]. Moreover, from Proposition 1.2 we have that *the following properties are equivalent.*

(j) *D is a (d, v)–Dedekind domain (= (d, v)–CICD).*
(jj) *D is a pseudo-Dedekind domain (i.e., A^v is invertible for all $A \in \boldsymbol{F}(D)$).*
(jjj) *$(AB)^{-1} = A^{-1}B^{-1}$ for all $A, B \in \boldsymbol{F}(D)$.*

Note that (d, v)–Dedekind domains were studied under the name of G(eneralized)-Dedekind domains by Zafrullah in 1986 [42, Theorem 1.1 and Lemma 1.2] and by D.D. Anderson and Kang [6] in 1989 under the name of pseudo-Dedekind domains used above. These domains include locally factorial Krull domains (e.g., UFD's), rank-one valuation domains with complete value group, the ring of entire functions, and domains whose groups of divisibility are complete lattice-ordered groups (cf. [42, Theorem 1.10, Example 2.1, Theorem 2.6] and [6, Theorem 2.8]). If D is a (d,v)–Dedekind domain, then $\boldsymbol{F}^v(D)$ coincides with the group $\mathrm{Inv}(D)$ of invertible ideals of D (cf. also Corollary 2.15 (c)); in the special case where A^v is principal for all $A \in \boldsymbol{F}(D)$, the set of nonzero fractional v-ideals $\boldsymbol{F}^v(D)$ forms a group which is isomorphic to the group of divisibility of D (Corollary 2.16).

Example 1.11. *Let $*$ be a star operation on an integral domain D.*

Case: $* = t$ or $* = w$.

The following properties are equivalent.

(i) *D is a t–Dedekind domain (= t–CICD).*
(ii) *D is a w–Dedekind domain (= w–CICD).*
(iii) *D is a Krull domain.*
(iv) $(AB^{-1})^{-1} = (A^{-1}B)^t$ *for all* $A, B \in \boldsymbol{F}(D)$.
(v) $(AB)^{-1} = (A^{-1}B^{-1})^t$ *and* $A^t = A^v$ *for all* $A, B \in \boldsymbol{F}(D)$.

The following properties are equivalent.

(j) *D is a (t,v)–Dedekind domain (= (t,v)–CICD).*
(jj) *D is a (t,w)–Dedekind domain (= (w,v)–CICD).*
(jjj) *D is a pre-Krull domain in the sense of* [43, Proposition 4.1] *(i.e., A^v is t-invertible for all $A \in \boldsymbol{F}(D)$).*
(jv) $(AB)^{-1} = (A^{-1}B^{-1})^w = (A^{-1}B^{-1})^t$ *for all* $A, B \in \boldsymbol{F}(D)$.

The statements (i)–(iii) and the statements (j)–(jv) are equivalent by Proposition 1.2 and from the fact that a t-invertible ideal is the same as a w-invertible ideal [5, Theorem 2.18]. Thus the $* = w$ case coincides with the $* = t$ case.

(i)⇔(iv) holds since (iv) is equivalent to condition (vii) of Proposition 1.4 when $* = t$.

The fact that (i) implies (v) follows from Propositions 1.2, 1.3, and 1.4 ((v)⇒(vi)). Conversely, in (v) take $B := A^{-1}$; then $D \subseteq (AA^{-1})^{-1} = (A^{-1}A^v)^t = (A^{-1}A^t)^t = (A^{-1}A)^t \subseteq D$, and hence A is t–invertible for all $A \in \boldsymbol{F}(D)$.

A (t,v)–Dedekind domain D is a particular Prüfer v–multiplication domain (for short, PvMD) or, equivalently, a t–Prüfer domain since $D = (F^vF^{-1})^t = (F^tF^{-1})^t = (FF^{-1})^t$ for all $F \in \boldsymbol{f}(D)$. Therefore (t,v)–Dedekind domains form a class of completely integrally closed PvMD's that contains the Krull domains (and, *a fortiori*, all the d–CICD's and the (d,v)–CICD's). Furthermore, we will show (Corollary 2.15) that for a (t,v)–Dedekind domain, the set $\boldsymbol{F}^v(D)$ of nonzero fractional v-ideals of D is a complete lattice-ordered group under t-multiplication.

The following result is a straightforward adaptation of Proposition 1.4.

Proposition 1.12. [7, Theorem 3.9] *Let $*$ be a star operation on an integral domain D. Then the following conditions are equivalent.*

(i) $(A : B)^{*_f} = (AB^{-1})^{*_f}$ *for all* $A, B \in \boldsymbol{F}(D)$.
(ii) $(A : B^{-1})^{*_f} = (AB)^{*_f}$ *for all* $A, B \in \boldsymbol{F}(D)$.
(iii) $(A^{*_f} : B) = (AB^{-1})^{*_f}$ *for all* $A, B \in \boldsymbol{F}(D)$.
(iv) $(A^{*_f} : B^{-1}) = (AB)^{*_f}$ *for all* $A, B \in \boldsymbol{F}(D)$.

(v) *D is a $*$–Dedekind domain.*
(vi) *D is a CICD and $A^{*_f} = A^t$ for all $A \in \boldsymbol{F}(D)$.*
(vii) *$(A^v : B^{-1}) = (A^v B)^{*_f}$ for all $A, B \in \boldsymbol{F}(D)$.*

The only difference between the above proposition and Theorem 3.9 of [7] is that in [7] what is called here a $*$–Dedekind domain is regarded there as a Krull domain, which is not correct (i.e., condition (8) of [7, Theorem 3.9] is weaker than the other conditions). For example, if $* = d$, a $*$–Dedekind domain is a Dedekind domain (Example 1.10), which is a very special kind of Krull domain. This leads to the following natural problems.

Problem 1.13. (a) *Prove or disprove: There is a finite character star operation $*$ that admits a $*$–Dedekind domain D such that* (1) *D is not Dedekind and* (2) *there is at least one non-Dedekind Krull domain R such that R is not $*$–Dedekind.*
(b) *Find an example of a Krull, but not a $*$–Dedekind domain, for some $* \neq d$, i.e., find a Krull domain with a star operation $*$ such that $d \lneq *_f \lneq t$ (and so $* \neq v$).*

The difficulty of problem (a) lies in the fact that, as soon as we consider the $*$-operation on a domain D, which is a $*$–Dedekind domain, this operation becomes the t-operation of the (Krull) domain D, as indicated in (v)$\Leftrightarrow$(vi) of Proposition 1.12. This problem is important because a positive answer would entail a procedure for finding finite character operations $*$ that admit $*$–Dedekind domains, and the existence of $*$–Dedekind domains that answer the problem would justify deeper study in terms of general $*$-operations. The negative answer, on the other hand, would give us what we can expect from a general study. That is, we shall know that a $*$-Dedekind domain is either a Krull domain or a Dedekind domain.

A positive answer to problem (b) follows from Example 5.3 of [16], where the authors give the construction of a star operation $*$ on a Krull domain such that $d \lneq * = *_f \lneq t = v$.

Let $*$ be a star operation on an integral domain D. Recall that $A \in \boldsymbol{F}(D)$ is called *$*$-finite* (respectively, *strictly $*$-finite*) if there exists an $F \in \boldsymbol{f}(D)$ (respectively, $F \in \boldsymbol{f}(D)$ and $F \subseteq A$) such that $A^* = F^*$ [43]. It is well known that if $*$ has finite character, then the notions of $*$-finite and strictly $*$-finite coincide. Moreover, for $A \in \boldsymbol{F}(D)$, A is $*_f$-invertible if and only if A is $*$-invertible and both A and A^{-1} are $*_f$-finite (for instance [17, Lemma 2.3 and Proposition 2.6], where this subject was handled in the semistar operation setting).

The characterizations of (t, v)–Dedekind domains (or pre-Krull domains) given in [43, Proposition 4.1] can be directly translated to the general star operation case as follows.

Proposition 1.14. *Let $*$ be a star operation on an integral domain D. Then the following conditions are equivalent.*

(i) *$(AB)^{-1} = (A^{-1}B^{-1})^{*_f}$ for all $A, B \in \boldsymbol{F}(D)$.*
(ii) *A^{-1} is $*_f$-invertible for all $A \in \boldsymbol{F}(D)$.*
(iii) *D is a $(*, v)$–Dedekind domain.*

(iv) *D is completely integrally closed and $(AB)^v = (A^vB^v)^{*_f}$ for all $A, B \in \boldsymbol{F}(D)$.*

Proof. (i)⇒(ii). For all $A \in \boldsymbol{F}(D)$, clearly we have $A^vA^{-1} \subseteq D$, and so $D \subseteq (A^vA^{-1})^{-1}$. Therefore, using (i), we have $D \subseteq (A^vA^{-1})^{-1} = (A^{-1}A^v)^{*_f} \subseteq D$.

(ii)⇒(iii). If A^{-1} is $*_f$-invertible, then $A^v = (A^{-1})^{-1}$ is also $*_f$-invertible.

(iii)⇒(i). For all $A, B \in \boldsymbol{F}(D)$, we have $A^{-1}B^{-1} \subseteq (AB)^{-1}$, and so $(A^{-1}B^{-1})^{*_f} \subseteq ((AB)^{-1})^{*_f} = (AB)^{-1}$. On the other hand, by assumption, $D = ((AB)^v(AB)^{-1})^{*_f}$ and clearly $((AB)^v(AB)^{-1})^{*_f} \supseteq (A^vB^v(AB)^{-1})^{*_f}$; thus $D \supseteq (A^vB^v(AB)^{-1})^{*_f}$. Multiplying both sides by $A^{-1}B^{-1}$ and applying $*_f$, we have $(A^{-1}B^{-1})^{*_f} \supseteq (A^vA^{-1}B^vB^{-1}(AB)^{-1})^{*_f} = ((AB)^{-1})^{*_f} = (AB)^{-1}$. We conclude that $(AB)^{-1} = (A^{-1}B^{-1})^{*_f}$.

(iii)⇒(iv). If $(A^vA^{-1})^{*_f} = D$, then also $(A^vA^{-1})^t = (A^vA^{-1})^v = (AA^{-1})^v = D$ for all $A \in \boldsymbol{F}(D)$. Therefore D is a CICD. For the remainder, since we have already proved (iii)⇒(i), for all $A, B \in \boldsymbol{F}(D)$ we have $(AB)^v = ((AB)^{-1})^{-1} = ((A^{-1}B^{-1})^{*_f})^{-1} = (A^{-1}B^{-1})^{-1} = ((A^{-1})^{-1}(B^{-1})^{-1})^{*_f} = (A^vB^v)^{*_f}$.

(iv)⇒(iii). Since D is a CICD, for all $A \in \boldsymbol{F}(D)$, we have $D = (AA^{-1})^v = (A^vA^{-1})^v$. By the equality in (iv), we conclude that $D = (A^vA^{-1})^v = (A^v(A^{-1})^v)^{*_f} = (A^vA^{-1})^{*_f}$. □

Obviously, as a $(*_f, v)$–CICD is a $(*, v)$–Dedekind domain, we can rewrite Proposition 1.5 as a set of quotient-based characterizations of $(*, v)$–Dedekind domains.

Remark 1.15. (a) An integral domain D is called $*$*–Noetherian* if A is strictly $*$-finite for all $A \in \boldsymbol{F}(D)$. It is known that *the following conditions are equivalent.*

(i) *D is $*$–Noetherian.*
(ii) *D is $*_f$–Noetherian.*
(iii) *D satisfies the ascending chain condition on $*$-ideals.*

For the proof of the previous statement and more details on this subject, cf. [11, Lemma 3.3 and Proposition 3.5]. Note also that, if D is $*$–Noetherian, then each ideal of D is $*$-finite, but the converse is false in general (cf. [3, page 29] and [20, Example 18]).

From the previous considerations, we easily deduce that a $*$–Dedekind domain is a $*$–Noetherian domain. Moreover, as observed above, a $*$–Dedekind domain is a P$*$MD. Note that the converse is also true, i.e., the $*$–Dedekind domains coincide with the $*$–Noetherian P$*$MD's. As a matter of fact, if D is $*$–Noetherian, then for all $A \in \boldsymbol{F}(D)$, there exists an $F \in \boldsymbol{f}(D)$ with $F \subseteq A$ such that $F^* = A^* = A^{*_f}$. Hence $F^v = (F^*)^v = (A^*)^v = A^v$; thus $F^{-1} = A^{-1}$. Therefore $(AA^{-1})^{*_f} = (A^{*_f}A^{-1})^{*_f} = (F^{*_f}F^{-1})^{*_f} = (FF^{-1})^{*_f}$. If we assume that D is also a P$*$MD, then $(FF^{-1})^{*_f} = D$; hence $(AA^{-1})^{*_f} = D$, i.e., D is $*$–Dedekind.

From the previous observations we can conclude that the notion of $*$–Dedekind domain, given here, coincides in the star operation case with the notion considered for semistar operations in [11, Proposition 4.1].

We can summarize some of the previous considerations by saying that *the following notions coincide.*

(i) $*$*–Dedekind* $(= *_f$*–Dedekind* $= *_f$*–CICD).*
(ii) $*_f$*–multiplication domain.*
(iii) $*$*–Noetherian and Prüfer* $*$*–multiplication domain.*
(iv) $*_f$*–Noetherian and* $*_f$*–Prüfer domain.*

(b) Note that Noetherian ideal systems are investigated in [23, Chapter 3]. In particular, the equivalent statements given in (a) are also proved in [23, Theorem 3.5] in the more general setting of ideal systems on monoids.

Given an ideal system r, r–Dedekind monoids are introduced and studied in [23, Chapter 23, §3]. However, the notion of $*$–Dedekind domain coincides with that in [23] in case $* = *_f$ or in the case of Krull domains, but they are different in general (e.g., v–Dedekind domains are precisely Krull domains, but v–Dedekind monoids are just completely integrally closed monoids).

We also note that, using the ideal systems approach on commutative monoids, Halter-Koch [25] has obtained a general version of Proposition 1.14.

2. Star Prüfer domains

Recall that an integral domain D is a *v–domain* if each $F \in \boldsymbol{f}(D)$ is v-invertible. We have already introduced a direct generalization of this definition when $*$ is a star operation on D by saying that D is a *$*$–Prüfer domain* if every $F \in \boldsymbol{f}(D)$ is $*$-invertible. Since a $*$-invertible ideal is always v-invertible, we observe that a $*$–Prüfer domain is always a v–domain. Note that $*$–Prüfer domains were recently introduced in the case of semistar operations $\star$ under the name of $\star$–domains [18, Section 2].

If $F \in \boldsymbol{f}(D)$ is $*$-invertible, then $F^* = F^v$. Since, in a $*$–Prüfer domain, this holds for all $F \in \boldsymbol{f}(D)$, we conclude that $*_f = t$ in a $*$–Prüfer domain.

Next, we can consider a weaker notion: call D a *$(*, v)$–Prüfer domain* if F^v is $*$-invertible for all $F \in \boldsymbol{f}(D)$. It is easy to see that a $(*, v)$–Prüfer domain is also a v–domain and that a $*$–Prüfer domain is a $(*, v)$–Prüfer domain. Clearly, if $*_1, *_2$ are two star operations on D and $*_1 \leq *_2$, then a $*_1$–Prüfer domain (respectively, $(*_1, v)$–Prüfer domain) D is a $*_2$–Prüfer domain (respectively, $(*_2, v)$–Prüfer domain).

We have already observed that, from the definitions, it follows immediately that the notions of $*_f$–Prüfer domain and P$*$MD (or P$*_f$MD) coincide. Therefore a P$*$MD is a $*$–Prüfer domain, but the converse is not true since there are v-domains (= v–Prüfer domains) that are not PvMD's [26]. Also note that for an ideal system r on a monoid, the notion of r–Prufer monoid, for a general r, was introduced in [23, Chapter 17]. However, most of the results on r–Prufer monoids in [23] were proved for r-finitary. Now r–Prufer monoids for finitary r coincide with $*$–Prüfer domains only in case $* = *_f$. That is, the r–Prüfer monoids studied in [23] were simply P$*$MD's in ring-theoretic terms.

Example 2.1. *Let $*$ be a star operation defined on an integral domain D.*

Case: $* = d$.

Clearly, from the definition, *the following notions coincide.*

(i) *D is a d–Prüfer domain.*
(ii) *D is a Prüfer domain.*
(iii) *Each $F \in \boldsymbol{f}(D)$ is invertible.*

The following notions coincide.

(i) *D is a (d, v)–Prüfer domain.*
(ii) *D is a generalized GCD (for short, GGCD) domain (i.e., the intersection of two invertible ideals is invertible).*
(iii) *F^v is invertible for all $F \in \boldsymbol{f}(D)$.*

Generalized GCD domains were introduced in [2], where the previous equivalence was also proven [2, Theorem 1].

Note that, while a Prüfer domain is a GGCD domain, there are examples of GGCD domains that are not Prüfer [2, Theorem 2 (2)]. So, while a $*$–Prüfer domain is a $(*, v)$–Prüfer domain, a $(*, v)$–Prüfer domain may not be a $*$–Prüfer domain.

Case: $* = t$ or $* = w$.

The following notions coincide.

(i) *D is a t–Prüfer domain.*
(ii) *D is a (t, v)–Prüfer domain.*
(iii) *D is a w–Prüfer domain.*
(iv) *D is a (w, v)–Prüfer domain.*
(v) *D is Prüfer v–multiplication domain.*

Since the maximal t-ideals coincide with the maximal w-ideals, the notions w-invertible and t-invertible coincide (cf. [5, Theorem 2.18] and [40, Section 5]), thus (iii)⇔(i)⇒(ii)⇔(iv) and, by definition, (v) coincides with (i). Finally, (ii)⇒(v), since $D = (F^vF^{-1})^t = (F^tF^{-1})^t = (FF^{-1})^t$ for all $F \in \boldsymbol{f}(D)$.

Case: $* = v$.

From the definitions, we immediately have that *the following notions coincide.*

(i) *D is a v–Prüfer domain.*
(ii) *D is a (v, v)–Prüfer domain.*
(iii) *D is a v–domain.*

Once we know that a Prüfer domain is just a special case of a $*$–Prüfer domain (and, in particular, of a $(*, v)$–Prüfer domain) we would like to see what ideal-theoretic characterizations of Prüfer domains can be translated to the framework of $*$–Prüfer and $(*, v)$–Prüfer domains. Here we point out some.

Theorem 2.2. *Let $*$ be a star operation defined on an integral domain D. Then the following properties are equivalent.*

(i) *D is a $*$–Prüfer domain.*
(ii) *Every (nonzero) two generated ideal of D is $*$-invertible.*
(iii$_{\boldsymbol{f}}$) *$((F \cap G)(F + G))^* = (FG)^*$ for all $F, G \in \boldsymbol{f}(D)$.*

(iii$_{\boldsymbol{F}}$) $((A \cap B)(A+B))^* = (AB)^*$ *for all* $A, B \in \boldsymbol{F}(D)$.
(iv$_{\boldsymbol{f}}$) $(F(G^* \cap H^*))^* = (FG)^* \cap (FH)^*$ *for all* $F, G, H \in \boldsymbol{f}(D)$.
(iv$_{\boldsymbol{fF}}$) $(F(A^* \cap B^*))^* = (FA)^* \cap (FB)^*$ *for all* $F \in \boldsymbol{f}(D)$ *and* $A, B \in \boldsymbol{F}(D)$.
(v) *If* $A, B \in \boldsymbol{F}(D)$ *are* $*$*-invertible, then* $A \cap B$ *and* $A + B$ *are* $*$*-invertible.*
(vi) *If* $A, B \in \boldsymbol{F}(D)$ *are* $*$*-invertible, then* $A + B$ *is* $*$*-invertible.*

Proof. The proof of (i)⇔(ii) follows from the reduction argument used in [33, Lemma 2.6] for showing that an integral domain is a v–domain if and only if every (nonzero) two generated ideal is v-invertible. For the sake of completeness, we give some details of the proof of (ii)⇒(i). Let $F \in \boldsymbol{f}(D)$; we want to show that F is $*$-invertible. We use induction on the number of generators of F. Let $F := (x_1, x_2, ..., x_n, x_{n+1})D$ with $n \geq 2$ and set $I := x_1D$, $J := (x_2, x_3, ..., x_n)D$, and $H := x_{n+1}D$. Then $F(IH+IJ+JH) = (J+H)(H+I)(I+J)$. Note that each of the factors on the right is generated by $k \leq n$ elements, and so is $*$-invertible by the induction hypothesis. This forces the factors on the left (and hence, in particular, F) to be $*$-invertible. (Note that this method of proof is essentially that used originally by H. Prüfer in [37, page 7] to show that D is a Prüfer domain if and only if every (nonzero) two generated ideal of D is invertible.)

(i)⇒(iii$_{\boldsymbol{f}}$). $((F \cap G)(F+G))^* \subseteq (FG)^*$ holds for all $F, G \in \boldsymbol{f}(D)$.

For the reverse containment, let $x \in FG$. Then $xG^{-1} \subseteq FGG^{-1} \subseteq F$ and $xF^{-1} \subseteq F^{-1}FG \subseteq G$. This gives $x(F^{-1} \cap G^{-1}) \subseteq F \cap G$. But $F^{-1} \cap G^{-1} = (F+G)^{-1}$. So we have $x(F+G)^{-1} \subseteq F \cap G$. Multiplying both sides by $F+G$ and applying $*$, we get $x \in ((F \cap G)(F+G))^*$. This gives $(FG)^* \subseteq ((F \cap G)(F+G))^*$.

(iii$_{\boldsymbol{f}}$)⇒(ii) is obvious because $((F \cap G)(F+G))^* = (FG)^*$ for all $F, G \in \boldsymbol{f}(D)$ implies that in particular $((xD \cap yD)(xD+yD))^* = xyD$ for all nonzero $x, y \in D$, which forces every (nonzero) two generated ideal of D to be $*$-invertible.

(iii$_{\boldsymbol{f}}$)⇒(iii$_{\boldsymbol{F}}$). Obviously, $((A \cap B)(A+B))^* \subseteq (AB)^*$ holds for all $A, B \in \boldsymbol{F}(D)$. For the reverse containment, it is enough to show that $AB \subseteq ((A \cap B)(A+B))^*$. For this, let $x \in AB$. Then $x \in FG$, where F and G are finitely generated with $F \subseteq A$ and $G \subseteq B$. But then, by (iii$_{\boldsymbol{f}}$), $x \in (FG)^* = ((F \cap G)(F+G))^* \subseteq ((A \cap B)(A+B))^*$. Thus $AB \subseteq ((A \cap B)(A+B))^*$.

(iii$_{\boldsymbol{F}}$)⇒(iii$_{\boldsymbol{f}}$) and (iv$_{\boldsymbol{fF}}$)⇒(iv$_{\boldsymbol{f}}$) are trivial.

(i)⇒(iv$_{\boldsymbol{fF}}$). Obviously $(F(A^* \cap B^*))^* \subseteq (FA)^* \cap (FB)^*$. For the reverse containment, note that F is $*$-invertible. Now consider $(F^{-1}((FA)^* \cap (FB)^*))^* \subseteq (F^{-1}(FA)^* \cap F^{-1}(FB)^*)^* \subseteq (F^{-1}(FA))^* \cap (F^{-1}(FB))^* = A^* \cap B^*$. So the inclusion $(F^{-1}((FA)^* \cap (FB)^*))^* \subseteq A^* \cap B^*$ gives, on multiplying by F and applying $*$ on both sides, the reverse containment.

(iv$_{\boldsymbol{f}}$)⇒(ii). Let $F := (a, b)$, $G := (\frac{1}{a})$ and $H := (\frac{1}{b})$, where a and b are two nonzero elements of D. Then, by assumption, we have $((a, b)((\frac{1}{a}) \cap (\frac{1}{b}))^* = ((\frac{1}{a})(a, b))^* \cap ((\frac{1}{b})(a, b))^*$. On the other hand, it is easy to see that $(a, b)(a, b)^{-1} = (a, b)((\frac{1}{a}) \cap (\frac{1}{b}))$. Therefore $D \supseteq ((a, b)(a, b)^{-1})^* = ((\frac{1}{a})(a, b)))^* \cap ((\frac{1}{b})(a, b))^* \supseteq D$, and so we conclude that $((a, b)(a, b)^{-1})^* = D$.

(v)⇒(vi)⇒(ii) are obvious (for the last implication note that a nonzero principal ideal is ($*$-)invertible).

(iii$_{\boldsymbol{F}}$)⇒(v). Since $A, B \in \boldsymbol{F}(D)$ are $*$-invertible if and only if AB is $*$-invertible, the conclusion follows from the equality $((A \cap B)(A+B))^* = (AB)^*$.

□

Remark 2.3. Let D be a $*$–Prüfer domain. If we assume that $*$ has finite character (hence, D is a P$*$MD), then (as we observed above) $A \in \boldsymbol{F}(D)$ is $*$-invertible if and only if A is (strictly) $*$-finite. In this case, (vi) of Theorem 2.2 reduces to "the sum of two $*$-finite ideals is $*$-finite". (This is a trivial statement since $(F^* + G^*)^* = (F+G)^*$ for all $F, G \in \boldsymbol{f}(D)$, [21, Proposition 32.2].)

However, if $*$ does not have finite character, a $*$-invertible ideal need not be strictly $*$-finite. In fact (for D a $*$–Prüfer domain), each $*$-invertible ideal is strictly $*$-finite precisely when D is a $*_f$–Prüfer domain (= P$*$MD). (If $F \in \boldsymbol{f}(D)$ is $*$-invertible, then F^{-1} is $*$-invertible, and so there exists $G \in \boldsymbol{f}(D)$ such that $G \subseteq F^{-1}$ and $F^{-1} = (F^{-1})^* = G^*$. Therefore $D = (FF^{-1})^* = (F(F^{-1})^*)^* = (FG^*)^* = (FG)^* = (FG)^{*_f} = (FF^{-1})^{*_f} \subseteq D$; hence $(FF^{-1})^{*_f} = D$.)

It is natural to ask whether it is possible to remove from statement (iv$_{\boldsymbol{fF}}$) of the previous theorem the condition that F is finitely generated. We do not have a complete answer to this question, however we have an interesting alternative described in the following proposition. Recall that a star operation $*$ is called *stable* (or, *distributes over finite intersections*) if $(A \cap B)^* = A^* \cap B^*$ for all $A, B \in \boldsymbol{F}(D)$ (cf. [15, page 174] and [4]). A star operation induced by a defining family of quotient rings of D is stable (cf. for instance [4, Proposition 2.2]).

Proposition 2.4. *Let $*$ be a star operation defined on an integral domain D. Then the following properties are equivalent.*

($\bar{\text{i}}$) *D is a $*$–Prüfer domain and $*$ is a stable star operation on D.*
($\overline{\text{iv}}$) *$(C(A \cap B))^* = (C(A^* \cap B^*))^* = (CA)^* \cap (CB)^*$ for all $A, B, C \in \boldsymbol{F}(D)$.*

Proof. ($\bar{\text{i}}$)⇒($\overline{\text{iv}}$). As seen above, in general we have $(C(A^* \cap B^*))^* \subseteq (CA)^* \cap (CB)^*$. Moreover, since $*$ is stable, $(C(A \cap B))^* = (C(A \cap B)^*)^* = (C(A^* \cap B^*))^*$. For the reverse containment, it is sufficient to show that $CA \cap CB \subseteq (C(A \cap B))^*$. For this, let $x \in CA \cap CB$. Then, in particular, there is an $F \in \boldsymbol{f}(D)$ such that $x \in FA \cap FB$ and $F \subseteq C$. So $x \in (FA \cap FB)^* = (FA)^* \cap (FB)^* = (F(A^* \cap B^*))^*$ (the last equality holds by Theorem 2.2 ((i)⇒(iv$_{\boldsymbol{fF}}$))). Again, by the stability of $*$, we have $(F(A^* \cap B^*))^* = (F(A \cap B))^*$. Thus we conclude that $x \in (FA \cap FB)^* = (F(A \cap B))^* \subseteq (C(A \cap B))^*$.

($\overline{\text{iv}}$)⇒($\bar{\text{i}}$). From $(C(A \cap B))^* = (CA)^* \cap (CB)^*$ for all $A, B, C \in \boldsymbol{F}(D)$, by setting $C = D$, we deduce that $*$ is stable. Moreover, as in the proof of Theorem 2.2 ((iv$_{\boldsymbol{f}}$)⇒(ii)), taking $C := (a, b)$, $A := (\frac{1}{a})$, and $B := (\frac{1}{b})$, where a and b are two nonzero elements of D, we obtain that $((a,b)(a,b)^{-1})^* = D$. □

As we have already observed, the notions of $*_f$–Prüfer domain and P$*$MD (or P$*_f$MD) coincide; moreover, for a P$*$MD the operation $*_f$ is stable [14, Theorem 3.1]. Furthermore, a $*$–CICD is a particular $*$–Prüfer domain, and for a $*$–CICD the operation $*$ is stable [4, Theorem 2.8]. Therefore, in order to find an example of a $*$–Prüfer domain for which $*$ is not stable, we have to consider the case of a $*$–Prüfer domain, not a $*$–CICD, with a star operation $*$ that is not of finite character. In case $* = v$, it is known that D is a v–domain if and only if D is integrally closed and v distibutes over finite intersections of finitely generated ideals (cf. [31, Theorem 1(2) and Theorem 2] or [4, Theorem 2.8]). Very recently, Mimouni has given an explicit example of a two-dimensional Prüfer domain (hence a v–domain, but not a (v–)CICD) with v not stable [32, Example 3.1].

Proposition 2.5. *Let $*$ be a star operation defined on an integral domain D. If D is a $*$–Prüfer domain, then:*

1. $(FG)^{-1} = (F^{-1}G^{-1})^*$ *for all* $F, G \in \boldsymbol{f}(D)$.
2. *If* $A, B \in \boldsymbol{F}(D)$ *are $*$-invertible, then* $A \cap B$ *is $*$-invertible.*
3. $*$ *is an a.b. star operation (i.e.,* $(FA)^* \subseteq (FB)^*$ *implies that* $A^* \subseteq B^*$ *for all* $F \in \boldsymbol{f}(D)$ *and* $A, B \in \boldsymbol{F}(D)$*). In particular, D is an integrally closed domain* [21, Corollary 32.8].

Proof. (1). If $F, G \in \boldsymbol{f}(D)$, then in particular G^{-1} is $*$-invertible. By Proposition 1.7, $(F^{-1}G^{-1})^* = (F^{-1} : G)^*$. The conclusion follows from the fact that $(F^{-1} : G)^* = ((D : F) : G)^* = (D : FG)^* = ((FG)^{-1})^* = (FG)^{-1}$.

That (2) holds was already observed in Theorem 2.2 ((i)⇒(v)).

(3). The proof is straightforward; multiply both sides of the relation $(FA)^* \subseteq (FB)^*$ by F^{-1} and apply the $*$-operation. □

Remark 2.6. (a) To see that statement (1) of Proposition 2.5 does not characterize $*$–Prüfer domains, recall that an integral domain D is a *pre-Schreier domain* if for all nonzero $x, y, z \in D$, $x|yz$ implies that $x = rs$, where $r|y$ and $s|z$. Pre-Schreier domains are a generalization of GCD domains (cf. [9] and [10, Theorem 1]). It is well known that a pre-Schreier domain satisfies (each of) the following equivalent properties.

($\boldsymbol{\alpha}$) For all $0 \neq a_i, b_j \in D$, with $1 \leq i \leq n,\ 1 \leq j \leq m$:

$$(\cap_{1 \leq i \leq n}(a_i))\,(\cap_{1 \leq j \leq m}(b_j)) = \cap_{1 \leq i \leq n, 1 \leq j \leq m}(a_i b_j).$$

($\boldsymbol{\beta}$) $(FG)^{-1} = F^{-1}G^{-1}$ for all $F, G \in \boldsymbol{f}(D)$.

(Cf. [41, Corollary 1.7] and [42, Proposition 1.6]; note that an integral domain satisfying these two equivalent conditions was called a $*$-domain by Zafrullah in [41]). Now there do exist pre-Schreier domains that are not integrally closed [41, page 1918]. Combining this piece of information with the fact that a $*$–Prüfer domain is integrally closed (Proposition 2.5 (3)), we easily conclude that $(FG)^{-1} = (F^{-1}G^{-1})^*$ for all $F, G \in \boldsymbol{f}(D)$ does not imply that D is a $*$–Prüfer domain.

(b) Statement (2) of Proposition 2.5 does not characterize $*$–Prüfer domains. For example, in a generalized GCD domain (Example 2.1), we have that $A \cap B$ is (d-)invertible for all (d-)invertible $A, B \in \boldsymbol{F}(D)$. However, a generalized GCD domain may not be a d–Prüfer domain (= Prüfer domain, Example 2.1) [2, Theorem 2 (2)]. (It may also be noted that for a mere pair of ideals $F, G \in \boldsymbol{f}(D)$, the fact that $F \cap G$ is $*$-invertible does not mean that F and G are both $*$-invertible. For instance, let k be a field and X, Y two indeterminates over k. Take $D := k[X, Y]$,

$F := (X,Y)$ and $G := (X)$, then $(X,Y) \cap (X)$ is principal, but (X,Y) is not invertible.)

(c) Statement (3) of Proposition 2.5 does not characterize $*$–Prüfer domains (cf. [18, Example 1 (2), page 150]). It is easy to show that an integral domain D is integrally closed if and only if there is an a.b. star operation $*$ defined on D. (The proof depends upon the fact that if D is integrally closed, then D is expressible as an intersection of valuation overrings of D [21, Theorems 19.8, 32.5 and Corollary 32.8].) Therefore, an integrally closed domain may not be a $*$–Prüfer domain for any star operation $*$ since there are integrally closed domains that are not v–domains (e.g., [21, page 429, Exercise 2]).

Bearing in mind Proposition 2.5 (1), we next give more precise relations among the notions coming into play.

Proposition 2.7. *Let $*$ be a star operation defined on an integral domain D, and consider the following statements:*

(a) *D is a $*$–Prüfer domain.*
(b) *$(AF)^{-1} = (A^{-1}F^{-1})^*$ for all $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$.*
(c) *D is a $(*, v)$–Prüfer domain.*

Then (a)⇒(b)⇔(c).

Proof. (a)⇒(b). Let $A \in \boldsymbol{F}(D)$, $F \in \boldsymbol{f}(D)$, and let $x \in A^{-1}F^{-1}$. Then $xF \subseteq A^{-1}F^{-1}F \subseteq A^{-1}$, and so $xAF \subseteq A^{-1}A \subseteq D$. Therefore $A^{-1}F^{-1} \subseteq (AF)^{-1}$. On the other hand, $(AF)^{-1}$ is a v-ideal (and so, in particular, a $*$-ideal); thus we have $(A^{-1}F^{-1})^* \subseteq ((AF)^{-1})^* = (AF)^{-1}$. Conversely, let $y \in (AF)^{-1}$. Then $yAF \subseteq (AF)^{-1}AF \subseteq D$, so $yF \subseteq A^{-1}$. Multiplying both sides by F^{-1}, applying $*$, and noting that $F \in \boldsymbol{f}(D)$, we get $y \in y(FF^{-1})^* \subseteq (A^{-1}F^{-1})^*$.

(b)⇒(c). Let $F \in \boldsymbol{f}(D)$, and set $A := F^{-1}$. Then $D \subseteq (F^{-1}F)^{-1} = (AF)^{-1} = (A^{-1}F^{-1})^* = (F^vF^{-1})^* \subseteq D$, and so D is a $(*, v)$–Prüfer domain.

(c)⇒(b). Let $x \in (AF)^{-1}$. We have $xAF \subseteq D$; so $xF \subseteq A^{-1}$, and hence $xF^v = (xF)^v \subseteq (A^{-1})^v = A^{-1}$. Therefore $xF^vF^{-1} \subseteq A^{-1}F^{-1}$. Since $(F^vF^{-1})^* = D$, we conclude that $(AF)^{-1} \subseteq (A^{-1}F^{-1})^*$. The reverse containment is straightforward since $((AF)^{-1})^* = (AF)^{-1}$. □

Note that we already observed that there are (d, v)–Prüfer domains that are not d–Prüfer domains (= Prüfer domains, Example 2.1), and so (c) of Proposition 2.7 does not imply (a). We will see later (Theorem 2.11 (c)) that (c) and (a) are equivalent under an additional condition.

For $*$–Prüfer domains, we have the following set of quotient-based characterizations.

Theorem 2.8. *Let $*$ be a star operation defined on an integral domain D. Then the following conditions are equivalent.*

(i) *D is a $*$–Prüfer domain.*
(ii) *For all $A \in \boldsymbol{F}(D)$ and $F \in \boldsymbol{f}(D)$, $A \subseteq F^*$ implies $A^* = (BF)^*$ for some $B \in \boldsymbol{F}(D)$.*

(iii) $(A:F)^* = (A^*:F) = (AF^{-1})^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.
(iv) $(A:F^{-1})^* = (A^*:F^{-1}) = (AF)^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.
(v) $(F:A)^* = (F^*:A) = (FA^{-1})^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.
(vi) $(F:A)^v = (F^v:A) = (FA^{-1})^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.
(vii) $(F^v:A^{-1}) = (FA^v)^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.
(viii) $((A+B):F)^* = ((A:F)+(B:F))^*$ *for all* $A, B \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.
(ix) $(A:(F\cap G))^* = ((A:F)+(A:G))^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F, G \in \boldsymbol{f}^*(D) := \{H \in \boldsymbol{f}(D) \mid H = H^*\}$.
(x) $(((a):_D(b))+((b):_D(a)))^* = D$ *for all nonzero* $a, b \in D$.

Proof. (i)⇒(ii). Set $B := (AF^{-1})^*$. Then clearly $(BF)^* = ((AF^{-1})^*F)^* = (A(FF^{-1})^*)^* = A^*$.

(ii)⇒(i). We show that every $F \in \boldsymbol{f}(D)$ is $*$-invertible. For this, let $0 \neq x \in F^*$, and set $A := (x)$. Then by assumption, there is a $B \in \boldsymbol{F}(D)$ such that $(x) = (x)^* = A^* = (BF)^*$, and so $D = ((x^{-1}B)F)^*$, which is equivalent to F being $*$-invertible.

(i)⇒(iii) follows from Proposition 1.7, since in the present situation $F \in \boldsymbol{f}(D)$ is $*$-invertible.

(i)⇒(iv). This implication can be proven in a similar fashion as (i)⇒(iii) using the fact that if F is $*$-invertible, then so is F^{-1}.

(i)⇒(v). Clearly $(F:A)^* \subseteq (F^*:A)$. Let $x \in (F^*:A)$. Then $xA \subseteq F^*$; so $xAF^{-1} \subseteq F^*F^{-1} \subseteq (F^*F^{-1})^* = D$, which gives $xF^{-1} \subseteq A^{-1}$. Now multiplying both sides by F and applying $*$, we get $x \in (FA^{-1})^*$. Next, to show that $(FA^{-1})^* \subseteq (F:A)^*$, let $y \in FA^{-1}$. Then $yA \subseteq FA^{-1}A \subseteq F$, which gives $y \in (F:A)$, and so $FA^{-1} \subseteq (F:A)$, which leads to $(FA^{-1})^* \subseteq (F:A)^*$. Now we have shown that $(F:A)^* \subseteq (F^*:A) \subseteq (FA^{-1})^* \subseteq (F:A)^*$, which establishes the equalities.

(i)⇒(vi). By the proof of (i)⇒(v), $(F:A)^v \subseteq (F^*:A)^v = (F^*:A) \subseteq (FA^{-1})^* \subseteq (F:A)^* \subseteq (F:A)^v$. This gives the required equations.

(i)⇒(vii). If we insert A^{-1} for A in (vi), then we get (vii).

Next we will show that each of the conditions (iii), (iv), (v), (vi) and (vii) implies that F is $*$-invertible for all $F \in \boldsymbol{f}(D)$.

(iii)⇒(i). In $(A^*:F) = (AF^{-1})^*$, set $A = F$ for $F \in \boldsymbol{f}(D)$. We have $(F^*:F) = (FF^{-1})^*$. Now note that $D \subseteq (F^*:F) = (FF^{-1})^* \subseteq D$.

(iv)⇒(i). In $(A^*:F^{-1}) = (AF)^*$, set $A = F^{-1}$ for $F \in \boldsymbol{f}(D)$.

(v)⇒(i). In $(F^*:A) = (FA^{-1})^*$, set $A = F$ for $F \in \boldsymbol{f}(D)$.

(vi)⇒(i). In $(F^v:A) = (FA^{-1})^*$, set $A = F$ for $F \in \boldsymbol{f}(D)$.

(vii)⇒(i). In $(F^v:A^{-1}) = (FA^v)^*$, set $A = F^{-1}$ for $F \in \boldsymbol{f}(D)$.

(iii)⇒(viii). Applying (iii), we have

$$\begin{aligned} ((A+B):F)^* = &\ ((A+B)F^{-1})^* \\ = &\ ((AF^{-1})^* + (BF^{-1})^*)^* = ((A:F)^* + (B:F)^*)^* \\ = &\ ((A:F)+(B:F))^*. \end{aligned}$$

(viii)⇒(i). Let $0 \neq a, b \in D$. Set $A := (a)$, $B := (b)$, and $F := (a, b)$. Then

$$\begin{aligned} D \subseteq \quad & ((a,b):(a,b))^* = (((a):(a,b)) + ((b):(a,b)))^* \\ = \quad & (((a):(a,b))^* + ((b):(a,b))^*)^* \\ = \quad & \left(\left(a(a,b)^{-1}\right)^* + \left(b(a,b)^{-1}\right)^*\right)^* \\ = \quad & \left(a(a,b)^{-1} + \ b(a,b)^{-1}\right)^* \\ = \quad & \left((a,b)(a,b)^{-1}\right)^* \\ \subseteq \quad & D \end{aligned}$$

which forces $((a,b)(a,b)^{-1})^* = D$. The conclusion follows from Theorem 2.2 ((i)⇔(ii)).

(i)⇒(ix). For all $A \in \boldsymbol{F}(D)$ and $F, G \in \boldsymbol{f}(D)$, note that $((A:F) + (A:G))^* = ((A:F)^* + (A:G)^*)^*$; moreover $(A:F)^* = (AF^{-1})^*$ and $(A:G)^* = (AG^{-1})^*$ by (i)⇒(iii). Therefore $((A:F)+(A:G))^* = ((AF^{-1})^* + (AG^{-1})^*)^* = (AF^{-1} + AG^{-1})^* = (A(F^{-1}+G^{-1}))^*$.

Since D is a $*$–Prüfer domain, F and G are $*$-invertible, and thus F^{-1} and G^{-1} are also $*$-invertible. Therefore, $F \cap G$ and $F^{-1} + G^{-1}$ are $*$-invertible by Theorem 2.2 ((i)⇒(v)). Moreover, since a $*$-invertible $*$-ideal is a v-invertible v-ideal [8, Proposition 3.1], for $F, G \in \boldsymbol{f}^*(D)$, we have in particular $F = F^* = F^v$, $G = G^* = G^v$, and $(F^{-1}+G^{-1})^* = (F^{-1}+G^{-1})^v$. On the other hand, $(F^{-1}+G^{-1})^{-1} = (D:(F^{-1}+G^{-1})) = (D:F^{-1}) \cap (D:G^{-1}) = F^v \cap G^v = F \cap G$. Therefore, $(A(F^{-1}+G^{-1}))^* = (A(F^{-1}+G^{-1})^*)^* = (A(F^{-1}+G^{-1})^v)^* = (A((F^{-1}+G^{-1})^{-1})^{-1})^* = (A(F\cap G)^{-1})^*$. Since $F \cap G$ is $*$-invertible, by Proposition 1.7 we have $(A(F\cap G)^{-1})^* = (A:(F\cap G))^*$. Then, putting it all together, we conclude that (ix) holds, i.e., $((A:F)+(A:G))^* = (A:(F\cap G))^*$.

(ix)⇒(x). Let $0 \neq a, b \in D$, and set $A := (a) \cap (b)$, $F := (a)$, $G := (b)$. By assumption, we have

$$(D \subseteq) \ \ (((a)\cap(b)):((a)\cap(b)))^* = ((((a)\cap(b)):(a)) + (((a)\cap(b)):(b)))^* .$$

On the other hand,

$$\begin{aligned} ((((a)\cap(b)):(a)) + (((a)\cap(b)):(b)))^* &= \left((((a)\cap(b))a^{-1}) + (((a)\cap(b))b^{-1})\right)^* \\ &= (((b):_D(a)) + ((a):_D(b)))^* \subseteq D\,. \end{aligned}$$

Therefore we conclude that (x) holds.

(x)⇒(i). Note that

$$\begin{aligned} \left((a,b)(a,b)^{-1}\right)^* &= \left(a(a,b)^{-1} + b(a,b)^{-1}\right)^* \\ &= \left(a(((a)\cap(b))(ab)^{-1}) + b(((a)\cap(b))(ab)^{-1})\right)^* \\ &= \left((((a)\cap(b))b^{-1}) + (((a)\cap(b))a^{-1})\right)^* \\ &= (((a):_D(b)) + ((b):_D(a)))^* \end{aligned}$$

and apply Theorem 2.2 ((ii)⇒(i)). □

Remark 2.9. (1) Note that from the proof of Theorem 2.8 it follows easily that the following conditions are equivalent to each of the conditions (i)–(x).

(iii′) $(A:F)^* = (AF^{-1})^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.

(iii″) $(A^*:F) = (AF^{-1})^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.

(iv′) $(A : F^{-1})^* = (AF)^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.
(iv″) $(A^* : F^{-1}) = (AF)^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.
(v′) $(F : A)^* = (FA^{-1})^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.
(v″) $(F^* : A) = (FA^{-1})^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.
(vi′) $(F : A)^v = (FA^{-1})^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.
(vi″) $(F^v : A) = (FA^{-1})^*$ *for all* $A \in \boldsymbol{F}(D)$ *and* $F \in \boldsymbol{f}(D)$.

As a by-product, we obtain a direct proof of [7, Corollary 4.3].

(2) In analogy with the equivalence (i)⇔(ii) of Theorem 2.8, it is straightforward to prove the following "multiplication-type" characterizations of the "Prüfer-like" classes of integral domains introduced above.

(a) *The following properties are equivalent.*
 (i) *D is a $*$–Prüfer domain.*
 (ii) *If* $F, G \in \boldsymbol{f}(D)$ *and* $F^* \subseteq G^*$, *then* $F^* = (GB)^*$ *for some* $B \in \boldsymbol{F}(D)$.
(b) *The following properties are equivalent.*
 (j) *D is a $(*, v)$–Prüfer domain.*
 (jj) *If* $F, G \in \boldsymbol{f}(D)$ *and* $F^* \subseteq G^v$, *then* $F^* = (GB)^*$ *for some* $B \in \boldsymbol{F}(D)$.
(c) *The following properties are equivalent.*
 ($\mathrm{i}_{\boldsymbol{f}}$) *$D$ is a $*_f$–Prüfer domain (= P$*$MD).*
 ($\mathrm{ii}_{\boldsymbol{f}}$) *If* $F, G \in \boldsymbol{f}(D)$ *and* $F^* \subseteq G^*$, *then* $F^* = (GH)^*$ *for some* $H \in \boldsymbol{f}(D)$.
(d) *The following properties are equivalent.*
 ($\mathrm{j}_{\boldsymbol{f}}$) *$D$ is a $(*_f, v)$–Prüfer domain.*
 ($\mathrm{jj}_{\boldsymbol{f}}$) *If* $F, G \in \boldsymbol{f}(D)$ *and* $F^* \subseteq G^v$, *then* $F^* = (GH)^*$ *for some* $H \in \boldsymbol{f}(D)$.

Remark 2.10. Referring to Theorem 6.6 in [30], which provides several characterizations of Prüfer domains, we can summarize that conditions (2), (5), (6), (7), (8), and (9) of that theorem have been modified in a canonical way (see, respectively, conditions (ii), ($\mathrm{iv}_{\boldsymbol{f}}$)&($\mathrm{iv}_{\boldsymbol{fF}}$), ($\mathrm{iii}_{\boldsymbol{f}}$)&($\mathrm{iii}_{\boldsymbol{F}}$) of Theorem 2.2 and conditions (ii), (viii), (ix) of Theorem 2.8) in order to obtain characterizations of $*$–Prüfer domains.

We have also observed that condition (3) (of [30, Theorem 6.6]) extends in a natural way to "$*$ is an a.b. star operation", and we have just seen that D being $*$–Prüfer implies that $*$ is an a.b. star operation on D, but not conversely (Proposition 2.5 (3) and Remark 2.6 (c)).

Moreover, there is no natural modification of condition (4) (of [30, Theorem 6.6]) which can provide a characterization of $*$–Prüfer domains: take, for instance, a one-dimensional quasi-local CICD (hence, v–domain) which is not a valuation domain [34, 35, 38]. We were unable to find an appropriate modification of condition (10) (of Theorem 6.6 in [30]) leading to a characterization of $*$–Prüfer domains.

Theorem 2.11. *Let D be an integral domain with quotient field K, let $*$ be a star operation on D, and let $\mathrm{Inv}^*(D)$ be the group of $*$-invertible $*$-ideals of D under $*$-multiplication (defined by $A * B := (AB)^*$ for all $A, B \in \mathrm{Inv}^*(D)$).*

(a) *D is a $*$–Prüfer domain if and only if $\mathrm{Inv}^*(D)$ is a lattice-ordered abelian group under the relation $A \leq B$ defined by $A \supseteq B$ for $A, B \in \mathrm{Inv}^*(D)$*

with $\sup(A,B) = A \cap B$ *and* $\inf(A,B) = (A+B)^* = (A+B)^v$ *for all* $A, B \in \mathrm{Inv}^*(D)$.

(b) *If* $\mathrm{Inv}^*(D)$ *is a lattice-ordered abelian group (under the relation* $\leq$ *defined above), then* $\sup(A,B) = A \cap B$ *and* $\inf(A,B) = (A+B)^v$ *for all* $A, B \in \mathrm{Inv}^*(D)$. *In this situation,* D *is a* $(*, v)$*–Prüfer domain.*

(c) D *is a* $*$*–Prüfer domain if and only if* D *is a* $(*, v)$*–Prüfer domain and* $(A+B)^* = (A+B)^v$ *for all* $A, B \in \mathrm{Inv}^*(D)$.

Proof. That $\mathrm{Inv}^*(D)$ is an abelian group was observed in [8, page 812]. That $\mathrm{Inv}^*(D)$ is a partially ordered group (under the partial ordered defined above) is easy to see because for $A, B \in \mathrm{Inv}^*(D)$ and for any nonzero fractional ideal X, $A \supseteq B$ implies $XA \supseteq XB$ and hence $(XA)^* \supseteq (XB)^*$. Thus, in particular, for all $X, A, B \in \mathrm{Inv}^*(D)$, $A \leq B$ implies $X{*}A = (XA)^* \leq (XB)^* = X{*}B$. So the relation $\leq$ is compatible with group multiplication, and hence $\mathrm{Inv}^*(D)$ is a partially ordered group [19, pages 61 and 107].

(a) Now suppose that D is a $*$–Prüfer domain. By Theorem 2.2 ((i)$\Rightarrow$(iii$_{\boldsymbol{F}}$)), $A \cap B$ $(= (A \cap B)^*)$ and $(A+B)^*$ both belong to $\mathrm{Inv}^*(D)$, whenever $A, B \in \mathrm{Inv}^*(D)$. Therefore, it is straightforward to verify that $A \cap B = \sup(A,B)$ and $(A+B)^* = \inf(A,B)$ for $A, B \in \mathrm{Inv}^*(D)$. Thus $\mathrm{Inv}^*(D)$ is a lattice-ordered group [19, page 107]. Note also that, since a $*$-invertible $*$-ideal is a v-ideal [29, Corollaire 1, page 24], $(A+B)^* = ((A+B)^*)^v = (A+B)^v$. Therefore $\inf(A,B) = (A+B)^* = (A+B)^v$ for $A, B \in \mathrm{Inv}^*(D)$.

Conversely, suppose that $\mathrm{Inv}^*(D)$ is a lattice-ordered group (under $\leq$ defined above) and that $\inf(A,B) = (A+B)^*$ for $A, B \in \mathrm{Inv}^*(D)$. In particular, $\inf(aD, bD)$ $= (aD + bD)^* \in \mathrm{Inv}^*(D)$ for all $0 \neq a, b \in D$; hence every two generated nonzero ideal is $*$-invertible, and thus D is a $*$–Prüfer domain by Theorem 2.2 ((ii)$\Rightarrow$(i)).

(b) We start by showing that under the present assumption $\inf(A,B) = (A+B)^v$ for $A, B \in \mathrm{Inv}^*(D)$. Since $\inf(A,B) \in \mathrm{Inv}^*(D)$ and, as we observed above, every $*$-invertible $*$-ideal is a v-ideal, clearly $\inf(A,B) \supseteq (A+B)^v$. For the reverse containment, for all $H \in \mathrm{Inv}^*(D)$ such that $H \supseteq A$ and $H \supseteq B$, we have that $H \supseteq \inf(A,B)$. Since $\mathrm{Inv}^*(D)$ contains all principal fractional ideals, in particular we have $\bigcap\{zD \mid 0 \neq z \in K,\ zD \supseteq A \text{ and } zD \supseteq B\} \supseteq \bigcap\{H \in \mathrm{Inv}^*(D) \mid H \supseteq A \text{ and } H \supseteq B\} \supseteq \inf(A,B)$. Since $\bigcap\{zD \mid 0 \neq z \in K,\ zD \supseteq A \text{ and } zD \supseteq B\} = (A+B)^v$, we conclude that $\inf(A,B) = (A+B)^v$.

Next we show that $\sup(A,B) = A \cap B$ for all $A, B \in \mathrm{Inv}^*(D)$. It is easy to verify that $(\sup(A,B)A^{-1}B^{-1})^* = \sup(A^{-1}, B^{-1})$ and that $\sup(A^{-1}, B^{-1}) = (\inf(A,B))^{-1}$ since, for all $A, B \in \mathrm{Inv}^*(D)$, $A^{-1}, B^{-1} \in \mathrm{Inv}^*(D)$ and $A \leq B$ if and only if $A^{-1} \geq B^{-1}$. Therefore we have $(\sup(A,B)\inf(A,B))^* = (AB)^*$ or, equivalently, $(\sup(A,B)\inf(A,B)A^{-1}B^{-1})^* = D$ for all $A, B \in \mathrm{Inv}^*(D)$. Replacing $\inf(A,B)$ by $(A+B)^v$ and applying the v-operation on both sides, we have $(\sup(A,B)(A+B)^v A^{-1}B^{-1})^v = ((\sup(A,B)(A+B)^v A^{-1}B^{-1})^*)^v = D$. So $D = (\sup(A,B)(A+B)^v A^{-1}B^{-1})^v = (\sup(A,B)(A+B)A^{-1}B^{-1})^v = (\sup(A,B)$ $(B^{-1}+A^{-1}))^v$, forcing $\sup(A,B) = (B^{-1}+A^{-1})^{-1} = A^v \cap B^v = A \cap B$ (since A and B are v-ideals, as observed above).

In order to show that D is a $(*, v)$–Prüfer domain, we start by showing that if F is a nonzero two generated ideal of D, then F^v is $*$-invertible. Let $F := aD + bD$, with $a, b \in D$ and $ab \neq 0$. We know that $abD = (\inf(aD, bD) \sup(aD, bD))^* = ((aD \cap bD)(aD+bD)^v)^*$, and thus $D = ((b^{-1}D \cap a^{-1}D)(aD+bD)^v)^*$, i.e., $(aD+bD)^v$ is $*$-invertible. The general case can be obtained by induction. Let F be a nonzero ideal of D generated by $n \geq 2$ elements and let $c \in D \setminus F$. By the previous arguments, we have that the ideal $F + cD$, generated by $(n+1)$ elements, is such that $(F + cD)^v = \inf(F, cD) \in \mathrm{Inv}^*(D)$.

(c) The "only if part" follows immediately from (a) and (b). For the "if part", let D be a $(*, v)$–Prüfer domain and let $F := aD + bD$, where $0 \neq a, b \in D$. Since $aD, bD \in \mathrm{Inv}^*(D)$, by assumption $F^v = (aD + bD)^v = (aD + bD)^* = F^*$. Therefore $D = (F^v F^{-1})^* = (F^* F^{-1})^* = (FF^{-1})^*$. The conclusion is an immediate consequence of Theorem 2.2 ((ii)⇒(i)). □

With the proof of Theorem 2.11, we have amply established the existence of a sort of GCD in $\mathrm{Inv}^*(D)$ for each pair of elements of $\mathrm{Inv}^*(D)$ when D is a $*$–Prüfer domain. However, the results are in terms of inf and sup of elements of the lattice-ordered group $\mathrm{Inv}^*(D)$. We now establish the existence of the ($*$-invertible $*$-ideal) GCD of $*$-invertible integral $*$-ideals using purely ring-theoretic means.

Before we do that, let us note that old masters such as van der Waerden regarded an integral ideal A of an integral domain D as a *divisor* of another ideal B of D if $A \supseteq B$, extending the well known property that, for $0 \neq a, b \in D$, $aD \supseteq bD$ if and only if $a|b$. In turn, the ideal B could be termed as a *multiple* of the ideal A. Now, given two integral ideals A, B of D, the ideal $A + B$ has the property that $A + B$ is a divisor of A and B and any common divisor C of A and B contains A and B, and hence $A + B$. In other words, any common divisor of A and B is a divisor of $A + B$. Thus $A + B$ fitted the bill as *the greatest common divisor* of A and B. In a similar fashion $A \cap B$ was regarded as *the least common multiple* of A and B [39, Vol. 2, page 119].

Now the trouble with this approach is that it is too general and so can only work in a very strict environment such as a PID or a Dedekind domain, the kind of rings the "ancients" worked with. Besides, there were other ways of looking at GCD's, such as generalizations of the GCD of two integers, which is an integer. Also, if we are dealing with $\mathfrak{I}^*(D)$, the set of integral $*$-ideals of D, and we want the GCD of two $*$-ideals $A, B \in \mathfrak{I}^*(D)$ to belong to $\mathfrak{I}^*(D)$, then in general $A + B$ would not deliver the "greatest common divisor" in $\mathfrak{I}^*(D)$, in the language of van der Waerden. So to find the GCD of $A, B \in \mathfrak{I}^*(D)$ inside $\mathfrak{I}^*(D)$, we need to consider $(A + B)^*$, which may be a proper divisor of $A + B$. In other words, we need GCD's from a pre-assigned set. Of course, we also need our GCD to be something like the GCD in Prüfer domains that we defined in the introduction. Having established what we want, we state a GCD-type characterization of $*$–Prüfer domains.

Proposition 2.12. *Let D be an integral domain, $*$ a star operation on D, and let $\mathrm{Inv}^*_{\mathfrak{I}}(D)$ be the set of integral $*$-invertible $*$-ideals of D.*

Assume that D is a $\ast$–Prüfer domain. If $A, B \in \mathrm{Inv}^\ast_{\mathfrak{J}}(D)$, then there is a unique ideal $C := (A+B)^\ast \in \mathrm{Inv}^\ast_{\mathfrak{J}}(D)$ such that $A = (A_1C)^\ast$, $B = (B_1C)^\ast$, where $(A_1+B_1)^\ast = D$. Conversely, if D is an integral domain such that for all $A, B \in \mathrm{Inv}^\ast_{\mathfrak{J}}(D)$, there is a unique ideal $C \in \mathrm{Inv}^\ast_{\mathfrak{J}}(D)$ such that $A = (A_1C)^\ast$, $B = (B_1C)^\ast$, where $(A_1+B_1)^\ast = D$, then D is a $\ast$–Prufer domain.

Proof. Let us first note that if I is an integral $\ast$-invertible $\ast$-ideal of a $\ast$–Prüfer domain D and J is an ideal contained in I, then $J^\ast = (IH)^\ast$ for some integral ideal H of D. This follows since $J \subseteq I$ implies $JI^{-1} =: H \subseteq D$. Now, multiplying both sides of the equality $JI^{-1} = H$ by I and applying $\ast$, we get $J^\ast = (IH)^\ast$.

Next, let $C := (A+B)^\ast$. Since D is a $\ast$–Prüfer domain, C is $\ast$-invertible (Theorem 2.2 ((i)⇒(vi)). Now as $A, B \subseteq C$, we have $(C^{-1}A)^\ast =: A_1 \subseteq D$ and $(C^{-1}B)^\ast =: B_1 \subseteq D$ so that $A = A^\ast = (CA_1)^\ast$ and $B = B^\ast = (CB_1)^\ast$. Now $C = ((CA_1)^\ast + (CB_1)^\ast)^\ast = (C(A_1+B_1))^\ast$. Multiplying both sides of the equality $C = (C(A_1+B_1))^\ast$ by C^{-1} and applying $\ast$, we get $(A_1+B_1)^\ast = D$.

The proof of the converse entails showing that for all $A, B \in \mathrm{Inv}^\ast_{\mathfrak{J}}(D)$, $(A+B)^\ast \in \mathrm{Inv}^\ast_{\mathfrak{J}}(D)$ (Theorem 2.2 (ii)⇒(i))). By assumption, we have $(A+B)^\ast = ((CA_1)^\ast + (CB_1)^\ast)^\ast = (C(A_1+B_1)^\ast)^\ast = C^\ast = C \in \mathrm{Inv}^\ast_{\mathfrak{J}}(D)$. □

As a consequence of Theorem 2.11, we have

Corollary 2.13. *Let D be an integral domain.*

(a) *D is a v–domain (respectively, a PvMD, a generalized GCD domain) if and only if $\mathrm{Inv}^v(D)$ (respectively, $\mathrm{Inv}^t(D)$, $\mathrm{Inv}(D)$) is a lattice-ordered abelian group (under $\leq$ defined in Theorem 2.11).*
(b) *Assume that $\mathrm{Inv}(D)$ is lattice-ordered and that $A+B = (A+B)^v$ for all $A, B \in \mathrm{Inv}(D)$. Then D is a Prüfer domain and, clearly, $\mathrm{Inv}(D) = \boldsymbol{f}(D)$.*

Proof. (a) The "if part" is a consequence of Theorem 2.11 (b) and Example 2.1. The "only if part" for v–domains and PvMD's follows from Theorem 2.11 (a) (and from Example 2.1). If D is a generalized GCD domain ($= (d,v)$–Prüfer domain), then it is well known that $\mathrm{Inv}(D)$ is lattice-ordered and $\mathrm{Inv}(D) = \{F^v \mid F \in \boldsymbol{f}(D)\}$ [2, Theorem 1 ((1)⇒(4))].

(b) is an easy consequence of Theorem 2.11 (c). □

Remark 2.14. The "PvMD part" of Corollary 2.13 gives back a classical characterization of these domains (see e.g. [29, page 55], [22, page 717] and [44, Proposition 2.4]). As we mentioned above, the "GGCD part" is well known [2]. On the other hand, the "v–domain part" is new.

Corollary 2.15. *Let D be an integral domain.*

(a) *Assume that D is a CICD. Then D is a v–domain with $\boldsymbol{F}^v(D) = \mathrm{Inv}^v(D)$, and moreover, $\boldsymbol{F}^v(D)$ is a complete lattice-ordered abelian group (under the order $\leq$ defined by $I \leq J :\Leftrightarrow I \supseteq J$ for all $I, J \in \boldsymbol{F}^v(D)$).*
(b) *Assume that D is a (t,v)–Dedekind domain (Example 1.11). Then D is a completely integrally closed PvMD with $\boldsymbol{F}^v(D) = \mathrm{Inv}^t(D)$, and moreover, $\boldsymbol{F}^v(D)$ is a complete lattice-ordered abelian group (under the order $\leq$ defined above).*

(c) *Assume that D is a pseudo-Dedekind domain (i.e., a (d,v)–Dedekind domain, Example 1.10). Then D is a generalized GCD domain (i.e., a (d,v)–Prüfer domain, Example 2.1) with $\boldsymbol{F}^v(D) = \mathrm{Inv}(D)$, and moreover, $\boldsymbol{F}^v(D)$ is a complete lattice-ordered abelian group (under the order $\leq$ defined above).*

Proof. (a) Since a CICD ($=$ v–CICD, by Example 1.9) is a v–domain, $\mathrm{Inv}^v(D)$ is a lattice-ordered group from Corollary 2.13. For the completeness, recall that a lattice-ordered group G is said to be complete if every nonempty subset of G that is bounded from below has a greatest lower bound (or, equivalently, if every nonempty subset of G that is bounded from above has a least upper bound). Note that when D is completely integrally closed, the lattice-ordered group $\mathrm{Inv}^v(D)$ coincides with $\boldsymbol{F}^v(D)$. Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a nonempty collection of ideals in $\mathrm{Inv}^v(D)$ bounded below in $\mathrm{Inv}^v(D)$, that is, there is $J \in \mathrm{Inv}^v(D)$ such that $A_\lambda \geq J$ for all $\lambda \in \Lambda$. In other words, $A_\lambda \subseteq J$ for all $\lambda \in \Lambda$. Then $\sum_\lambda A_\lambda \subseteq J$, and hence $(\sum_\lambda A_\lambda)^v \subseteq J^v = J$. This gives $A_\lambda \subseteq (\sum_\lambda A_\lambda)^v \subseteq J$, which translates (in $(\mathrm{Inv}^v(D), \leq)$) to $A_\lambda \geq (\sum_\lambda A_\lambda)^v \geq J$. Since $(\sum_\lambda A_\lambda)^v \in \boldsymbol{F}^v(D) = \mathrm{Inv}^v(D)$, we conclude that $(\sum A_\lambda)^v$ is a lower bound and, more precisely, it is easy to verify that $(\sum A_\lambda)^v$ is in fact the greatest lower bound of $\{A_\lambda \mid \lambda \in \Lambda\}$.

(b) In this case, it is clear that $\boldsymbol{F}^v(D) = \mathrm{Inv}^t(D)$. Since "$t$-invertible" implies "$v$-invertible", we have that $\boldsymbol{F}^v(D) = \mathrm{Inv}^v(D)$ ($= \mathrm{Inv}^t(D)$), and thus D is completely integrally closed. Moreover a (t,v)–CICD is a particular (t,v)–Prüfer domain, which is a PvMD (Example 2.1). We conclude by (a).

(c) The proof is similar: in this case, $\boldsymbol{F}^v(D) = \mathrm{Inv}(D)$ and hence, in particular, $\boldsymbol{F}^v(D) = \mathrm{Inv}^v(D) = \mathrm{Inv}(D)$. □

Note that the converse of part (c) of the previous corollary is also true. In fact, it is known that an integral domain is pseudo-Dedekind if and only if $\mathrm{Inv}(D)$ is a complete lattice-ordered abelian group [6, Theorem 2.8].

Corollary 2.16. *Let D be an integral domain with quotient field K. Assume that D is a pseudo-principal domain (i.e, A^v is principal for all $A \in \boldsymbol{F}(D)$). Then D is a GCD domain such that $\boldsymbol{F}^v(D)$ is isomorphic to the group of divisibility of D. Moreover, $\boldsymbol{F}^v(D)$ is a complete lattice-ordered abelian group (under the order $\leq$ defined by $I \leq J$ if $I \supseteq J$).*

Proof. Recall that the group of divisibility of D is the multiplicative abelian group $G(D) := K^\times/\mathcal{U}(D)$, where $K^\times := K \setminus \{0\}$ and $\mathcal{U}(D)$ is the group of units of D, endowed with a partial order defined by $x\mathcal{U}(D) \leq y\mathcal{U}(D)$ if $yx^{-1} \in D$. It is easy to see that the group of divisibility of D is canonically isomorphic to $\mathrm{Prin}(D) := \{zD \mid 0 \neq z \in K\}$ with a partial order $\leq$ defined by $xD \leq yD$ if $xD \supseteq yD$ [21, page 172].

Since a GCD domain is characterized by the fact that F^v is principal for all $F \in \boldsymbol{f}(D)$ (cf. for instance [36, Proposition 1.19] or [1, Remark 2.2]), it is straightforward from the assumption that D is a GCD domain and $\boldsymbol{F}^v(D) = \mathrm{Prin}(D)$, with identical definitions of partial order.

Next, for every nonempty subset $\mathfrak{S}$ of principal fractional ideals of D bounded below under $\leq$, let A be the fractional ideal of D generated by the ideals in $\mathfrak{S}$. Then A^v is principal by assumption, and $A^v \supseteq sD$ for all $sD \in \mathfrak{S}$. Thus $A^v \leq sD$ (in $(\boldsymbol{F}^v(D), \leq)$) for all $sD \in \mathfrak{S}$. It is routine to show that the principal fractional ideal A^v is in fact the greatest lower bound of the family $\mathfrak{S}$. □

References

[1] D.D. Anderson, *GCD domains, Gauss' Lemma and contents of polynomials*, *in* Non-Noetherian Commutative Ring Theory, 1–31, Math. Appl., vol. 520, Kluwer Acad. Publ., Dordrecht, 2000.

[2] D.D. Anderson and D.F. Anderson, *Generalized GCD domains*, Comment. Math. Univ. St. Pauli 28 (1979), 215–221.

[3] D.D. Anderson and D.F. Anderson, *Some remarks on star operations and the class group*, J. Pure Appl. Algebra 51 (1988), 27-33.

[4] D.D. Anderson and S.M. Clarke, *Star-operations that distribute over finite intersections*, Comm. Algebra 33 (2005), 2263–2274.

[5] D.D. Anderson and S.J. Cook, *Two star-operations and their induced lattices*, Comm. Algebra 28 (2000), 2461–2475

[6] D.D. Anderson and B.G. Kang, *Pseudo-Dedekind domains and divisorial ideals in* $R[X]_T$, J. Algebra 122 (1989), 323–336.

[7] D.D. Anderson, J. Mott and M.Zafrullah, *Some quotient based characterizations of domains of multiplicative ideal theory*, Bull. Math. Ital. 3-B (1989), 455–476.

[8] D.F. Anderson, *A general theory of class groups*, Comm. Algebra 16 (1988), 805–847.

[9] P.M. Cohn, *Bezout rings and their subrings*, Proc. Cambridge Philos. Soc. 64 (1968), 251–264.

[10] D.M. Dribin, *Prüfer ideals in commutative rings*, Duke Math. J. 4 (1938), 737–751.

[11] S. El Baghdadi, M. Fontana and G. Picozza, *Semistar Dedekind domains*, J. Pure Appl. Algebra 193 (2004), 27–60.

[12] S. El Baghdadi, S. Gabelli and M. Zafrullah, *Unique representation domains II,* J. Pure Appl. Algebra (to appear).

[13] J. Escoriza and B. Torrecillas, *Divisorial multiplication rings*, in *Rings, Modules, and Abelian Groups,* (A. Facchini, E. Houston, and L. Salce Editors), Lecture Notes Pure Appl. Math., Marcel Dekker, New York, 2004, vol. 236, 161–176.

[14] M. Fontana, P. Jara and E. Santos, *Prüfer star-multiplication domains and semistar operations*, J. Algebra Appl. 2 (2003), 21–50.

[15] M. Fontana and J. Huckaba, *Localizing systems and semistar operations*, *in* Non-Noetherian Commutative Ring Theory, 169–197, Math. Appl., vol. 520, Kluwer Acad. Publ., Dordrecht, 2000.

[16] M. Fontana and K.A. Loper, *Nagata rings, Kronecker function rings and related semistar operations*, Comm. Algebra 31 (2003), 4775–4806.

[17] M. Fontana and G. Picozza, *Semistar invertibility on integral domains*, Algebra Colloq. 12 (2005), 645–664.

[18] M. Fontana and G. Picozza, *Prüfer $\star$-multiplication domains and $\star$-coherence*, Ricerche Mat. 55 (2006), 145–170

[19] L. Fuchs and L. Salce, Modules over non-Noetherian domains, Math. Surveys and Monographs, Vol. 84, Am. Math. Soc., 2001.

[20] J.M. García, P. Jara and E. Santos, *Prüfer $*$-multiplication domains and torsion theories*, Comm. Algebra 27 (1999), 1275–1295.

[21] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, New York, 1972.

[22] M. Griffin, *Some results on v–multiplication rings*, Canad. J. Math. 19 (1967), 710–722.

[23] F. Halter-Koch, Ideal Systems. An Introduction to Multiplicative Ideal Theory, M. Dekker, New York, 1998.

[24] F. Halter-Koch, *Characterization of Prüfer multiplication monoids and domains by means of spectral localizing systems*, Monatsh. Math. 139 (2003), 19–31.

[25] F. Halter-Koch, Private communication, October 2007.

[26] W. Heinzer and J. Ohm, *An essential domain which is not a v-multiplication ring*, Canad. Math. J. 25 (1973), 856–861.

[27] E. Houston, S. Malik, and J. Mott, *Characterizations of ∗-multiplication domains*, Canad. Math. Bull. 27(1984), 48-52.

[28] E. Houston and M. Zafrullah, *Integral domains in which every t-ideal is divisorial*, Michigan Math. J. 35 (1988), 291–300.

[29] P. Jaffard, Les Systèmes d'Idéaux, Dunod, Paris, 1960.

[30] M.D. Larsen and P.J. McCarthy, Multiplicative Theory of Ideals, Academic Press, New York, 1971.

[31] R. Matsuda and A. Okabe, *On an AACDMZ question*, Math. J. Okayama Univ. 35 (1993), 41–43.

[32] A. Mimouni, *Note on the star operations over polynomial rings*, (preprint).

[33] J. Mott, B. Nashier and M. Zafrullah, *Contents of polynomials and invertibility*, Comm. Algebra 18 (1990), 1569–1583

[34] M. Nagata, *On Krull's conjecture concerning valuation rings*, Nagoya Math. J. 4 (1952), 29–33.

[35] M. Nagata, *Corrections to my paper "On Krull's conjecture concerning valuation rings"*, Nagoya Math. J. 9 (1955), 209–212.

[36] G. Picavet, *About GCD domains*, Lecture Notes Pure Appl. Math., Marcel Dekker, New York, 1999, vol. 205, 501–519.

[37] H. Prüfer, *Untersuchungen über Teilbarketiseigenschaftenin Körpern*, J. Reine Angew. Math. 168 (1932), 1–36.

[38] P. Ribenboim, *Sur une note de Nagata relative à un problème de Krull*, Math. Z. 64 (1956), 159–168.

[39] B.L. van der Waerden, Algebra (Fifth Ed.), F. Ungar, New York, 1970.

[40] F.G Wang and R.L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra 25 (1997), 1285–1306.

[41] M. Zafrullah, *On a property of pre-Schreier domains*, Comm. Algebra 15(1987), 1895–1920.

[42] M. Zafrullah, *On generalized Dedekind domains*, Mathematika, 33 (1986), 285–296.

[43] M. Zafrullah, *Ascending chain condition and star operations*, Comm. Algebra 17 (1989), 1523–1533.

[44] M. Zafrullah, *Putting t-invertibility to use*, in Non-Noetherian Commutative Ring Theory, 429–457, Math. Appl., vol. 520, Kluwer Acad. Publ., Dordrecht, 2000.

[45] M. Zafrullah, *A Question-Answer Session on v-domains*, http://www.lohar.com.

D.D.A.: Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA.
E-mail address: dan-anderson@uiowa.edu

D.F.A.: Department of Mathematics, University of Tennessee at Knoxville, Knoxville, TN 37996-1300, USA.
E-mail address: anderson@math.utk.edu

M.F.: Dipartimento di Matematica, Università degli Studi "Roma Tre", 00146 Rome, Italy.
E-mail address: fontana@mat.uniroma3.it

M.Z.: Department of Mathematics, Idaho State University, Pocatello, ID 83209, USA.
E-mail address: mzafrullah@usa.net