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Globalizing Local Properties of Prüfer Domains*

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1. INTRODUCTION

A property Π is said to be a *globalizing property* for a class $\mathscr{C}$ of integral domains if for each integral domain R in $\mathscr{C}$, the property Π holds for the localizations R_M of R at each maximal ideal M of R. For example, if Π is the property that each non-zero ideal is contained in at most finitely many maximal ideals, then clearly Π holds for every local integral domain, and hence is a globalizing property for the class of integral domains. In this paper we focus on several globalizing properties and indicate how these properties arise in the consideration of different classes of Prüfer domains. We are particularly interested in when local divisoriality and invertibility properties can be globalized.

Globalizing properties allow information about the localizations of an integral domain to be transferred to the domain itself. For example, it will be shown in Section 4 that an integrally closed domain R is SV-stable (i.e., every ideal I is an invertible ideal of $(I : I)$) if and only if each non-zero ideal of R is contained in at most finitely many maximal ideals of R and R_M is locally SV-stable for all maximal ideals M of R. This reduces consideration of integrally closed SV-stable domains to the local case, which, in this instance, is well understood: A local domain is integrally closed and SV-stable if and only if it is a valuation domain containing no non-zero idempotent prime ideals.

H-local domains provide a prominent example of a globalizing property. An integral domain R is defined to be *h-local* provided each non-zero

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ideal of R is contained in at most finitely maximal ideals of R and each non-zero prime ideal of R is contained in a unique maximal ideal of R. H-local domains arise in many contexts and have a surprising variety of characterizations. Examples of these domains include Noetherian domains of Krull dimension one, almost maximal domains, and integral domains for which every ideal is divisorial (i.e., every ideal I has the property that $I = (I^{-1})^{-1}$).

In Section 3 we give different characterizations of h-local integral domains and then restrict to h-local Prüfer domains. If I is a finitely generated ideal of an integral domain R, then for each maximal ideal M of R, it is the case that $(I^{-1})_M = (I_M)^{-1}$. Thus every Noetherian domain has the property that the localization of the dual of an ideal is the dual of the localization, but this fact does not hold true for arbitrary integral domains. Since we are interested in globalizing divisoriality and invertibility properties of Prüfer domains and both of these properties involve the dual of an ideal, it makes sense to determine when the dual of an ideal localizes. It has been shown recently that if an integral domain R is h-local, then every ideal I of R satisfies $(I^{-1})_M = (I_M)^{-1}$. We show in Section 3 that the converse is true for Prüfer domains.

We also consider several weaker globalizing properties for Prüfer domains, including (##) and the separation property. In Section 2 we characterize the separation property in different ways, showing how this property globalizes local divisoriality properties of non-maximal prime ideals of Prüfer domains. This is used to show that if the prime spectrum of a Prüfer domain R is Noetherian, then the duals of radical ideals of R localize, in the sense discussed above. We also indicate in Section 3 how for Prüfer domains the requirement that each ideal be contained in at most finitely many maximal ideals in the h-local criterion can be replaced by weaker properties such as (##) and the radical trace property.

There has been recent interest in Prüfer domains having no non-zero idempotent prime ideals. These domains are known as *strongly discrete* Prüfer domains. The richness of this class of domains is due to their strong local properties. In fact, the property of being strongly discrete is solely a local property. In Sections 4 and 5, we characterize different classes of strongly discrete Prüfer domains using some of the globalizing properties developed in the previous sections. The characterization of integrally closed SV-stable domains has been mentioned above. If R is a strongly discrete valuation domain and R' is an overring of R, then R' has the property that every prime ideal of R' is a divisorial ideal of R'. This property in fact characterizes strongly discrete valuation domains. We show the strongly discrete Prüfer domains which preserve this property are precisely those satisfying (##). This class is well known to coincide with the class of generalized Dedekind domains, for which we give several other characterizations in terms of stability and divisoriality properties.

It is not hard to show that strongly discrete valuation domains can be characterized by the property that each prime ideal P is a cancellation ideal of $(P:P)$. We show that the class of Prüfer domains that possess this property is the same as the class of strongly discrete Prüfer domains that have the separation property. The remaining case, h-local strongly discrete Prüfer domains, has been classified by Bazzoni and Salce. An integral domain R is said to be a *divisorial domain* if every ideal of R is divisorial. If every overring of R is divisorial, then R is a *totally divisorial domain*. Bazzoni and Salce have shown that an integrally closed domain is totally divisorial if and only if it is an h-local strongly discrete Prüfer domain.

In the final section we focus first on a representation theorem for ideals of totally divisorial domains. S. Gabelli and N. Popescu, in a recent paper, proved a representation theorem for divisorial ideals of generalized Dedekind domains. They showed that a Prüfer domain R is a generalized Dedekind domain if and only if every divisorial ideal of R can be written as a product of an invertible ideal and finitely many comaximal prime ideals. Using the results of the previous sections, we show that every ideal of a Prüfer domain R is of the form $JP_1P_2 \cdots P_n$ for some invertible ideal J of R and pairwise comaximal prime ideals $P_1, P_2, \dots, P_n$ of R if and only if R is strongly discrete and h-local.

To close the final section, we list several existence results for the integral domains studied in Section 4. A. Facchini has shown that any Noetherian tree with least element can be realized as the prime spectrum of a generalized Dedekind domain. Using his result and the results of Section 4, we indicate how successively restricting Noetherian trees with least element guarantees that they can be realized by integrally closed SV-stable domains and integrally closed totally divisorial domains.

I thank Professor Gabelli for making a preprint of [10] available to me. I am also indebted to the referee for several suggestions that have helped streamline the paper.

Terminology and notation are standard throughout, with a few exceptions that are noted below. By *integral domain* we mean a commutative ring with identity having no zero-divisors. To avoid vacuous assertions, all integral domains are assumed not to be fields. If R is an integral domain with quotient field Q and X and Y are submodules of Q, then $(Y:X)$ will denote $\{q \in Q : qX \subseteq Y\}$. We will sometimes write $E(X)$ for $(X:X)$ when X is a submodule of Q and we wish to emphasize the ring structure of $(X:X)$. The choice of notation is explained by the fact that $E(X)$ can be identified with $\mathrm{End}_R(X)$. An ideal I of R is *SV-stable* (in the sense of Sally and Vasconcelos [22]) if I is an invertible ideal of $E(I)$. The set of maximal ideals of an integral domain R is denoted $\mathrm{Max}(R)$; the set of prime ideals, Spec(R). We will often have occasion to refer to the ring $\cap\{R_N : N \in \text{Max}(R) \text{ and } N \neq M\}$, where M is some maximal ideal of R. Following Matlis in [19], we give this ring a name, "[M]." We will view R_M as a submodule of the quotient field Q of R and if X is a submodule of the quotient field Q, then we will often write X_M for XR_M. Finally, we write $\subset$ for proper inclusion.

For more on generalized Dedekind domains, SV-stable domains, strongly discrete Prüfer domains, and integral domains satisfying (##), see the recent monograph [6].

2. (##) AND RELATED PROPERTIES

An integral domain R is said to satisfy (#) if for any two distinct subsets Δ_1 and Δ_2 of Max(R), it is the case that $\bigcap_{M \in \Delta_1} R_M \neq \bigcap_{N \in \Delta_2} R_N$. If every overring of R satisfies (#), then R is said to satisfy (##). If M is a maximal ideal of R, define $[M] = \{R_N : N \in \text{Max}(R) \text{ and } N \neq M\}$. Then an integral domain R satisfies (#) if and only if $[M] \nsubseteq R_M$ for each $M \in \text{Max}(R)$. Prüfer domains satisfying (#) were shown in Theorem 1 of [12] to be precisely those Prüfer domains which can be represented uniquely as an intersection of a family $\{V_\alpha\}$ of valuation overrings of R having no containment relations among the V_α's. Thus (#) is a restriction on how R can be assembled as an intersection of valuation domains.

Before restricting to Prüfer domains, we note the following interpretation of (#) in terms of divisorial ideals, stating first a simple lemma that will be needed often.

LEMMA 2.1. *Let R be an integral domain with quotient field Q. If M is a maximal ideal of R and $q \in Q$, then $q \in R_M$ if and only if $R \cap Rq^{-1} \nsubseteq M$.*

Proof. Assume first that $q \in R_M$. Then there exist $a, b \in R$ such that $b \notin M$ and $q = a/b$. Therefore, $b \in R \cap Rq^{-1}$ and $b \notin M$. Conversely, suppose that there exists $b \in R \cap Rq^{-1}$ such that $b \notin M$. Then there exists $a \in R$ such that $b = aq^{-1}$. It follows that $q \in R_M$. ∎

PROPOSITION 2.2. *The following are equivalent for an integral domain R with maximal ideal M.*

(1) $[M] \nsubseteq R_M$.

(2) *There exists a divisorial ideal D of R such that the only maximal ideal containing D is M.*

(3) *There exists an ideal J of R such that $J^{-1} \neq R$ and the only maximal ideal containing J is M.*

Proof. (1) ⇒ (2). Assuming (1), there exists $q \in [M] \setminus R_M$. Define $D = R \cap Rq^{-1}$. By Lemma 2.1, the only maximal ideal containing the divisorial ideal D is M.

(2) ⇒ (3). Assuming (2), there exists a divisorial ideal D of R such that the only maximal ideal containing D is M. Since D is divisorial, it cannot be the case that $D^{-1} = R$.

(3) ⇒ (2). Assuming (3), there exists an ideal J of R such that $(R:J) \neq R$. Since $(R:J) \neq R$ and M is the only maximal ideal containing J, then $(R:(R:J)) \subseteq M$, and M is the only maximal ideal containing the divisorial ideal $(R:(R:J))$.

(2) ⇒ (1). Assuming (2), there exists a divisorial ideal D such that the only maximal ideal containing D is M. Write $D = \bigcap Rq_\alpha$ for some collection $\{q_\alpha\}$ of elements of Q, the quotient field of R. Then there exists β such that $R \cap Rq_\beta$ is properly contained in R. Since the only maximal ideal containing D is M, it must be the case that M is the only maximal ideal of R containing $R \cap Rq_\beta$. By Lemma 2.1, $[M] \not\subseteq R_M$. ∎

It follows that a Prüfer domain R satisfies (#) if and only if for each maximal ideal M of R there exists a finitely generated ideal I of R such that the only maximal ideal containing I is M [11, Theorem 3]. Prüfer domains satisfying (##) admit a similar description.

LEMMA 2.3 [12, Theorem 3]. *A Prüfer domain R satisfies (##) if and only if for each prime ideal P of R, there exists a finitely generated ideal I contained in P such that each maximal ideal containing I contains P; equivalently, $\bigcap_\alpha R_{M_\alpha} \not\subseteq R_P$ for each non-zero prime ideal P and collection $\{M_\alpha\}$ of maximal ideals not containing P.*

As Lemma 2.3 suggests, whether or not a Prüfer domain satisfies (##) depends on the prime ideals of R. Proposition 2.5 will make this explicit. Another lemma is needed first.

LEMMA 2.4 [7, Corollary 4.15]. *If I is a radical ideal of a Prüfer domain R, then*

$$E(I) = \left(\bigcap_\alpha R_{P_\alpha}\right) \cap \left(\bigcap_\beta R_{M_\beta}\right),$$

where $\{P_\alpha\}$ is the collection of minimal prime ideals of I and $\{M_\beta\}$ is the collection of all maximal ideals of R not containing I.

PROPOSITION 2.5. *A Prüfer domain R satisfies (##) if and only if $E(P)$ satisfies (#) for every non-zero prime ideal P of R.*

Proof. Clearly if R satisfies (##), then so does $E(P)$ for every prime ideal P of R. To prove the converse, assume that $E(P)$ satisfies (#) for all $P \in \mathrm{Spec}(R)$. Since $E(M) = R$ for every maximal ideal M of R, this means R satisfies (#). Suppose P is a non-zero non-maximal prime ideal of R and $\{M_\alpha\}$ is the collection of maximal ideals of R not containing P. By Lemma 2.4, $E(P) = R_P \cap (\bigcap_\alpha R_{M_\alpha})$. By Theorem 1 of [12], $E(P)$ can be written uniquely as an intersection of valuation overrings of V_β of $E(P)$ having no containment relations between the V_β's. Therefore, it cannot be the case that $\bigcap_\alpha R_{M_\alpha} \subseteq R_P$. This fact and the fact that R satisfies (#) implies by Lemma 2.3 that R satisfies (##). ▮

In the next section we characterize those Prüfer domains R for which every ideal I of R has the property that $(R : I)_M = (R_M : I_M)$ for all $M \in \operatorname{Max}(R)$. This property, as will be seen, is considerably stronger than (##). In the present section we show that a weaker version of this localization property is valid for Prüfer domains with the radical trace property, namely, that the duals of products of prime ideals localize. From this it will follow that the duals of radical ideals of Prüfer domains with Noetherian prime spectrum localize. (Recall that an integral domain R has the *radical trace property* provided that for every ideal I of R, II^{-1} is either a radical ideal of R or is R itself.)

Non-maximal prime ideals of valuation domains are divisorial, and every overring of a valuation domain inherits this property. We characterize in Lemma 2.7 those Prüfer domains which preserve this property. An integral domain R satisfies the *separation property* if for each pair of distinct prime ideals P and Q of R such that $P \subset Q$, there exists a finitely generated ideal I of R such that $P \subseteq I \subseteq Q$. In [15] it is shown that R has the separation property if and only if $E(P)Q = E(P)$ for all prime ideals P and Q of R such that $P \subset Q$.

LEMMA 2.6 [15, Theorems 3.2 and 3.8 and Proposition 3.9]. *If P is a non-maximal prime ideal of a Prüfer domain R, then $P^{-1} = E(P)$.*

LEMMA 2.7. *The following are equivalent for a Prüfer domain R.*

(1) *R has the separation property.*

(2) *If R' is an overring of R, then every non-zero non-maximal prime ideal of R' is a divisorial ideal of R'.*

(3) *For each non-zero prime ideal P of R, non-zero non-maximal prime ideals of $E(P)$ are divisorial.*

(4) *For each non-zero prime ideal P of R, P is a maximal ideal of $E(P)$.*

(5) *For each non-zero prime ideal P of R, $(P : P)_M = (P_M : P_M)$ for all $M \in \operatorname{Max}(R)$.*

(6) *For each non-zero radical ideal I of R, $(I : I)_M = (I_M : I_M)$ for all $M \in \operatorname{Max}(R)$.*

Proof. (1) ⇔ (4). This follows from Lemma 10 and Theorem 11 of [5].

(1) ⇒ (5). Let P be a non-zero prime ideal of R. It suffices to check that $(P : P)_M = (P_M : P_M)$ for maximal ideals M containing P. Let M be

such a maximal ideal and note that $E(P)R_M$ is a valuation ring contained in R_P. Thus there exists a prime ideal Q such that $P \subseteq Q \subseteq M$ and $E(P)R_M = R_Q$. If $Q \neq P$, then by (1), $E(P)Q = E(P)$. Hence $QR_Q = R_Q$, a contradiction.

(5) ⇒ (6). Assume (5) and let I be a non-zero radical ideal of R. By Lemma 2.4, $(I : I) = (\bigcap_\alpha R_{P_\alpha}) \cap (\bigcap_\beta R_{M_\beta})$, where $\{P_\alpha\}$ is the collection of minimal prime ideals of I and $\{M_\beta\}$ is the collection of all maximal ideals of R not containing P. Suppose first that $N \in \{M_\beta\}$. Then $R_N \subseteq (I : I)_N \subseteq (I_N : I_N) = R_N$. Suppose next that $N \in \operatorname{Max}(R)$ but $N \notin \{M_\alpha\}$. Since the set of prime ideals contained in N is linearly ordered, it is the case that there exists a unique α_0 such that $P_{\alpha_0} \in \{P_\alpha\}$ and $P_{\alpha_0} \subseteq N$. Write $N_{\alpha_0} = N$. For each $\alpha \neq \alpha_0$, there is a maximal ideal N_α containing P_α. It follows that if $\alpha \neq \beta$, then $N_\alpha \neq N_\beta$ and for all $\alpha \neq \alpha_0$, no N_α contains P_{α_0}. Defining $P = P_{\alpha_0}$, we have $(I : I)_N = R_P \cap (\bigcap_{\alpha \neq \alpha_0} R_{P_\alpha})_N \cap (\bigcap_\beta R_{M_\beta})_N$. By (5) and Lemma 2.4, $R_P = (P : P)_N \subseteq (\bigcap_{\alpha \neq \alpha_0} R_{N_\alpha})_N \cap (\bigcap_\beta R_{M_\beta})_N \subseteq (\bigcap_{\alpha \neq \alpha_0} R_{P_\alpha})_N \cap (\bigcap_\beta R_{M_\beta})_N$. Therefore, $(I : I)_N = R_P$, and by Lemma 2.4, $(I_N : I_N) = R_P$. This proves (6).

(6) ⇒ (5). This is clear.

(5) ⇒ (1). Assuming (5), let P and Q be distinct prime ideals of R such that $P \subset Q$. To show that there exists a finitely generated ideal I of R such that $P \subseteq I \subseteq Q$, it suffices by the criterion mentioned before the statement of the lemma to show that $QE(P) = E(P)$. By assumption P is a non-maximal prime ideal of R. Thus, for every $M \in \operatorname{Max}(R)$ such that $P \subset M$, $(P : P)_M = R_P = R_P Q = Q(P : P)_M$, and so $E(P) = QE(P)$.

(2) ⇒ (3). This is clear.

(3) ⇒ (4). Assuming (3), suppose that P is a non-maximal prime ideal of R that is also a non-maximal prime ideal of $E(P)$. Then by Lemma 2.6, $(E(P) : P) = E(P)$. By (3), P is a divisorial ideal of $E(P)$, which implies $P = E(P)$, a contradiction that forces P to be a maximal ideal of $E(P)$.

(5) ⇒ (2). Note that the equivalence of (1) and (5) has already been established. Assume that (5) holds. Then R satisfies the separation property. This property is inherited by every overring of R; likewise statement (5) holds for every overring of R. Now let R' be an overring of R and assume P' is a non-zero non-maximal prime ideal of R'. Then by Lemma 2.6, $(R' : P') = (P' : P')$, and so $(R' : E(P'))_M \subseteq (R'_M : R'_{P'}) = P'R'_M$ for every $M \in \operatorname{Max}(R')$ containing P'. Hence $(R' : E(P')) = P'$ and (2) holds. ∎

If R is a Prüfer domain that satisfies (##), then R has the radical trace property [18, Corollary 25]. Furthermore, every Prüfer domain that has the radical trace property has the separation property [18, Theorem 27].

LEMMA 2.8. *If R is a Prüfer domain satisfying the radical trace property, then every non-idempotent prime ideal of R is SV-stable and divisorial.*

Proof. Assuming R has the radical trace property, let P be a non-idempotent prime ideal of R and suppose that P is not an invertible ideal of $E(P)$. If P is a maximal ideal of R, then $(P : P) = R$; otherwise, it follows from Lemma 2.7 and the fact that R has the separation property that P is a maximal ideal of $E(P)$. Thus $(R : P^2) = ((R : P) : P) = E(P)$ and $P^2(R : P^2) = P^2$, which is not a radical ideal of R. This contradiction implies P is an invertible ideal of $E(P)$. If P is maximal, then it is clearly divisorial; if P is non-maximal, then P is divisorial by Lemma 2.7. ∎

THEOREM 2.9. *If R is a Prüfer domain having the radical trace property and $P_1, P_2, \ldots, P_n$ are non-zero prime ideals of R, then $(R : P_1P_2 \cdots P_n)_M = (R_M : (P_1P_2 \cdots P_n)_M)$ for all $M \in \mathrm{Max}(R)$.*

Proof. The proof is by induction on n. Suppose that R is a Prüfer domain having the radical trace property and that N is a maximal ideal of R. If $N = N^2$, then N is not an invertible ideal of R and N_N is not an invertible ideal of R_N. Thus $(R_N : N_N) = R_N = (R : N)_N$. Otherwise, if $N \neq N^2$, Lemma 2.8 applies and N must be an invertible ideal of R. Hence the conclusion holds for all maximal ideals of R. Since R has the separation property it follows that if P is a non-zero non-maximal prime ideal of R and $M \in \mathrm{Max}(R)$, then by Lemma 2.7 and Lemma 2.6, $(R : P)_M = (P : P)_M = (P_M : P_M) = (R_M : P_M)$. Hence the claim holds true for $n = 1$.

Suppose next that the statement of the corollary is true for all products of no more than n prime ideals. Let $P_1, P_2, \ldots, P_n$ be non-zero prime ideals of R, $I = P_1P_2 \cdots P_n$, and $N \in \mathrm{Max}(R)$. Suppose first that N is invertible and that $M \in \mathrm{Max}(R)$. Then $(R : IN)_M = ((R : I) : N)_M = ((R : I)_M : N_M) = ((R_N : I_N) : N_N) = (R_N : IN_N)$, by the induction hypothesis. Finally, if P is a non-invertible maximal ideal or a non-zero non-maximal prime ideal of R, then by Lemma 2.6, $(R : P) = E(P)$. Thus $(R : IP)_M = (E(P) : E(P)I)_M$. Since R is a Prüfer domain, $E(P)I$ must be a product of prime ideals of $E(P)$ [15, Proposition 2.4]. Therefore, since $E(P)$ has the radical trace property [18, Corollary 24], the induction hypothesis applies and $(E(P) : E(P)I)_M = (E(P)_M : E(P)I_M) = (E(P_M) : I_M) = ((R_M : P_M) : I_M) = (R_M : PI_M)$. By induction, the proof is complete. ∎

COROLLARY 2.10. *If R is a Prüfer domain with Noetherian prime spectrum and I is a non-zero radical ideal of R, then $(R : I)_M = (R_M : I_M)$ for all $M \in \mathrm{Max}(R)$.*

Proof. An integral domain R has Noetherian prime spectrum if and only if each radical ideal of R is the radical of a finitely generated ideal of

R [20, Proposition 2.1]. Assuming then that R is a Prüfer domain with Noetherian prime spectrum, we have that R satisfies (##) (Lemma 2.3). Since R satisfies (##), every finitely generated ideal of R has only finitely many minimal primes [12, Proposition 3]. It follows that every radical ideal has finitely many minimal prime ideals and thus can be written as a product of finitely many prime ideals. Since R satisfies (##), R has the radical trace property and Theorem 2.9 applies. ∎

COROLLARY 2.11. *If R is a Prüfer domain having Noetherian prime spectrum, then every locally SV-stable radical ideal of R is SV-stable and divisorial.*

Proof. Let I be a locally SV-stable radical ideal. As noted in the proof of Corollary 2.10, $I = P_1 P_2 \cdots P_n$ for some pairwise comaximal prime ideals $P_1, P_2, \ldots, P_n$ of R. Since R is a Prüfer domain, I is a radical ideal of $E(I)$ and $E(I)$ has Noetherian prime spectrum. It follows then from Corollary 2.10 that I is SV-stable. Similarly, $(R:(R:I)) = \bigcap_{M \in \mathrm{Max}(R)}(R_M:(R:I)_M) = \bigcap_{M \in \mathrm{Max}(R)}(R_M:(R_M:I_M)) = \bigcap_{M \in \mathrm{Max}(R)} I_M = I$, which proves the claim. ∎

3. H-LOCAL PRÜFER DOMAINS

Recall that an integral domain R is *h-local* provided each non-zero ideal of R is contained in only finitely many maximal ideals of R and each non-zero prime ideal of R is contained in a unique maximal ideal of R. Although we focus mainly on ideal theoretic characterizations of this property, the reader may refer to [19, Theorem 22] for several interesting module-theoretic interpretations of the h-local criterion. Recall that if R is an integral domain and $M \in \mathrm{Max}(R)$, then $[M] = \bigcap\{R_N : N \in \mathrm{Max}(R)$ and $N \neq M\}$.

PROPOSITION 3.1. *The following statements are equivalent for an integral domain R with quotient field Q.*

(1) *R is an h-local domain.*

(2) $[M]R_M = Q$ *for each* $M \in \mathrm{Max}(R)$.

(3) *If* $\{X_\alpha\}$ *is a collection of submodules of Q having non-trivial intersection, then* $(\bigcap_\alpha X_\alpha)_M = \bigcap_\alpha (X_\alpha)_M$ *for each* $M \in \mathrm{Max}(R)$.

(4) *If* $\{I_\alpha\}$ *is a collection of ideals of R having non-trivial intersection and M is a maximal ideal of R such that* $\bigcap_\alpha I_\alpha \subseteq M$, *then there exists* β *such that* $I_\beta \subseteq M$.

Proof. (1) ⇔ (2). See Theorem 22 of [19].

(2) ⇒ (3). Let $\{X_\alpha\}$ be a collection of submodules of Q having non-trivial intersection and let $M \in \text{Max}(R)$. Then $(\bigcap_\alpha X_\alpha)_M = \bigcap_\alpha (X_\alpha)_M \cap (\bigcap_\alpha \bigcap_{N \neq M} (X_\alpha)_N)_M \subseteq \bigcap_\alpha (X_\alpha)_M \cap (\bigcap_\alpha (\bigcap_{N \neq M} (X_\alpha)_N)_M)$. Let $0 \neq x \in \bigcap_\alpha X_\alpha$. Then $Q = [M]R_M \subseteq x^{-1} \bigcap_\alpha (\bigcap_{N \neq M} (X_\alpha)_N)_M$, which means $(\bigcap_{N \neq M} (X_\alpha)_N)_M = Q$. Thus $(\bigcap_\alpha X_\alpha)_M = \bigcap_\alpha (X_\alpha)_M \cap \bigcap_\alpha (\bigcap_{N \neq M} (X_\alpha)_N)_M = \bigcap_\alpha (X_\alpha)_M$.

(3) ⇒ (4). Let $\{I_\alpha\}$ be a collection of ideals of R having non-trivial intersection and let $M \in \text{Max}(R)$ be such that $\bigcap_\alpha I_\alpha \subseteq M$. Then $(\bigcap_\alpha I_\alpha)_M \subseteq M_M$. By (3), $(\bigcap_\alpha I_\alpha)_M = \bigcap_\alpha (I_\alpha)_M$. If there does not exist I_β such that $I_\beta \subseteq M$, then $(I_\alpha)_M = R_M$ for all α and $(\bigcap_\alpha I_\alpha)_M = R_M$, a contradiction.

(4) ⇒ (1). Examination of the proofs of Theorems 2.4 and 2.5 in [13] shows that they depend only on statement (4). ∎

Several elementary characterizations of h-local Prüfer domains follow readily from Proposition 3.1.

COROLLARY 3.2. *The following are equivalent for a Prüfer domain R.*

(1) *R is an h-local domain.*

(2) *$[M]$ is not a fractional ideal of R for each $M \in$ Max(R).*

(3) *If $M \in$ Max(R), then $[M] \not\subseteq R_P$ for any prime ideal P contained in M.*

Proof. (1) ⇒ (2). Suppose that R is h-local, $M \in \text{Max}(R)$, and $[M]$ is a fractional ideal of R. Then $[M]R_M$ is a fractional ideal of R_M. But by Proposition 3.1, $Q = [M]R_M$, where Q is the quotient field of R, and so Q is a fractional ideal of R_M, a contradiction.

(2) ⇒ (3). Suppose that $M \in \text{Max}(R)$ and P is a prime ideal contained in M. If $[M] \subseteq R_p$, then $[M]R_M \subseteq R_P$. Since R_M is a valuation domain, this means $[M]R_M$ is a fractional ideal of R_M and there exists $r \in R$ such that $r[M]R_M \subseteq R_M$. Thus $r[M] \subseteq R_M \cap [M] = R$ and $[M]$ is a fractional ideal of R, contradicting (2).

(3) ⇒ (1). Assume (3) and suppose that R is not h-local. Then there exists $M \in \text{Max}(R)$ such that $[M]R_M \neq Q$. Since R_M is a valuation domain, every overring of R_M is of the form R_P for some prime ideal P of R. There exists a prime ideal P of R such that $[M] \subseteq [M]R_M \subseteq R_P$, which is a contradiction to (3). ∎

Corollary 3.2 allows for a useful weakening of the defining criteria of h-local domains in the case of Prüfer domains. A result due to Gilmer and Heinzer is needed first.

LEMMA 3.3 [12, Theorem 5]. *If R is a Prüfer domain for which each ideal of R is contained in only finitely many maximal ideals of R, then R satisfies* (##).

Under the additional assumption that each prime ideal of R is contained in a unique maximal ideal of R, the converse of Lemma 3.3 is true, as the next proposition shows.

PROPOSITION 3.4. *The following statements are equivalent for a Prüfer domain R such that each non-zero prime ideal of R is contained in a unique maximal ideal of R.*

(1) *R is an h-local domain.*

(2) *Each non-zero ideal of R is contained in at most finitely many maximal ideals of R.*

(3) *R satisfies (##).*

(4) *R has the radical trace property.*

(5) *Every non-zero ideal of R has finitely many minimal prime ideals.*

Proof. (1) ⇔ (2). This is clear.

(2) ⇒ (3). See Lemma 3.3.

(3) ⇒ (1). Assume (3) and let $M \in \mathrm{Max}(R)$. By Corollary 3.2, it is enough to show that there does not exist a non-zero prime ideal P of R contained in M such that $[M] \subseteq R_P$. Suppose that such a P does exist. By assumption the only maximal ideal of R containing P is M. Since $R_P \cap [M]$ satisfies (#) and $R_N \not\subseteq R_P$ for all maximal ideals $N \neq M$, it cannot be the case that $[M] \subseteq R_P$, contradicting the choice of P.

(3) ⇒ (4). See [18, Corollary 25].

(4) ⇒ (3). Assuming (4), let M be a maximal ideal of R. Define $I = R_M m \cap R$, where $m \in M$ and $M_M \neq R_M m$. Then if $N \in \mathrm{Max}(R)$ and $N \neq M$, $I_N = R_M R_N m \cap R_N = R_N$, since the fact that $M \cap N$ contains no non-zero prime ideals implies that $R_M R_N = Q$, where Q is the quotient field of R [19, Theorem 19]. Thus the only maximal ideal of R containing I is M. If $I^{-1} = R$, then I is a radical ideal of R since R has the radical trace property. Thus $mR_M = I_M$ is a principal prime ideal of R_M which implies $MR_M = mR_M$, a contradiction. By Proposition 2.2 we can conclude that R satisfies (#). Because the radical trace property is inherited by overrings of Prüfer domains [18, Corollary 24], as is the property that every non-zero prime ideal is contained in a unique maximal ideal, it is clear how to extend the argument to show that every overring of R satisfies (#).

(5) ⇔ (2). This is clear. ∎

Statements (3), (4), and (5) are closely related and often coincide. For example, when R is a strongly discrete Prüfer domain, all three conditions are equivalent and serve to characterize generalized Dedekind domains [8, Théorème 2.7].

We proceed now to consideration of Prüfer domains for which $(I^{-1})_M = (I_M)^{-1}$ for all ideals I of R and $M \in \mathrm{Max}(R)$. The concept of divisoriality will play a key role. For an integral domain R with quotient field Q, if X

and Y are submodules of Q, then X is *Y-divisorial* provided $(Y:(Y:X)) = X$. It can be checked that this occurs precisely when $X = \bigcap\{Yq : q \in Q$ and $X \subseteq Yq\}$.

LEMMA 3.5. *Let R be an integral domain with quotient field Q. The following statements are equivalent for a proper submodule Y of Q.*

(1) $(Y:X)_M = (Y_M : X_M)$ *for all non-zero submodules X of Q such that $X \subseteq Y$ and $M \in \mathrm{Max}(R)$.*

(2) *If $\{X_\alpha\}$ is a collection of Y-divisorial submodules of Q having non-trivial intersection, then $(\bigcap_\alpha X_\alpha)_M = \bigcap_\alpha (X_\alpha)_M$ for each $M \in \mathrm{Max}(R)$.*

Proof. (1) ⇒ (2). Assume (1) and let $\{X_\alpha\}$ be a collection of Y-divisorial submodules of Q having non-trivial intersection. For each α there exists a submodule W_α of Q such that $X_\alpha = (Y:W_\alpha)$. Thus if $M \in \mathrm{Max}(R)$, $(\bigcap_\alpha X_\alpha)_M = (\bigcap_\alpha (Y:W_\alpha))_M = (Y:\Sigma_\alpha W_\alpha)_M = (Y_M : \Sigma_\alpha (W_\alpha)_M) = \bigcap_\alpha (Y_M : (W_\alpha)_M) = \bigcap_\alpha (Y:W_\alpha)_M = \bigcap_\alpha (X_\alpha)_M$.
(2) ⇒ (1). Assume (2) and let X be a submodule of Y. If $M \in \mathrm{Max}(R)$, then $(Y:X)_M = (Y:\Sigma_{x\in X} Rx)_M = (\bigcap_{x \in X} Yx^{-1})_M = \bigcap_{x\in X} Y_M x^{-1} = \bigcap_{x \in X}(Y_M : R_M x) = (Y_M : \Sigma_{x\in X} R_M x) = (Y_M : X_M)$. ∎

LEMMA 3.6. *The following statements are equivalent for an integral domain R.*

(1) $(R:I)_M = (R_M : I_M)$ *for all ideals I of R and $M \in \mathrm{Max}(R)$.*

(2) *If $\{D_\alpha\}$ is a collection of divisorial ideals of R having non-trivial intersection, $M \in \mathrm{Max}(R)$, and $\bigcap_\alpha D_\alpha \subseteq M$, then $D_\alpha \subseteq M$ for some α.*

Proof. (1) ⇒ (2). Assume (1). Let $\{D_\alpha\}$ be a collection of divisorial ideals of R having non-trivial intersection. Suppose $M \in \mathrm{Max}(R)$, $\bigcap_\alpha D_\alpha \subseteq M$, and $D_\alpha \nsubseteq M$ for all α. By Lemma 3.5, $(\bigcap_\alpha D_\alpha)_M = \bigcap_\alpha (D_{\alpha M}) = \bigcap_\alpha R_M = R_M$. Thus $R_M = (\bigcap_\alpha D_\alpha)_M \subseteq M_M$, which is a contradiction.
(2) ⇒ (1). Assume (2) and suppose I is an ideal of R and $M \in \mathrm{Max}(R)$. It suffices to show $(R_M : I_M) \subseteq (R:I)_M$. To this end, let $q \in (R_M : I_M)$. Then $qi \in R_M$ for all $i \in I$, from which it follows by Lemma 2.1 that $R \cap R(qi)^{-1} \nsubseteq M$ for each $i \in I$. Thus, by (2), $\bigcap_{i\in I}(R \cap R(qi)^{-1}) \nsubseteq M$. But $\bigcap_{i \in I}(R \cap R(qi)^{-1}) = R \cap q^{-1}(\bigcap_{i\in I} Ri^{-1}) = R \cap (R:qI)$, and so there exists $b \in R \setminus M$ such that $bqI \in R$. Therefore, $q = (bq)(b^{-1}) \in (R:I)_M$. ∎

LEMMA 3.7. *The following statements are equivalent for an integral domain R.*

(1) *Every non-zero ideal of R is contained in only finitely many maximal ideals of R.*

(2) *If $\{M_\alpha\}$ is a collection of maximal ideals of R having non-trivial intersection, $N \in \mathrm{Max}(R)$, and $\bigcap_\alpha M_\alpha \subseteq N$, then $N = M_\alpha$ for some α.*

Proof. (1) ⇒ (2). Let $\{M_\alpha\}$ be a collection of maximal ideals of R having non-trivial intersection. By (1), $\{M_\alpha\}$ must be a finite collection. Thus if $N \in \operatorname{Max}(R)$ and $\bigcap_\alpha M_\alpha \subseteq N$, it follows that $N = M_\alpha$ for some α.

(2) ⇒ (1). The proof is essentially that of Lemma 2.5 in [13]. Let I be a non-trivial ideal of R and $\{M_\alpha\}$ be the collection of ideals containing I. For each α, define $J_\alpha = \bigcap_{\beta \neq \alpha} M_\alpha$ and let $J = \sum_\alpha J_\alpha$. By (2), $J_\alpha \not\subseteq M_\alpha$ for each α. Therefore, $I \subseteq J$ and $J \not\subseteq M_\alpha$ for any choice of α. This means $J = R$, which implies $R = \sum_{k=1}^n J_k$ for some $J_1, \dots, J_n \in \{J_\alpha\}$. If M_α is distinct from the elements of $\{M_1, \dots, M_n\}$, then $R = \sum_{k=1}^n J_k \subseteq M_\alpha$, which is a contradiction. ∎

The proof of the next lemma can be found in [3, Lemma 2.3]. It also follows from Lemma 3.5 and Proposition 3.1.

LEMMA 3.8. *If R is an h-local integral domain with quotient field Q and X and Y are submodules of Q such that $X \subseteq Y$, then $(Y : X)_M = (Y_M : X_M)$ for all $M \in \operatorname{Max}(R)$.*

LEMMA 3.9. *The following statements are equivalent for an integral domain R such that every non-zero prime ideal of R is contained in a unique maximal ideal of R.*

(1) *R is h-local.*

(2) *If I is an ideal of R such that $E(I) = R$, then $(R : I)_M = (R_M : I_M)$ for all maximal ideals M of R.*

(3) *Every locally invertible ideal of R is invertible.*

Proof. (1) ⇒ (2). See Lemma 3.8.

(2) ⇒ (3). This is clear.

(3) ⇒ (1). Assume (3) and let $\{M_\alpha\}$ be a collection of maximal ideals of R having non-empty intersection. Suppose M is a maximal ideal of R such that $\bigcap_\alpha M_\alpha \subseteq M$ and $M \notin \{M_\alpha\}$. Let m be a non-zero element of $\bigcap_\alpha M_\alpha$ and define for each α, $I_\alpha = R \cap R_{M_\alpha} m$. If N and N' are maximal ideals of R such that $N \neq N'$, Then $R_N R_{N'} = Q$, where Q is the quotient field of R, since by assumption $N \cap N'$ contains no non-zero prime ideals of R [19, Theorem 19]. Thus for each α, $(I_\alpha)_N = R_N$ for all maximal ideals $N \neq M_\alpha$. It follows that for each α, the only maximal ideal of R containing I_α is M_α. By assumption, since I_α is a locally invertible ideal of R, we have that I_α is an invertible ideal of R for each α. Define next $J = \sum_\alpha (R : I_\alpha)$. Then $m \in \bigcap_\alpha I_\alpha = (R : J)$ and so J is a fractional ideal of R. Now if $N \in \operatorname{Max}(R)$, then since $J_N = \sum_\alpha (R_N : (I_\alpha)_N)$, it is the case that $J_N = R_n$ if $N \notin \{M_\alpha\}$ and $J_N = (R_N : R_N m) = R_N m^{-1}$ if $N = M_\beta$ for some β. Therefore J is a fractional invertible ideal of R, and so by assumption, J is a fractional invertible ideal and $R_M = (R_M : R_M) =$

$R_M : \Sigma_\alpha(R_M : (I_\alpha)_M)) = (R_M : J_M) = (R : J)_M = (\bigcap_\alpha I_\alpha)_M \subseteq (\bigcap_\alpha M_\alpha)_M \subseteq M_M$, which is a contradiction. Hence $M \in \{M_\alpha\}$. By Lemma 3.7, each non-zero ideal of R is contained in at most finitely many maximal ideals of R. ∎

THEOREM 3.10. *The following statements are equivalent for a Prüfer domain R with quotient field Q.*

(1) *R is h-local.*

(2) *$(Y : X)_M = (Y_M : X_M)$ for all $M \in \mathrm{Max}(R)$ and non-zero submodules X and Y of Q such that $X \subseteq Y$.*

(3) *$(R : I)_M = (R_M : I_M)$ for each non-zero ideal I of R and $M \in \mathrm{Max}(R)$.*

Proof. (1) ⇒ (2). See Lemma 3.8.

(2) ⇒ (3). This is clear.

(3) ⇒ (1). Observe first that from (3) it follows easily that an ideal A of R is a divisorial ideal of R if and only if A_M is a divisorial ideal of R_M for each $M \in \mathrm{Max}(R)$. Secondly, if I and J are divisorial ideals of R, then so are $I + J$ and $I \cap J$. This follows from the first remark and the fact that for each $M \in \mathrm{Max}(R)$, $I_M \subseteq J_M$ or $J_M \subseteq I_M$, since R_M is a valuation domain.

To show that every prime ideal of R is contained in a unique maximal ideal, let P be a non-maximal prime ideal of R and suppose that M and N are two distinct maximal ideals of R such that $P \subseteq M \cap N$. Note that if $L \in \mathrm{Max}(R)$, P_L is either R_L or is a non-maximal prime ideal of R_L. Since valuation domains have the separation property, non-maximal prime ideals of valuation domains are divisorial. It follows from the above observations that P is a divisorial ideal of R and that $P + Rr$ is a divisorial ideal of R for each $r \in R$. Define $\{D_\alpha\}$ to be the collection of divisorial ideals D_α of R such that $P \subseteq D_\alpha \not\subseteq M$. The proof now proceeds similarly to that of [13, Theorem 2.4]. Define $D = \bigcap_\alpha D_\alpha$. By Lemma 3.6, $D \not\subseteq M$, so let $d \in D \setminus M$. Then $P + Rd^2$ is a divisorial ideal of R and $P + Rd^2 \not\subseteq M$; hence $P + Rd^2 \in \{D_\alpha\}$. Thus $d \in P + Rd^2$ and there exists $r \in R$ such that $d(1 - rd) \in P$. Since $P \subseteq M$, $d \notin P$, and so $1 - rd \in P$. Observe next that $D \subseteq N$: If $n \in N \setminus M$, then $P + Rn \in \{D_\alpha\}$, which means $D \subseteq N$. Thus $rd \in N$ and $1 - rd \in N$, a contradiction from which we can conclude any prime ideal of R is contained in a unique maximal ideal of R. The conclusion now follows from Lemma 3.9. ∎

4. STRONGLY DISCRETE PRÜFER DOMAINS

A Prüfer domain R is *strongly discrete* if $P \neq P^2$ for every non-zero prime ideal P of R. It is easy to check that whether or not a Prüfer domain is strongly discrete is determined locally, i.e., that a Prüfer domain R is strongly discrete if and only if R_M is strongly discrete for all $M \in \mathrm{Max}(R)$. From this it follows that a Prüfer domain R is strongly discrete if and only if R has no non-zero idempotent ideals. In this section we successively weaken globalizing properties on strongly discrete Prüfer domains and characterize the integral domains that result at each stage.

Multiplicative properties of the ideals of strongly discrete valuation domains are well-understood. Several characterizations are collected in the next proposition.

PROPOSITION 4.1 [3, Proposition 7.6]. *The following statements are equivalent for a valuation domain R.*

(1) *R is strongly discrete.*

(2) *Every ideal of R is isomorphic to a prime ideal of R.*

(3) *Every non-zero prime ideal P of R is principal over R_P.*

(4) *R is SV-stable.*

(5) *R is totally divisorial.*

Thus strongly discrete Prüfer domains are locally totally divisorial and locally SV-stable. As mentioned in the Introduction, to globalize the property of being totally divisorial, the h-local hypothesis is necessary. While the following proposition is not stated explicitly in [3], its proof is implicit in the statement of several other propositions in their work.

PROPOSITION 4.2 [3]. *An integral domain R is integrally closed and totally divisorial if and only if R is an h-local strongly discrete Prüfer domain.*

Recall that an ideal I of an integral domain is *SV-stable* if I is an invertible ideal of $E(I)$ and that an integral domain R is an *SV-stable domain* if every ideal of R is SV-stable. Interest in Noetherian SV-stable domains can be traced back at least to Bass [2]. Recently there has been interest in the integrally closed case [1, 9]. Before giving a characterization of these domains, we establish several lemmas.

LEMMA 4.3. *Let R be an integral domain such that every non-zero ideal of R is contained in at most finitely many maximal ideals of R. An ideal I of R is SV-stable if and only if I_M is an SV-stable ideal of R_M for all $M \in \mathrm{Max}(R)$.*

Proof. Assume that I is locally SV-stable. We prove first that for each maximal ideal N of R, $(I : I)_N = (I_N : I_N)$. Let $t \in (I_N : I_N)$. Then since each non-zero ideal of R is contained in at most finitely many maximal ideals of R, there exists at most finitely many maximal ideals M of R such that $R_M \neq R_M t$. Denote this collection by $\{M_1, M_2, \dots, M_n\}$. Since I_{M_i} is SV-stable for each i, it follows that there exists a finitely generated ideal

A_i of R such that $A_i \subseteq I$ and $I_{M_i} = E(I_{M_i})A$. By assumption, $tA_i \in I_N$, and so there exists, since A is finitely generated, a non-zero element $d_i \in R \setminus N$ such that $d_i tA_i \in I$. Define $d = d_1 d_2 \cdots d_n$. Then since $dtA_i \in I$, it follows that $dtI_{m_i} = dtE(I_{M_i})A_i \subseteq E(I_{M_i})I = I_{M_i}$. Hence, for all i, $dtI_{M_i} \subseteq I_{M_i}$. Also, if M is a maximal ideal of R such that $M \notin \{M_1, M_2, \dots, M_n\}$, then $dtI_M = dI_M \subseteq I_M$. It follows that $dtI \subseteq I$. Therefore, $t = (dt)d^{-1} \in (I : I)_N$.

Define $S = E(I)$ and assume $M \in \operatorname{Max}(R)$. Using a similar argument, we prove next that $(S : I)_M = (SR_M : I_M)$. To see this let $q \in (SR_M : I_M)$. Clearly $qI \subseteq SR_M$. Since every non-zero ideal of R is contained in at most finitely many maximal ideals of R, there exists at most finitely many maximal ideals N such that $R_N \neq R_N q$. Denote this collection by $\{N_1, N_2, \dots, N_k\}$. Since I_{N_j} is SV-stable for each $j = 1, 2, \dots, k$, there exists for each j, a finitely generated ideal B_j of R such that $B_j \subseteq I$ and $I_{N_j} = E(I_{N_j})B_j$. By assumption, $qB_j \in SR_M$ for each j. Thus, since B_j is finitely generated for each j, there exists $b_j \in R \setminus M$ such that $b_j qB_j \in S$. Define $b = b_1 b_2 \cdots b_k$ and observe that since $bqr_j \in S$, it follows that $bqI_{N_j} = bqE(I_{N_j})B_j \subseteq SE(I_{N_j}) = S(I : I)R_{N_j} = SR_{N_j}$. As above, $bqI \subset S$ and so $q \in (S : I)_M$.

We show finally that I is an invertible ideal of S. Since $(I : I)_M = (I_M : I_M)$ for all $M \in \operatorname{Max}(R)$, we have by assumption that I_M is an invertible ideal of SR_M for each maximal ideal M of R. Hence $I(S : I)_M = I_M(SR_M : I_M) = SR_M$ for all $M \in \operatorname{Max}(R)$, and I must be an invertible ideal of S.

To see that the converse is true, assume that I is SV-stable, and note that for each ideal I of R and $M \in \operatorname{Max}(R)$, $(I : I)_M = (I_M : I_M)$, since I is a finitely generated $E(I)$-module. It follows that I_M must be SV-stable for each $M \in \operatorname{Max}(R)$.

LEMMA 4.4. *Let R be a Prüfer domain satisfying (#) and let Q be its quotient field. If X is a non-zero submodule of Q such that $E(X) = R$, then for each $M \in$* Max(R), $(X : X)_M = (X_M : X_M)$.

Proof. Let R and X be as in the claim and let M be a maximal ideal of R. Then $R_M = (X : X)_M = (X_M : X_M) \cap (\bigcap_{N \neq M} (X_N : X_N))_M$, where N ranges over all maximal ideals of R distinct from M. Since R satisfies (#), it cannot be the case that $R_M = (\bigcap_{N \neq M} (X_N : X_N))_M$, and since R_M is a valuation domain, this means $R_M = (X : X)_M$. ∎

The next lemma can be compared to a characterization of integrally closed divisorial domains due to Heinzer. He proved that an integral domain R is integrally closed and divisorial if and only if R is an h-local Prüfer domain with invertible maximal ideals [13, Theorem 5.1]. Observe that the requirement that the maximal ideals be invertible can be weakened to the requirement that the maximal ideals be non-idempotent, which is a restriction only on R_M for each $M \in \operatorname{Max}(R)$. This is because a maximal ideal of a valuation domain is invertible if and only if it is not idempotent. By Lemma 3.8, the h-local assumption implies that locally invertible ideals are invertible.

Dropping the restriction on prime ideals in the definition of h-local, we have:

LEMMA 4.5. *The following are equivalent for an integral domain R.*

(1) *R is integrally closed and each ideal I of R such that $E(I) = R$ is invertible.*

(2) *R is integrally closed and each ideal I of R such that $E(I) = R$ is divisorial.*

(3) *R is a Prüfer domain such that every non-zero ideal of R is contained in at most finitely many maximal ideals of R and maximal ideals of R are not idempotent.*

Proof. (1) ⇒ (2). This is clear.

(2) ⇒ (1). Assuming (2), let I be an ideal of R such that $E(I) = R$. Then $(R : II^{-1}) = ((R : I^{-1}) : I) = (I : I) = R$. Therefore, $E(II^{-1}) = R$ and by assumption, $II^{-1} = (R : (R : II^{-1})) = (R : R) = R$, proving that I is invertible.

(1) ⇒ (3). Assume (1). Since R is integrally closed, if I is a finitely generated ideal, then $E(I) = R$. By (1), I is invertible and R must be a Prüfer domain. If M is a maximal ideal of an integrally closed domain R, then $E(M) = R$. It follows that maximal ideals of R are invertible and hence not idempotent. It remains to show that every non-zero ideal is contained in at most finitely many maximal ideals of R. We do this via Lemma 3.7. Suppose there exists a collection $\{M_\alpha\}$ of maximal ideals of R having non-empty intersection and such that $\bigcap_\alpha M_\alpha \subseteq M$, for some maximal ideal $M \notin \{M_\alpha\}$. Each maximal ideal is invertible; therefore, $\bigcap_\alpha M_\alpha = \bigcap_\alpha (R : (R : M_\alpha)) = (R : \Sigma_\alpha (R : M_\alpha))$. Define $J = \Sigma_\alpha (R : M_\alpha)$ and let $M_\beta \in \{M_\alpha\}$. Then (again using the fact that each maximal ideal is invertible) $J_{M_\beta} = \Sigma_\alpha (R_{M_\beta} : (M_\alpha)_{M_\beta}) = (R_{M_\beta} : (M_\beta)_{M_\beta})$, which is an invertible fractional ideal of R_{M_β}. A similar argument shows that if N is a maximal ideal of R such that $N \notin \{M_\alpha\}$, then $J_N = R_N$. This means J is a locally invertible fractional ideal of R and, as such, it must be the case that $E(J) = R$. By (1), J is an invertible fractional ideal of R. Thus $(\bigcap_\alpha M_\alpha)_M = (R : J)_M = (R_M : J_M)$. But as we have already noted, $J_M = R_M$, since $M \notin \{M_\alpha\}$. Therefore, it cannot be the case that $\bigcap_\alpha M_\alpha \subseteq M$. By Lemma

3.7, we conclude that every non-zero ideal of R is contained in at most finitely maximal ideals of R.

(3) $\Rightarrow$ (1). By (3), each maximal ideal of R is not idempotent. As in [13, Lemma 5.2], this means the valuation domain R_M is a divisorial domain for each $M \in \text{Max}(R)$ (see the remark preceding this lemma). Arguing as in the proof of (2) $\Rightarrow$ (1), it follows that each ideal I_M of R_M such that $E(I_M) = R_M$ is invertible. If I is an ideal of R such that $E(I) = R$, then by Lemmas 3.3 and 4.4, $E(I)_M = R_M$. Since I_M is an SV-stable ideal of R_M for each $M \in \text{Max}(R)$, Lemma 4.3 applies and I must be an invertible ideal of R. ∎

In [1, Proposition 2.10] it is shown that for semilocal Prüfer domains, the property of being strongly discrete is equivalent to the domain in question being SV-stable. Theorem 4.6 generalizes this fact.

THEOREM 4.6. *The following are equivalent for an integral domain R.*

(1) *R is an integrally closed SV-stable domain.*

(2) *R is a strongly discrete Prüfer domain such that each non-zero ideal of R is contained in at most finitely many maximal ideals of R.*

(3) *R is a Prüfer domain such that every ideal I of R can be generated by two elements as an ideal over $E(I)$.*

Proof. (1) $\Rightarrow$ (2). Assume (1). By Lemma 4.5, R is a Prüfer domain and each non-zero ideal of R is contained in at most finitely many maximal ideals of R. If P is a non-zero prime ideal of R, then clearly $P \neq P^2$, since P is an invertible ideal of $E(P)$.

(2) $\Rightarrow$ (1). Assuming (2), R_M is by Proposition 4.1 an SV-stable domain for all $M \in \text{Max}(R)$. By Lemma 4.3, (1) holds.

(1) $\Rightarrow$ (3). We have shown (2) is equivalent to (1). Thus, in assuming (1) it must be true that every non-zero ideal of R is contained in at most finitely many maximal ideals of R. If I is an ideal of R, then by Proposition 4.1, $I_M \cong E(I_M)$ for all $M \in \text{Max}(R)$. Since by (1), I is a finitely generated $E(I)$-module, this means $I_M \cong E(I)_M$ for all $M \in \text{Max}(R)$. By [19, Theorem 26], I can be generated by two elements as an $E(I)$-module.

(3) $\Rightarrow$ (1). This is clear. ∎

We consider a larger class of strongly discrete Prüfer domains next, those for which every radical ideal is the radical of a finitely generated ideal of R. These domains are known as *generalized Dedekind domains*, and were first defined in [21]. In [8], a Prüfer domain R is shown to be a generalized Dedekind domain if and only if R is a strongly discrete Prüfer domain satisfying (##). It follows from Lemma 3.3 and Theorem 4.6 that integrally closed SV-stable domains are generalized Dedekind domains.

This implication will also be clear from the next theorem. S. Gabelli and N. Popescu have investigated characterizations of generalized Dedekind domains involving stability and divisoriality conditions [9, 10]. In the next theorem, we give several other such characterizations. A proof of the equivalence of statements (1) and (3) can also be found in [9, Theorem 5].

THEOREM 4.7. *The following statements are equivalent for a Prüfer domain R.*

(1) *R is a generalized Dedekind domain.*

(2) *Every radical ideal of R is SV-stable.*

(3) *Every prime ideal of R is SV-stable.*

(4) *If R' is an overring of R, then every radical ideal of R' is a divisorial ideal of R'.*

(5) *If R' is an overring of R, then every maximal ideal of R' is an invertible ideal of R'.*

Proof. (1) ⇒ (2) and (4). The prime spectrum of a generalized Dedekind domain is a Noetherian space [8, Thèorém 2.7]. Statements (2) and (4) now follow from Proposition 4.1 and Corollary 2.11.

(2) ⇒ (3). This is clear.

(3) ⇒ (1). Suppose that P_0 is a non-zero prime ideal of R and $P_0 = P_0^2$. By (2), $E(P_0) = P_0(E(P_0): P_0) = P_0(P_0 : P_0^2) = P_0$, a contradiction. Thus R is strongly discrete. Let P' be an ideal of an overring R' of R. Then since R is a Prüfer domain, $P' = R'P$ for some prime ideal P of R. Thus $P'(E(P') : P') = R'P((R'P : R'P) : R'P)$. Since P is $E(P)$-projective, $(R'P : R'P) = (R'P : P) = R'E(P)$. Thus $P'(E(P'): P') = P(R'E(P): P) = R'E(P) = E(P')$. Hence P' is SV-stable. It follows that every prime ideal of every overring of R is SV-stable. Now let M' be a maximal ideal of the overring R'. Since M' is SV-stable and $E_{M'} = R'$, it follows that M' is a divisorial ideal of R' and, since M' is maximal, it must be the case that $M' = R' \cap R'q$ for some element q in the quotient field of R. By Lemma 2.1, $q^{-1} \in [M']$, but $q^{-1} \notin R'_{M'}$. Thus R' satisfies (#) and so R satisfies (##). As remarked before this theorem, this means R' is a generalized Dedekind domain.

(4) ⇒ (5). This follows from the fact that every divisorial maximal ideal of a Prüfer domain is invertible.

(5) ⇒ (1). Suppose that P is a non-zero prime ideal of R such that $P^2 = P$. Then $P_P^2 = P_P$ also. But P_P is a maximal ideal of R_P and by assumption divisorial. Since R_P is a valuation domain and P_P is divisorial, the maximality of P_P implies that P_P is a principal ideal of R_P. Therefore, it cannot be the case that $P_P^2 = P_P$, contradicting the choice of P. Thus R is strongly discrete. Arguing as in the proof of (3) ⇒ (1), it follows from

the fact that every maximal ideal of every overring of R is divisorial that R satisfies (##) and hence R must be a generalized Dedekind domain. ∎

Before proceeding to the case of strongly discrete Prüfer domains that have the separation property, we note the following proposition since it anticipates our characterization of these domains. Recall that an ideal I of an integral domain R is a *cancellation ideal* if $IJ = IK$, where J and K are ideals of R, implies that $J = K$.

PROPOSITION 4.8. *If R is a generalized Dedekind domain, then every non-zero ideal I of R is a cancellation ideal of $E(I)$.*

Proof. Let R be a generalized Dedekind domain and I be a non-zero ideal of R such that $IJ = IK$ for two non-zero ideals J and K of $E(I)$. Define $R' = E(I)$ and let $M \in \mathrm{Max}(R')$. Then $I_M J_M = I_M K_M$. Since every overring of a generalized Dedekind domain is a generalized Dedekind domain, R' must be a strongly discrete Prüfer domain satisfying (##). By Proposition 4.1, $I_M = E(I_M)i$ for some $i \in I_M$. By Lemma 4.4, $E(I_M) = R'_M$. Since J and K are ideals of R', this means $J_M = K_M$. This holds for all maximal ideals M of R'. We conclude that $J = K$. ∎

The converse of Proposition 4.8 is not true. If R is an almost Dedekind domain that is not a Dedekind domain, then R does not satisfy (#) [11, Theorem 3], and so R is not a generalized Dedekind domain. Yet every ideal of R is a cancellation ideal of R [17, Theorem 9.4].

PROPOSITION 4.9. *The following are equivalent for a Prüfer domain R.*

(1) *R is a strongly discrete Prüfer domain that has the separation property.*

(2) *Every non-zero radical ideal I of R is a cancellation ideal of $E(I)$.*

(3) *Every non-zero prime ideal P of R is a cancellation ideal of $E(P)$.*

Proof. (1) ⇒ (2). Assume (1) and let I be a radical ideal of R. Since R is a Prüfer domain, R has the property that if q is an element of the quotient field of R and $q^n \in R$ for some integer n, then $q \in R$. Therefore, I is also a radical ideal of $E(I)$. Suppose that $IJ = IK$ for two ideals J and K of $E(I)$. Define $R' = E(I)$ and let $M \in \mathrm{Max}(R')$. Then $I_M J_M = I_M K_M$, and by Proposition 4.1, $I_M \cong E(I_M)$. By Lemma 2.7 and the fact that since R has the separation property, so does R', $E(I_M) = (I : I)_M = R'_M$. Thus I_M is a principal ideal of R'_M and it follows that $J_M = K_M$. We can conclude that $J = K$.

(2) ⇒ (3). This is clear.

(3) ⇒ (1). Assuming (3), suppose that P is a non-zero prime ideal of R such that $P^2 = P$. Since P is a cancellation ideal of $E(P)$, this means $P = E(P)$, a contradiction that implies R is a strongly discrete Prüfer domain. Now let P and Q be two distinct prime ideals of R such that

$P \subset Q$. To show that R has the separation property we must show there exists a finitely generated ideal I of R such that $P \subseteq I \subseteq Q$. To prove this, it is enough to show that $E(P)Q = E(P)$ (see the remarks before Lemma 2.7). We first show that $E(P)Q = (PQ : P)$. Observe that $(PQ : P)P \subseteq PQ \subseteq P(PQ : P)$. Therefore, $PQ = P(PQ : P)$. Since P is a cancellation ideal of $E(P)$, this means $E(P)Q = (PQ : P)$, as desired. Now note that if M is a maximal ideal of R containing P, then since $E(P_M) = R_P$ (Lemma 2.4), $P_M = P_P = PQ_P = PQ_M$. If M is a maximal ideal not containing P, then $P_M = R_M = P_M Q_M = PQ_M$. From this we conclude that $PQ = P$, and so $E(P)Q = (PQ : P) = E(P)$, which means R has the separation property. ∎

5. FURTHER RESULTS ON STRONGLY DISCRETE PRÜFER DOMAINS

Ideals of Dedekind domains can be written as products of prime ideals. Integral domains possessing this property, ZPI-rings, must be one dimensional and Noetherian [17, Theorem 9.10]. Thus the only integrally closed domains for which every ideal is the product of prime ideals are Dedekind domains. Gabelli and Popescu have, however, discovered a similar phenomenon for the divisorial ideals of generalized Dedekind domains. They proved that an integral domain R is a generalized Dedekind domain if and only if the divisorial ideals of R are precisely those of the form $JP_1 P_2 \cdots P_n$, where J is an invertible ideal of R and $P_1, P_2, \ldots, P_n$ are pairwise comaximal prime ideals of R [10, Theorem 3.3].

It follows from the results of the previous section that integrally closed totally divisorial domains are generalized Dedekind domains. Thus each ideal of an h-local strongly discrete Prüfer domain is the product of an invertible ideal and finitely many pairwise comaximal prime ideals of R. In Theorem 5.2, we show that for Prüfer domains the converse is true, thus characterizing the Prüfer domains for which every ideal can be represented in this way.

LEMMA 5.1. *If I and J are comaximal ideals of an integral domain R, then $(K : I \cap J) = (K : I) + (K : J)$ for all ideals K of R.*

Proof. Let I and J be comaximal ideals of R. Then there is an exact sequence $0 \to I \cap J \to I \oplus J \to I + J \to 0$. Since $I + J = R$, this sequence splits. Let K be an ideal of R. Applying $\mathrm{Hom}(-, K)$ to the sequence yields a commutative diagram,

$$\begin{array}{ccccccc} \mathrm{Hom}(R,K) & \longrightarrow & \mathrm{Hom}(I \oplus J, K) & \longrightarrow & \mathrm{Hom}(I \cap J, K) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ (K:I) \cap (K:J) & \longrightarrow & (K:I) \oplus (K:J) & \longrightarrow & (K:I) + (K:J) & \longrightarrow & 0. \end{array}$$

Observe that $(K : I + J) = (K : I) \cap (K : J)$, so the left vertical mapping is an isomorphism. Since the middle vertical mapping is also an isomorphism, the claim follows. ∎

THEOREM 5.2. *A Prüfer domain R is strongly discrete and h-local if and only if every ideal of R is of the form $JP_1P_2 \cdots P_n$, where J is an invertible ideal of R and $\{P_1, P_2, \dots, P_n\}$ is a non-empty collection of prime ideals of R that are pairwise comaximal if $n > 1$.*

Proof. As noted above, sufficiency follows from Theorem 3.3 of [10]. So with the goal of establishing necessity, we prove first that R is h-local. If P is a non-maximal prime ideal of R, then by Lemma 2.6, $P(R:P) = P(P:P) = P$. If, on the other hand, P is a maximal ideal of R, then either $PP^{-1} = P$ or $PP^{-1} = R$. Suppose now that $A = P_1P_2 \cdots P_n$ is a product of pairwise comaximal prime ideals of R and that $n \geq 2$. We will show that AA^{-1} is either a radical ideal of R or is R itself. For each $k = 1, 2, \dots, n$, define $A_k = \prod_{i \neq k} P_i$. If P_k is invertible for some k, then $AA^{-1} = A_kP_k((R:A_k):P_k) = A_k(R:A_k)$, and if each P_i is invertible, then clearly $AA^{-1} = R$. We may thus assume without loss of generality that for all i, P_i is not invertible. Observe that since the P_i's are pairwise comaximal, $A = P_1 \cap P_2 \cap \cdots \cap P_n$. It follows from Lemma 5.1 that $(R : P_1P_2 \cdots P_n) = (R : P_1) + (R : P_2) + \cdots + (R : P_n)$. Thus $AA^{-1} = AP_1^{-1} + AP_2^{-1} + \cdots + AP_n^{-1}$. By assumption, no P_i is invertible. Hence $P_i^{-1} = (P_i : P_i)$ for each i, and so $AP_i^{-1} = A$ for each i. Therefore, $AA^{-1} = A$ and AA^{-1} is a radical ideal of R.

Now let I be a non-zero ideal of R. By assumption $I = JB$, where J is an invertible ideal and B is a product of pairwise comaximal prime ideals of R. Thus $II^{-1} = BJ((R:B):J) = B(R:B)$, which, as was established above, is a radical ideal of R or is R itself. Hence R has the radical trace property. To complete the argument that R is h-local, let $M \in \mathrm{Max}(R)$ and I' be an ideal of R. By assumption, $I' = J'C$, where J' is an invertible ideal of R and C is a product of pairwise comaximal prime ideals. Thus $(R: I')_M = ((R:C)_M : J'_M)$. By Theorem 2.9, $(R:C)_M = (R_M : C_M)$. It follows that $(R:I')_M = (R_M : I'_M)$. The choice of I' was arbitrary, as was that of M. Therefore, by Theorem 3.10, R is h-local.

It remains to show that R has no non-zero idempotent prime ideals. We prove this for maximal ideals first. Let M be a maximal ideal of R. Since

h-local domains satisfy (##), there exists by Lemma 2.3 an invertible ideal B such that only maximal ideal containing B is M. By assumption, there exists an invertible ideal J and prime ideals $P_1, \dots, P_n$ such that $B = JP_1 \cdots P_n$. Note that this means for each i, $E(P_i) \subseteq E(B) = R$. Since R satisfies (##), it follows by Lemma 2.4 that $E(P_i) = R$ if and only if P_i is a maximal ideal of R. Therefore, each P_i is a maximal ideal of R. But the only maximal ideal containing B is M, so $l = 1$ and $B = JM$. This means $M = BJ^{-1}$ is invertible and clearly not idempotent. Now suppose P is a non-maximal non-zero prime ideal of R and let p be a non-zero element of P. Then by assumption $E(P)p = KQ_1Q_2 \cdots Q_m$, for some comaximal prime ideals $Q_1, Q_2, \dots, Q_m$ of R and invertible ideal K. There exists a unique maximal ideal M of R containing P. If no Q_i is contained in M, then $E(P)R_M p = K_M$, and it follows (using Theorem 3.10) that $R_P = (E(P_M) : E(P_M)) = (E(P)R_M p : E(P)R_M p) = (K_M : K_M) = R_M$, a contradiction. Thus there exists (a necessarily unique) j such that Q_j is contained in M. Hence, $R_P p = K_M(Q_j)_M$. Since K is invertible, it follows that $R_P = (K(Q_j)_M : K(Q_j)_M) = ((Q_j)_M : (Q_j)_M) = R_{Q_j}$, and so $Q_j = P$. Therefore, P_M is an invertible ideal of $R_P = E(P_M)$ and so P cannot be idempotent. ∎

A. Facchini has shown that every Noetherian tree with least element arises as the prime spectrum of a generalized Dedekind domain. We indicate next how this result can be used to derive similar existence theorems for integrally closed SV-stable domains and integrally closed totally divisorial domains. A *tree* is a partially ordered set $(X, \leq)$ with the property that for every $x \in X$, the set $B_x = \{y \in X : y \leq x\}$ is a chain. The tree X is *Noetherian* if every ascending chain $x_1 \leq x_2 \leq x_3 \leq \dots$ of elements of X is stationary.

PROPOSITION 5.3 [4, Theorem 5.3]. *The following statements are equivalent for a partially ordered set X.*

(1) *X is a Noetherian tree with a least element.*

(2) *There exists a generalized Dedekind domain R whose prime spectrum is order isomorphic to X.*

PROPOSITION 5.4. *The following statements are equivalent for a partially ordered set $(X, \leq)$.*

(1) *X is a Noetherian tree with a least element x_0 such that every element of X except possibly x_0 is contained in at most finitely many maximal elements of X.*

(2) *There exists an integrally closed SV-stable domain R whose prime spectrum is order isomorphic to X.*

Proof. (1) ⇒ (2). Assume X is as in (1). Then by Proposition 5.3, there exists a generalized Dedekind domain R whose prime spectrum is order isomorphic to X. Therefore, each non-zero ideal has at most finitely many minimal primes [8, Thèoréme 2.7]. Since the prime spectrum of R is order isomorphic to X, each non-zero prime ideal of R is contained in at most finitely many maximal ideals of R. Thus each non-zero ideal of R is contained in at most finitely many maximal ideals of R. By Theorem 4.6, R is an SV-stable domain.

(2) ⇒ (1). If (2) holds, then by Proposition 5.3, X is a Noetherian tree with least element. The additional restriction on X follows from Theorem 4.6. ∎

PROPOSITION 5.5. *The following statements are equivalent for a partially ordered set* $(X, \leq)$.

(1) *X is a Noetherian tree with a least element x_0 and every element of X except possibly x_0 is contained in a unique maximal element of X.*

(2) *There exists an integrally closed domain R that is totally divisorial and whose prime spectrum is order isomorphic to X.*

Proof. (1) ⇒ (2). Assuming (1), there exists by Proposition 5.3 a generalized Dedekind domain R whose prime spectrum is order isomorphic to X. Thus, by (1), each prime ideal of R is contained in a unique maximal ideal of R. By Propositions 4.2 and 3.4, R is totally divisorial.

(2) ⇒ (1). Let R be as in (2). Then by Theorem 4.7, R is clearly a generalized Dedekind domain. By Proposition 5.3 it follows that the prime spectrum of R is a Noetherian tree ordered by $\subseteq$ and having a least element. Since R is h-local (Proposition 4.1), every non-zero prime ideal of R is contained in a unique maximal ideal of R. Statement (1) follows. ∎

Gabelli has given an example of an integrally closed SV-stable domain that is not a generalized Dedekind domain [9]. While Propositions 5.3 and 5.4 guarantee the existence of many such examples, we briefly indicate a direct way to construct examples of generalized Dedekind domains that are not SV-stable. Let R be a generalized Dedekind domain with infinitely many maximal ideals. If Q is the quotient field of R and $Q[X]$ is the polynomial ring of Q in one variable X, then the integral domain $S = R + XQ[X]$ is also a generalized Dedekind domain [8, Thèorém 4.1]. Since the prime ideal $XQ[X]$ of S is contained in every maximal ideal of S, Theorem 4.6 implies that S is a generalized Dedekind domain that is not SV-stable.

Similarly, Propositions 5.4 and 5.5 guarantee that the class of integrally closed SV-stable domains is distinct from the class of integrally closed totally divisorial domains. In fact, it follows from these last two propositions that even for semi-local domains the two classes do not coincide.

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