MR2063800 (2006a:13003) 13A15 (13A18)

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Semistar linkedness and flatness, Pfüfer semistar multiplication domains. (English summary)

Comm. Algebra **32** (2004), *no. 3,* 1101–1126.

Let D be a domain with quotient field K. A mapping $*\colon \overline{F}(D) \to \overline{F}(D)\colon E \mapsto E^*$ on the set $\overline{F}(D)$ of nonzero D-submodules of K is a semistar operation on D if, for $x \in K$, $x \neq 0$, and $E, F \in \overline{F}(D)$, the following properties hold: (1) $(xE)^* = xE^*$; (2) $E \subseteq F \Rightarrow E^* \subseteq F^*$; (3) $E \subseteq E^*$ and $E^* = (E^*)^*$. When also $D^* = D$, then $*$ restricted to the class of fractional ideals of K is the classical star operation; see sections 32 and 34 of [R. Gilmer, *Multiplicative ideal theory*, Dekker, New York, 1972; MR0427289 (55 #323)]. If $*$ is a semistar operation on D, the semistar operation $*_f$ of finite type associated to D is defined by setting $E \mapsto E^{*_f} := \bigcup\{F^* \colon F$ is a nonzero finitely generated D-submodule of $K\}$. If T is an overring of D and $*'$ is a semistar operation on T, then T is $(*, *')$-linked to D if $F^* = D^* \Rightarrow (FT)^{*'} = T^{*'}$ for each nonzero finitely generated integral ideal F of D.

The authors give a thorough treatment of semistar operations, and focus especially on semistar linked overrings and a notion of semistar flat overrings. As an application, Prüfer semistar multiplication domains are characterized in terms of semistar linked overrings and semistar flatness. (Here a domain D with semistar operation $*$ is a Prüfer $*$ multiplication domain if for each nonzero finitely generated D-submodule F of K, $(FF^{-1})^{*_f} = D^*$.) These characterizations extend to the semistar setting the characterizations of a Prüfer domain D as a domain for which every overring is integrally closed [E. D. Davis, Trans. Amer. Math. Soc. **110** (1964), 196–212; MR0156868 (28 #111) (Theorem 1)], or a domain for which every overring is D-flat [F. Richman, Proc. Amer. Math. Soc. **16** (1965), 794–799; MR0181653 (31 #5880) (Theorem 4)]. The authors also provide a number of useful examples and constructions involving semistar operations.

Reviewed by *Bruce Olberding*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.