Journal of Pure and Applied Algebra 29 (1983) 285-296
North-Holland

ANALOGUES OF A THEOREM OF COHEN FOR OVERRINGS*

Marco FONTANA**
Istituto Matematico "Guido Castelnuovo", Universitá di Roma, I, Piazzale A. Moro, 00185 Roma, Italia

T.M. VISWANATHAN***
Departamento de Matemática, Universidade Estadual de Campinas, Caixa Postal 6155, 13.100-Campinas, S.P., Brazil

Communicated by H. Bass
Received 1 October 1982

1. Introduction

An integral domain R has *the finite* [*finite presentation*] *overring property*, if every overring of R in the quotient field K of R is a finitely generated [finitely presented] R-algebra. Papick in [24] (Proposition 23) has given a characterization of one-dimensional domains with the finite presentation overring property. In this paper, we remove the restriction on the dimension. This is achieved in Theorem 18 using a characterization of domains with the finite overring property (Theorem 14).

An important contribution in obtaining these results comes from a consideration of an analogue for overrings of the following theorem of Cohen [17, Theorem 7]. Let I be an ideal of a commutative unitary ring R. Assume that I is not finitely generated and is maximal with respect to this property. Then I is a prime ideal of R.

Studies on overrings of integral domains [12, 13, 14, 18, 28] suggest that, in multiplicative domains, overrings of domains play a role analogous to ideals in commutative rings and that valuation overrings may be thought of as analogues of prime ideals. Looking for an analogue of the above theorem of Cohen, we are led to introduce the following definition: Let R be a G-domain, with quotient field K. We say that an overring S of R in K is a *Cohen overring* of R, if S is not a finitely generated R-algebra and S is maximal with respect to this property. We show that Cohen overrings are always local rings (Proposition 2). We exhibit examples showing that they may not be valuation rings. However, we point out that there are

* Communicated at the 6th Latin American School of Mathematics at Oaxtepec, Mexico in July, 1982.

** Work done under the auspices of the CNR, Italy (Gruppo Nazionale per le Strutture Algebriche, Geometriche e loro Applicazioni).

*** Part of the present work was done while the second author was visiting the University of Rome in 1981, supported by the CNR of Italy.

important classes of domains for which all Cohen overrings are valuation rings (Proposition 4). Thus our theory reinforces the analogy between valuation rings and prime ideals and suggests that Cohen overrings may be particularly useful in the study of non-integrally closed overrings.

As is to be expected, our work is closely related to Papick's paper [24]. In that spirit, in Section 3, we consider the corresponding problem using non-finite presentation instead of non-finite generation.

2. Cohen overrings and domains with the finite overring property

Recall that R is a *G-domain* if the quotient field K of R is a finitely generated R-algebra or equivalently K is of the form $K=R[1/t]$ for some $t\in K^*$. R is a *strong G-domain* [28] if every overring of R is of the form $R[1/t]$ or equivalently R is a finite intersection of valuation rings of well-ordered rank (with respect to inclusion of prime ideals). Notice that R has the finite overring property if and only if every non-empty family of overrings of R has a maximal element or equivalently the ascending chain condition holds for overrings of R.

We begin with a result of Papick [24] giving a proof based on the abstract Riemann surface X of the integral domain R [30, p. 110].

Recall that X is the space of all valuation overrings of R and that a basis for the open sets of its topology is given by the family of sets $E(x_1, x_2, \dots, x_t) = \{V\colon V$ is a valuation overring of $R[x_1, x_2, \dots, x_t]\}$, as $x_1, x_2, \dots, x_t$ run through the quotient field of R. It is well known that X is a quasi-compact space.

Theorem 1 (Papick). *Let R be a G-domain with quotient field K. If every valuation overring of R is a finitely generated R-algebra, then the integral closure $\bar{R}$ of R is a strong G-domain.*

Proof. Let $\{V_i : i\in I\}$ be the family of minimal valuation overrings of R. By hypothesis, $V_i = R[x_{i1}, x_{i2}, \dots, x_{it_i}]$ with $x_{ij}\in K$. By definition $\{E(x_{i1}, x_{i2}, \dots, x_{it_i} : i\in I)\}$ is an open covering of the quasi-compact space X. Thus I must be a finite set. Hence $\bar{R}$ is a finite intersection of valuation rings of K, say $V_1, V_2, \dots, V_k$. We remark that each V_i has the finite overring property and so is of well-ordered rank [28, Theorem 3.3. and Proposition 6.1]. We conclude that $\bar{R}$ is a strong G-domain.

This paper originated from an attempt to strengthen Theorem 1. Our basic consideration is that of Cohen overrings.

Proposition 2. *Let R be a G-domain. Then every Cohen overrings of R is a local domain.*

Proof. Let S be a Cohen overring of the G-domain R with K as its quotient field.

We notice that S has the finite overring property. Hence for every $\mathfrak{p} \in \mathrm{Spec}(S)$, there exists $f \in S \setminus \mathfrak{p}$ such that the canonical map $S_f \to S_{\mathfrak{p}}$ is the identity map [28, Proposition 6.1]. Furthermore S is semilocal, since $\bar{S}$ is a strong G-domain (Theorem 1). Let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_r$ be the maximal ideals of S. Assume $r \geq 2$. Then $S_{\mathcal{M}_i} = S_{f_i}$ for some $f_i \in S$, $1 \leq i \leq r$ and so is a finitely generated R-algebra, by the maximality property of S, since $r \geq 2$. Thus the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a locally finitely generated morphism of affine schemes [15, Definition I.6.2.1, p. 297]. Hence S itself is a finitely generated R-algebra [15, Proposition I.6.2.5, p. 299] - a contradiction. Thus we must have $r = 1$, so that S is local.

Corollary 3. *Let R be a G-domain and S a Cohen overring of R. Then S is a valuation ring if and only if S is integrally closed.*

Proof. Since S has the finite overring property the conclusion follows from Theorem 3.3 of [20] and proposition above.

Proposition 4. *If R is a noetherian G-domain or a Prüfer G-domain, then every Cohen overring of R is a valuation ring. More generally the same conclusion holds, if R and all the overrings of R are coherent G-domains.*

Proof. It is known that if R is a ring in the first statement, then R and the overrings of R are coherent G-domains ([13, (33.9) and (33.8), pp. 481 and 480] and [7, (11-10)(ii), p. 76]). Hence it is enough to prove the last statement. Let S be a cohen overring of R. By Corollary 3, it is enough to show that S is integrally closed. Assume $R \subsetneq S \subsetneq \bar{S}$. Since $\bar{S}$ is a finitely generated R-algebra, there exists an R-algebra $T = R[x_1, x_2, \ldots, x_n] \subseteq S$ such that $\bar{S}$ is a finite T-module (the generators of $\bar{S}$ over R are integral over S; take x_i to be the coefficients of the equations of integral dependence). Now T and each overring between T and S are coherent by hypothesis. Hence S is a finitely generated (even finitely presented) T-module [24, Corollary 14, p. 167]. This means S is a finitely generated R-algebra - a contradiction.

We next give an example showing that a Cohen overring may not be a valuation ring.

Example 5. Let $G = (\mathbb{R}, +)$ be the additive group of real numbers and $k = \mathbb{Q}(\sqrt{2})$. Denote by G_+ the set of non-negative elements of G. Let K be the field of all functions $f: G_+ \to k$ with well-ordered support in the ordering of G. The elements of K can also be written as formal power series in one indeterminate X with exponents in G and coefficients in k. Let V be the subring of K consisting of those series with exponents in G_+. It is well known that V is a rank-1 non-discrete valuation ring of K, having value group order-isomorphic to G. The maximal ideal $\mathcal{M}$ of V consists of those power series with constant term zero and the residue field is k. We denote by v the corresponding valuation. Let $G=I_1 \cup I_2$, where $I_1=[0,1)$ is the half-open interval and $I_2=[1,\infty)$. Let R be the subring of V of those functions $f \in V$ such that f restricted to I_1 takes values in $\mathbb{Q}$. It is easy to see that R has K as its quotient field. Notice that $V=R[\sqrt{2}]$, because every function $f \in V$ can always be written as $f=g_1+\sqrt{2}g_2$ with $g_1, g_2 \in R$. Thus $\bar{R}=V$ and R is local. Denote by $\mathfrak{m}=\mathcal{M} \cap R$ the maximal ideal of R. Consider the subring $S=k+\mathcal{M}$ of V. Clearly $R \subsetneq S$, since the power series $\sqrt{2}X^{1/2}$ is in $\mathcal{M}$, but not in R. Also $\mathfrak{m}V=\mathcal{M}$. Furthermore $\mathfrak{m}\mathcal{M}=\mathfrak{m}\mathcal{M}V=\mathcal{M}^2=\mathcal{M}$, since V is non-discrete. Hence $\mathfrak{m}S=\mathfrak{m}R+\mathfrak{m}\mathcal{M}=\mathcal{M}$ so that $S=R+\mathfrak{m}S$. Now if S were a finitely generated R-algebra, then integral dependence would make it a finitely generated R-module and by Nakayama's Lemma S would coincide with R - a contradiction. Thus S is not a finitely generated R-algebra. It is also clear that every proper overring of S in K contains V. Thus S is a Cohen overring of R, but S is not a valuation ring.

The next result shows that in the above example S is the only Cohen overring of R, as the ideal $\mathcal{M}$ has to be contained in any Cohen overring of R. One would imagine that Cohen overrings are rather large and that their conductors in integral closures are quite big. We can substantiate it for example when the minimal valuation overrings are non-discrete. Recall that a *valuation ring* is *discrete*, if its maximal ideal is principal.

Proposition 6. *Let R be an integral domain with the finite overring property and $\bar{R}$ the integral closure of R. Assume that $R \subsetneq \bar{R}$ and that $\mathfrak{f}$ is the conductor of R in $\bar{R}$. Then we have the following:*

(i) *If all the minimal valuation overrings of R are non-discrete, then the Jacobson radical $\mathcal{J}(\bar{R})$ of $\bar{R}$ is contained in R. The diagram*

$$\begin{array}{ccc} R & \twoheadrightarrow & R/\mathcal{J}(R) \\ \downarrow & & \downarrow \\ \bar{R} & \twoheadrightarrow & \bar{R}/\mathcal{J}(\bar{R}) \end{array}$$

is a pullback diagram and $R/\mathcal{J}(R) \to \bar{R}/\mathcal{J}(\bar{R})$ is a finite extension of reduced artinian rings.

(ii) *If the minimal valuation overrings of R are independent two by two and if no one of them is rank 1 discrete, then $R/\mathfrak{f}$ and $\bar{R}/\mathfrak{f}$ are noetherian semilocal rings of dimension at most 1. More precisely $\bar{R}/\mathfrak{f}$ is a finite direct product of artinian local rings and rank 1 discrete valuation rings.*

We will need the following result for the proof of Proposition 6.

Lemma 7. *Let R be a domain with the finite overring property. If V is a non-discrete valuation overring of R with maximal ideal $\mathcal{M}$ and $\mathfrak{m}=\mathcal{M} \cap R$, then $\mathfrak{m}V=\mathcal{M}$.*

Proof. Let v be the Krull valuation associated to V. Consider the family of overrings of R of the form $R+xV$ with $x\in\mathcal{M}$. By the noetherian property for overrings, there exists a maximal overring of the form $R+xV$ with $x\in\mathcal{M}$. Since $\mathcal{M}$ is not principal, $xV\subsetneq\mathcal{M}$. For every $y\in\mathcal{M}\setminus xV$, we have $xV\subseteq yV$ and so $R+xV=R+yV$. Thus $y=r+xa$ with $a\in V$ and $r\in R$. Hence $v(y)=v(r)$ and $yV=rV$, whence $r\in\mathfrak{m}$ and $\mathfrak{m}V=\mathcal{M}$.

Proof of Proposition 6. (i) Consider the ring $T=R+\mathcal{J}(\bar{R})$. Assume $R\subsetneq T$. T is a finitely generated R-module. Let $\mathcal{M}_i, 1\le i\le r$ be the maximal ideals of $\bar{R}$ and $\mathfrak{m}_1,\mathfrak{m}_2,\ldots,\mathfrak{m}_t$, $1\le t\le r$ their restrictions to R. We have $\mathcal{J}(R)=\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_t$ and by Lemma 7 $\mathcal{J}(R)\bar{R}_{\mathcal{M}_i}=\mathcal{M}_i\bar{R}_{\mathcal{M}_i}$. Hence $\mathcal{J}(R)\bar{R}=\mathcal{J}(\bar{R})$. Also $(\mathcal{M}_i\bar{R}_{\mathcal{M}_i})^2=\mathcal{M}_i\bar{R}_{\mathcal{M}_i}$ because of non-discreteness so that $\mathcal{J}(R)\mathcal{J}(\bar{R})=\mathcal{J}(\bar{R})$. Thus $T=R+\mathcal{J}(R)T$. Hence by Nakayama's Lemma $T=R$. So $\mathcal{J}(\bar{R})\subseteq R$. The other statements of (i) are easily proven, if we observe that $\bar{R}$ is a strong G-domain.

((ii) Let $\mathcal{M}_i$, $1\le i\le r$ be as before the maximal ideals of $\bar{R}$. Because of the hypothesis, each $\mathcal{M}_i$ contains a largest prime ideal $\mathcal{P}_i$ which is not finitely generated [13, p. 269]. Notice that $\mathcal{P}_i=\mathcal{M}_i$ if and only if $\bar{R}_{\mathcal{M}_i}$ is a non-discrete valuation ring. The hypothesis on the independence of the minimal valuation overrings of R guarantees that each $\mathcal{P}_i$ is contained in exactly one maximal ideal $\mathcal{M}_i$. Hence the $\mathcal{P}_i$ are comaximal two by two. We claim that

$$\mathcal{Q}=\bigcap_{i=1}^{r}=\mathcal{P}_i=\mathcal{P}_1\mathcal{P}_2\cdots\mathcal{P}_r\subseteq R.$$

For, let $T=R+\mathcal{Q}$; then we have $\mathcal{J}(R)\bar{R}=\mathcal{M}_1^{e_1}\mathcal{M}_2^{e_2}\cdots\mathcal{M}_r^{e_r}$ with $e_i\ge 1$, $1\le i\le r$ [17, Theorem 97]; also $\mathcal{J}(R)\mathcal{Q}=\mathcal{Q}$, since $\mathcal{M}_i\mathcal{P}_i\bar{R}_{\mathcal{M}_i}=\mathcal{P}_i\bar{R}_{\mathcal{M}_i}$, $i=1,2,\ldots,r$, since $\mathcal{P}_i\bar{R}_{\mathcal{M}_i}$ is not finitely generated in $\bar{R}_{\mathcal{M}_i}$. Again we conclude by Nakayama's Lemma that $\mathcal{Q}\subseteq R$ and hence $\mathcal{Q}\subseteq\mathfrak{f}$; we have $\mathcal{Q}\bar{R}_{\mathcal{M}_i}=\mathcal{P}_i\bar{R}_{\mathcal{M}_i}\subseteq\mathfrak{f}\bar{R}_{\mathcal{M}_i}$ for every $1\le i\le r$. If $\mathfrak{f}_i=\mathfrak{f}\bar{R}_{\mathcal{M}_i}\cap\bar{R}$, then $\bigcap_{i=1}^{r}\mathfrak{f}_i=\mathfrak{f}$. We have $\mathcal{P}_i\subseteq\mathfrak{f}_i\subseteq\bar{R}$, $1\le i\le r$. This shows that the $\mathfrak{f}_i$'s are comaximal two by two and that $\bar{R}/\mathfrak{f}_i$ is either an artinian local ring or a rank 1 discrete valuation ring or the zero ring. Since $\bar{R}/\mathfrak{f}\simeq\prod_{i=1}^{r}\bar{R}/\mathfrak{f}_i$ and $\bar{R}/\mathfrak{f}$ is a finitely generated $R/\mathfrak{f}$-module, we get the stated result.

Next we give some examples showing that the above description of $\mathfrak{f}$ is the best possible in some sense.

Example 8. We present an example of a G-domain R such that the integral closure $\bar{R}$ is a finitely generated R-module and is a valuation ring with principal maximal ideal $\mathcal{M}$; $\bar{R}$ contains a Cohen overring S of R with $\mathcal{M}\not\subseteq S$.

Let $K=k((X,Y))$ be the field of formal power series in two independent indeterminates over a field k. Let $W=k((X))[[Y]]=k((X))+\mathcal{M}_W$ with $\mathcal{M}_W=YW$. Let $V=k[[X]]+\mathcal{M}_W$. Then W and V are valuation rings of rank 1 and 2 respectively. Let

$$\mathcal{Q}=Yk((X^2))+Y^2k((X)),\qquad R=k[[X^2]]+\mathcal{Q}$$

and

$$S=k[[X^2,X^3]]+\mathcal{M}_W=R[X^3,Y/X,Y/X^3,\dots,Y/X^{2n+1},\dots].$$

R and S are two-dimensional local domains. Furthermore $R\subset S\subset V\subset W$ have the same field of quotients K. It is clear that $R[X]=V$ and that the integral closure $\bar{R}$ of R is V. However S is not a finitely generated R-algebra, if we observe that

$$\frac{Y}{X^{2n+3}}\notin R\left[X^3,\frac{Y}{X},\frac{Y}{X^3},\dots,\frac{Y}{X^{2n+1}}\right].$$

S is in fact a Cohen overring of R ([3], Theorem 3.1) and $\mathcal{M}_V=XV\not\subseteq S$. If we indicate by $\mathfrak{f}$ the conductor of S in $\bar{S}$, then $S/\mathfrak{f}$ is a field and $\bar{S}/\mathfrak{f}$ is a non-reduced artinian local ring.

Example 9. We show now that the ring $\bar{R}/\mathfrak{f}$ of Proposition 6 need not be artinian. Let $F=Q(\sqrt{2})$ and $K=F((X,Y))$. Consider $W=F((X))[[Y]]=F((X))+\mathcal{M}_W$ where $\mathcal{M}_W=YW$ and $V=F[[X]]+\mathcal{M}_W$ and $R=Q[[X]]+\mathcal{M}_W$. Clearly $R\subset V\subset W$ have the same quotient field K. We have $\bar{R}=V$ and that V is a finitely generated R-module. It is easy to see that R satisfies the finite overring property ([13, Theorem 3.1] or Theorem 14 below). The conductor $\mathfrak{f}$ of R in $\bar{R}$ is $\mathcal{M}_W$ which is a height 1 prime ideal of R and $\bar{R}$. Thus $\bar{R}/\mathfrak{f}$ and $R/\mathfrak{f}$ are rank-1 discrete valuation rings (with different quotient fields).

Example 10. We will construct yet another example, this time of a non-local noetherian G-domain R satisfying the finite overring property. The integral closure $\bar{R}\supsetneq R$ will be a Dedekind semilocal domain, the conductor $\mathfrak{f}$ of R in $\bar{R}$ will contain $\mathcal{J}(R)$ properly and $R/\mathfrak{f}$ and $\bar{R}/\mathfrak{f}$ will be reduced semilocal artinian rings.

Take $K=k((X_1,X_2,X_3))$ the field of formal power series in three indeterminates. Let

$$V_1=k((X_2,X_3))[[X_1]],\qquad V_2=k((X_1,X_3))[[X_2]],$$
$$V_3=k((X_1,X_2))[[X_3]].$$

We have three rank-1 discrete valuation rings of K incomparable two by two. Consider

$$A_1=k((X_2^2,X_3))+X_1V_1\quad\text{and}\quad A_2=k((X_1^2,X_3))+X_2V_2$$

and denote by R the integral domain $R=A_1\cap A_2\cap V_3$. It is easily seen that R is a noetherian one-dimensional G-domain with three maximal ideals and that $\bar{R}=V_1\cap V_2\cap V_3$. We have the following pullback diagram:

$$\begin{array}{ccc} R & \longrightarrow\!\!\!\!\rightarrow & k_1^*\times k_2^* \\ \cap\downarrow & & \cap\downarrow v \\ \bar{R} & \underset{u}{\longrightarrow\!\!\!\!\rightarrow} & k_1\times k_2 \end{array}$$

Here υ is the product of the canonical embeddings

$$k_1^* = k((X_2^2, X_3)) \to k_1 = k((X_2, X_3)), \qquad k_2^* = k((X_1^2, X_3)) \to k_2 = k((X_1, X_3))$$

and u is the natural projection from $\bar{R}$ to the product of the residue fields of two of its three maximal ideals. The results we claim follow easily from the general theory concerning pullback diagrams [9, §1].

Remark 11. Other examples may be constructed using the $k+(M_1 \cap M_2)$ construction in [14] or [18]. If we choose $k+M_1$ and $k+M_2$ to be strong G-domains there, then $D = k+(M_1 \cap M_2)$ is local with $\bar{D}$ not local. it is easy to check that D has the finite overring property. The ring D^* of Proposition 4.5 in [14] is again a ring with the finite overring property. This ring D^* is not local.

We want now to characterize domains with the finite overring property. The following result will be needed in this connection. Recall that R is a *locally pqr ring* if for every $\mathfrak{p} \in \operatorname{Spec} R$, the localization $R_\mathfrak{p} = R_t$ for some $t \in R$ [28].

Proposition 12. *Let R be an integral domain with integral closure $\bar{R}$. Assume that for every multiplicative subset $\mathcal{S}$ of R, there exists $t \in \bar{R}$ such that $\bar{R}_\mathcal{S} = \bar{R}_t$. Then R is a locally pqr domain. In particular, the conclusion holds if $\bar{R}$ is a strong G-domain.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. We want to show that there exists $f \in R \setminus \mathfrak{p}$ such that $R_\mathfrak{p} = R_f$. Let $\mathcal{S} = R - \mathfrak{p}$. By hypothesis, there exists $t \in \bar{R}$ such that $\bar{R}_\mathcal{S} = \bar{R}_t$. Now $t^{-1} \in \bar{R}_\mathcal{S}$ so that $t^{-1} = x/f$ with $x \in \bar{R}$ and $f \in \mathcal{S}$. Then $1 = tt^{-1} = tx/f$. Hence $f = tx \in R \setminus \mathfrak{p}$. We claim that $R_f = R_\mathfrak{p}$. Clearly $R_f \subseteq R_\mathfrak{p}$. Now take any $s \in R \setminus \mathfrak{p}$; we will show that $1/s = r/f^m$ for some $r \in R$ and $m \geq 1$. Notice that $t \in \mathfrak{q}$ for every $\mathfrak{q} \in \operatorname{Spec}(\bar{R})$ with $\mathfrak{q} \cap \mathcal{S} \neq \emptyset$. Thus $f = tx \in \mathfrak{q} \cap R$ for every such prime $\mathfrak{q}$. If $\mathfrak{q}' \in \operatorname{Spec}(R)$ and $s \in \mathfrak{q}'$, then $\mathfrak{q}' \nsubseteq \mathfrak{p}$. Let now $\mathfrak{q}$ be a prime ideal of $\bar{R}$ over $\mathfrak{q}'$. Hence $\mathfrak{q} \cap \mathcal{S} \neq \emptyset$. Then $f \in \mathfrak{q}' = \mathfrak{q} \cap R$ so that $\operatorname{rad}(sR) \ni f$. This shows that there exist $m \geq 1$ and $r \in R$ such $f^m = sr$, as we wanted to prove.

With simple topological techniques, we can prove a more general result:

Proposition 13. *Let $R \to T$ be an integral extension of integral domains. If T is a locally pqr domain, then so is R.*

Proof. By [11, §3], it is enough to show that $\operatorname{Spec}(R)$ is an Alexandroff discrete topological space; i.e. that for every family of closed sets $\{F_i \mid i \in I\}$ in $\operatorname{Spec}(R)$ the union $\bigcup_{i \in I} F_i$ is still a closed set of $\operatorname{Spec}(R)$. Denote by $\varphi : \operatorname{Spec}(T) \to \operatorname{Spec}(R)$, the continuous map cononically associated to the given integral homomorphism $R \to T$. Clearly φ is surjective and closed [2, Theorem 5.1 and Ex. 1, p. 67]. Now $\varphi^{-1}(\bigcup F_i) = \bigcup \varphi^{-1}(F_i)$ is a closed set of $\operatorname{Spec}(T)$, since T is locally *pqr*. Thus $\bigcup F_i = \varphi(\varphi^{-1}(\bigcup F_i))$ is a closed set of $\operatorname{Spec}(R)$.

Recall that $R \subseteq T$ is called a *very finite extension* of rings, it T and all the intermediate rings between R and T are finitely generated as R-modules [24, p. 166]. Examples of such extensions are the finite minimal homomorphisms in the sense of Ferand–Olivier [8], the finitely generated integral ring extensions of noetherian rings and extensions arising via Gilmer's $D+M$ constructions. As one would expect, this is the key to the characterization of integral domains with the finite overring property.

Theorem 14. *Let R be an integral domain. Then the following statements are equivalent:*

(1) *R has the finite overring property.*

(2) *The integral closure $\bar{R}$ of R is a strong G-domain and $R \subseteq \bar{R}$ is a very finite extension.*

Proof. (1)⇒(2) is clear from Theorem 1.

(2)⇒(1). If (1) is not valid, let S be a Cohen overring of R. If S is integrally closed, then S is a valuation ring by Corollary 3 and there exists $f \in \bar{R}$ such that $S = \bar{R}_f$ [28, Theorem 3.5]. Since $\bar{R}$ is a finite R-module, we see that S is a finitely generated R-algebra, contradicting the assumption that S is a Cohen overring. Hence S is properly contained in its integral closure $\bar{S}$ and $\bar{S}$ is a finitely generated R-algebra. Let T be the integral closure of R in $\bar{S}$. Then $\bar{S}$ is a finitely generated T-algebra. Hence we can find a ring T^*, $T \subseteq T^* \subseteq S$, such that T^* is a finitely generated T-algebra and $\bar{S}$ is a finite T^*-module. Now T is integrally closed in T^*. Let $\mathcal{M}$ be the maximal ideal of S and let $\mathfrak{m}^* = \mathcal{M} \cap T^*$ and $\mathfrak{m} = \mathcal{M} \cap T$. Observe that T^* is a local domain with maximal ideal $\mathfrak{m}^*$. Since $\bar{R}$ and T are Prüfer domains, we deduce [22, Proposition 2.26] that $T \subseteq T^*$ is quasi-finite at $\mathfrak{m}^*$ (see [29, p. 40] for the definition). Since T^* is local, we get immediately from Proposition 4 of [29, p. 43] that $T^* = T_{\mathfrak{m}}$. Since T is integrally closed in S, so is $T_{\mathfrak{m}}$. Thus $T^* = S$. Hence $S = T_{\mathfrak{m}} = T_f$ for some $f \in T \setminus \mathfrak{m}$ (Proposition 12). Since $T \subseteq \bar{R}$, T is a finitely generated R-module by hypothesis. Hence S is a finitely generated R-algebra, again leading to a contraction. We conclude that no Cohen overring of R can be found and that R has indeed the finite overring property.

3. Non-finite presentation

It is natural to attempt to force the conclusion of Proposition 4 by stronger assumptions. A variation on the theme is the notion of *presentation* in the place of generation. I. Papick in [24] has already successfully explored this in a similar setting.

Let R be a G-domain. By a *Cohen overring of R with respect to presentation*, we mean an overring S of R with is not finitely presented as an R-algebra and is maximal with respect to this property. One curious phenomenon is the following:

A Cohen overring S with respect to presentation may not have the finite presentation overring property! See Example 5 (contd.) below: While a proper overring of S is finitely presented as an R-algebra, it may not be so as an S-algebra! However S has the finite overring property and an analogue of Proposition 2 holds.

Proposition 15. *Let R be a G-domain. Then every Cohen overring of R with respect to presentation is a local ring.*

Proof. Let S be one such ring. As remarked above, S has the finite overring property. Hence we can apply the argument in Proposition 2. Thus S is semilocal and if S were not local, then for every maximal ideal $\mathcal{M}$ of S, $S_{\mathcal{M}}$ would be a finitely presented R-algebra. Moreover for every $\mathfrak{p} \in \mathrm{Spec}(S)$, there exists $f \in S \setminus \mathfrak{p}$ such that $S_{\mathfrak{p}} = S_f$. Hence the canonical map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a locally finitely presented morphism of schemes [15, Definition I.6.2.1, p. 297]. We conclude [15, Proposition I.6.2.9, p. 302] that S itself is a finitely presented R-algebra – a contradiction. Hence S is local.

As in the finite generation case, a Cohen overring with respect to presentation need not be integrally closed.

Example 5 (contd.) We reconsider Example 5. We claim that the ring S is also a Cohen overring of R with respect to presentation. First we observe that the integral closure $\bar{R}$ is a finitely presented R-algebra. Indeed, if Z is an indeterminate over R, then the canonical homomorphism $\upsilon : R[Z] \to R[\sqrt{2}] = V$ sending Z to $\sqrt{2}$ has its kernel generated by $Z^2 - 2$, $\sqrt{2}XZ - 2X$ and $XZ - \sqrt{2}X$. Since there are no intermediate rings between S and V, our claim is established. Notice that S does not have the finite presentation property, since S is not coherent (Remark following Proposition 9 of [24, p. 66] and Theorem 3 of ([6]). In fact, something even stranger happens: the ring $\bar{S} = V$ is not even a finitely presented S-algebra or S-module ([15, Proposition I.6.2.10] and [24, Lemma 6]), even though it is a finitely generated S-module. Finally, notice that the ring R is coherent, while the Cohen overring S is not.

It may happen that a Cohen overring (with respect to generation) may be properly contained is a Cohen overring with respect to presentation.

Example 8 (contd.) We reconsider Example 8. We claim that $\bar{R} = V$ is not a finitely presented R-algebra. To see this, observe that $\bar{R}$ is a finitely presented S-algebra, since S is coherent ([6, Theorem 3] and proof of Corollary 1.4 of [16]). Now if $\bar{R}$ were finitely presented R-algebra, then S would be finitely generated over R – a contradiction (Lemma 13 of [24]). Thus the two types of Cohen overrings are distinct. In fact V in this case is a Cohen overring of R with respect to presentation.

We pass to the analogue of Proposition 4. We first consider a preliminary result suggested by the proof of Proposition 9 in [24].

Proposition 16. *Let R be a coherent G-domain. Suppose that the integral closure $\bar{R}$ of R is a Prüfer domain. Then local overrings of R which are finitely generated as R-algebras are also finitely presented as R-algebras.*

Proof. Let $(S, \mathcal{M})$ be a local overing of R finitely generated as an R-algebra. Let T be the integral closure of R in S and $\mathfrak{m} = \mathcal{M} \cap T$. Then reasoning as in the proof of Theorem 14, we see via Proposition 2.26 of [22] that $T \subseteq S$ is a quasi-finite extension at $\mathfrak{m}$. We deduce that $S = T_{\mathfrak{m}}$ [29, Proposition 4, p. 43]. Since S is a finitely generated T-algebra, we can find elements $t_1, t_2, \ldots, t_n \in T$ and $x_1, x_2, \ldots, x_n \in T \setminus \mathfrak{m}$ such that $S = T[t_1/x_1, t_2/x_2, \ldots, t_n/x_n]$. Let $R^* = R[t_1, t_2, \ldots, t_n, x_1, x_2, \ldots, x_n]$. Then $S = R^*[1/x_1, 1/x_2, \ldots, 1/x_n]$. This shows that S is a finitely presented R^*-algebra. Furthermore, R^* is a finitely generated torsion-free module over the coherent domain R and so R^* is a finitely presented R-module [16, Proof of Corollary 1.4]. Then R^* is also a finitely presented R-algebra [15, Proposition I.6.2.10, p. 302]. By transitivity ([15, p. 135] or [19, F.2, p. 49]), we conclude that S is a finitely presented R-algebra.

Proposition 17. *Let R be a coherent G-domain such that all the overrings of R are coherent. then every Cohen overring of R with respect to presentation is a valuation ring.*

Proof. Let S be one such Cohen overring. By Theorem 1 of [25], we know that the integral closure $\bar{R}$ is a Prüfer domain. Thus applying Proposition 16, we conclude that S can not be a finitely generated R-algebra. Hence S is also a Cohen overring of R with respect to generation. The result follows from Proposition 4.

The following characterization of domains with the finite presentation overring property generalizes Proposition 23 of Papick in [24], removing the restriction on the dimension.

Theorem 18. *The following conditions on a G-domain are equivalent:*

(1) *R has the finite presentation overring property.*

(2) *R is coherent and has the finite overring property.*

(3) *R is coherent, the integral closure $\bar{R}$ is a strong G-domain and $R \subseteq \bar{R}$ is a very finite extension.*

Proof. The equivalence of (1) and (2) is proved in Proposition 9 of [24] and the remark following it. That (2) and (3) are equivalent follows from Theorem 14.

We close our study by posing two relevant questions:

Questions 19. What are those G-domains R for which all Cohen overrings are valuation rings? What are those G-domains for which the two types of Cohen overrings that we have defined coincide?

Acknowledgements

The authors are grateful to R. Gilmer and I.J. Papick for their stimulating correspondence. They wish to thank R.C. Cowsik and Amit Roy for discussions on some of the examples in this work. The second author wishes to thank the Tata Institute of Fundamental Research, Bombay for their hospitality during the final phase of this work.

References

[1] D.D. Anderson and D.E. Dobbs, Pairs of rings with the same prime ideals, Canad. J. Math. 32 (1980) 451-463.

[2] M.F. Atiyah and I.G. MacDonald, Introduction to Commutative Algebra (Addison-Wesley, New York, 1969).

[3] E. Bastida and R. Gilmer, Overrings and divisorial ideals of rings of the form $D+M$, Michigan Math. J. 20 (1974) 79-95.

[4] N. Bourbaki, Algèbre Commutative (Hermann, Paris, 1961-1965).

[5] J.W. Brewer and D. Costa, Seminormality and projective modules over polynomial rings, J. Algebra 58 (1979) 217-226.

[6] D.E. Dobbs and I.J. Papick, When is $D+M$ coherent?, Proc. Amer. Math. Soc. 56 (1976) 51-54.

[7] O. Endler, Valuation Theory (Springer, Berlin, 1972).

[8] D. Ferrand and J.P. Olivier, Homomorphismes minimaux d'anneaux, J. Algebra 16 (1970) 461-471.

[9] M. Fontana, Topologically defined classes of commutative rings, Annali Mat. Pura Appl. 123 (1980) 331-355.

[10] M. Fontana, Carrés cartesiens et anneaux de pseudo-valuation, Publ. Math. Univ. Lyon 17 (1981) 57-95.

[11] M. Fontana and P. Maroscia, Sur les anneaux de Goldman, Boll. Un. Mat. Ital. 13-B (1976) 743-759.

[12] R. Gilmer, Overrings of Prüfer domains, J. Algebra 4 (1966) 331-340.

[13] R. Gilmer, Multiplicative Ideal Theory, Queen's Papers on Pure and Applied Mathematics, No. 12 (Queen's University, Kingston, Ontario, 1968).

[14] R. Gilmer and W. Heinzer, Intersections of quotient rings of an integral domain, J. Math. Kyoto Univ. 7 (1967) 133-150

[15] A. Grothendieck and J.A. Dieudonné, Eléments de Géométrie Algébrique I, Die Grundlehren der Math. Wissenschaften, Band 166 (Springer, Berlin, 1971).

[16] M.E. Harris, Some results on coherent rings, Proc. Amer. Math. Soc. 17 (1966) 474-479.

[17] I. Kaplansky, Commutative Rings (Allyn and Bacon, Boston, 1970).

[18] T. Kikuchi, Some remarks on S-domains, J. Math. Kyoto Univ. 6 (1966) 49-60.

[19] B. Iversen, Generic Local Structure in Commutative Algebra (Springer, Berlin, 1973).

[20] P. Maroscia, Topological properties of some classes of G-domains, Boll. Un. Mat. Ital. 15-A (1978) 688-692.

[21] P. Maroscia, Some results on G-domains, Boll. Un. Mat. Ital., to appear.
[22] I.J. Papick, Topologically defined classes of going-down domains, Trans. Amer. Math. Soc. 219 (1976) 1-37.
[23] I.J. Papick, Coherent overrings, Canad. Math. Bull. 22 (1979) 331-337.
[24] I.J. Papick, Finite type extensions and coherence, Pacific J. Math. 78 (1978) 161-172.
[25] I.J. Papick, A note on proper overrings, Preprint.
[26] R.L. Pendleton, A characterization of Q-domains, Bull. Amer. Math. Soc. 72 (1966) 499-500.
[27] G. Picavet, Sur les anneaux commutatifs dont tout idéal premier est de Goldman, C.R. Acad. Sci. Paris A 280 (1975) A1719-A1721.
[28] R. Ramaswamy and T.M. Viswanathan, Overring properties of G-domains, Proc. Amer. Soc. 58 (1976) 59-66.
[29] M. Raynaud, Anneaux Locaux Henséliens (Springer, Berlin, 1970).
[30] O. Zariski and P. Samuel, Commutative Algebra, Vol. II (Von Nostrand, Princeton, 1960).