Locally Pseudo-Valuation Domains (*).

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Sunto. – *In questo lavoro vengono studiate due controparti globali della nozione di dominio di pseudo-valutazione (abbreviato,* PVD), *un tipo di dominio quasi-locale diviso introdotto da Hedstrom-Houston (Pac. J. Math.,* **75** (1978), pp. 137-147). *La più larga di queste controparti è la classe dei domini localmente di pseudo-valutazione (abbr.,* LPVD), *la quale è contenuta tra la classe dei domini di Prüfer e quella dei domini seminormali localmente divisi. L'altra è quella dei domini globalmente di pseudo-valutazione (abbr.,* GPVD); *ciascun dominio R di tale classe è un* LPVD *con un sopraanello di Prüfer* unibranched *canonicamente associato. Per i domini quasi-locali, le nozioni di* LPVD, GPVD *e* PVD *coincidono.* E*ntrambe le classi (dei* GPVD *e dei* LPVD) *sono stabili in relazione a svariate costruzioni ed operazioni della teoria dei domini. Ciascun sopraanello di un dominio R è un* LPVD *se, e solamente se, R è un* LPVD *la cui chiusura integrale è un dominio di Prüfer. Se R è un* PVD *avente V come sopraanello di valutazione canonicamente associato e se R^* (risp., V^*) è la chiusura integrale di R (risp., V) in un campo contenente R, allora R^* è un* GPVD, *avente V^* come sopraanello di Prüfer canonicamente associato. Numerosi esempi di* LPVD *e* GPVD *vengono costruiti.*

1. – Introduction.

One of the most fruitful recent generalizations of the concept of a valuation domain is that of a pseudo-valuation domain, or PVD. This type of quasi-local domain, introduced by HEDSTROM-HOUSTON [17] and studied extensively thereafter ([6], [18], [1], [11]), is particularly interesting since any pseudo-valuation domain R admits a canonically associated valuation overring V with the same set of prime ideals as R, such that R may be recovered from its residue-class field and V by a pullback construction [1, Proposition 2.6]. It happens that any PVD is a divided domain, a type of quasi-local going-down ring introduced in [5] (cf. also [12]), and one now has at hand a theory of the so-called locally divided domains ([7], [12]). Accordingly, it is natural to ask if PVD's also admit a global counterpart, forming a class of seminormal domains intermediate between Prüfer domains and locally divided domains, possibly such that each of these new domains has an associated unibranched Prüfer overring from which its structure may be recovered a by pullback. This paper answers the above question by finding two such counterparts.

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The first, and larger, of these counterparts is the class of locally pseudo-valuation domains, or LPVD's, introduced in section 2. A domain R is naturally defined to be a LPVD in case R_P is a PVD for each prime ideal P of R. The class of LPVD's clearly contains all Prüfer domains and all PVD's; it contains an abundance of other semi-quasi-local domains, by applying the construction in [14, section 3] (see Example 2.5 and Remarks 2.11 (*b*), (*c*)); and it is stable under several domain-theoretic constructions (see Remarks 2.4 (*e*) and Propositions 2.6 and 2.7). Perhaps the most interesting result in section 2 is the following part of Theorem 2.9: if R is a LPVD, then each overring of R is an LPVD if and only if the integral closure of R is a Prüfer domain. While this statement may be viewed as a globalization of [18, Proposition 2.7], its proof uses recent work on seminormality [2].

Section 3 treats the class of GPVD's, a second global counterpart to PVD's. Roughly speaking, a domain R is a GPVD in case R is an LPVD whose localizations have pullback descriptions arising from a canonically associated unibranched Prüfer overring of R which has the same Jacobson radical as R. (For a more precise statement, see the characterization of GPVD's in Theorem 3.1.) Examples of Noetherian GPVD's are provided by the rings $\mathbb{Z}[\sqrt{d}]$, where d is a square-free integer such that $d \equiv 5 \pmod 8$ (see [8] for details). By using the material on K-rings in [21], Example 3.2 (*b*) presents a one-dimensional non-Noetherian GPVD whose associated Prüfer overring is the Prüfer domain with uncountable maximal spectrum constructed in [13, Example 1]. Example 3.4 presents an LPVD which, although not a GPVD, has Prüfer integral closure. (Its construction depends on some lemmas of independent interest concerning locally finite intersections; their proofs, patterned after [19], may be found in the appendix.) Nevertheless, the class of GPVD's is quite extensive, containing in particular all Prüfer domains and all PVD's, as well as the semi-quasi-local LPVD's mentioned earlier. It also behaves well under integral closure (Proposition 3.5), and Noetherian GPVD's can be characterized by a global counterpart of [11, Corollaire 1.6] (Proposition 3.6). Perhaps the following is the most natural way for GPVD's to arise. Let R be a GPVD (for instance, a PVD) with associated Prüfer domain T, let F be a field containing T, and let R^* (resp., T^*) be the integral closure of R (resp., T) in F. It is a classic result of Prüfer that T^* is a Prüfer domain; Corollary 3.9 asserts that R^* is a GPVD, with associated Prüfer domain T^*. Results in the same vein for more general classes of going-down rings have been scarce (cf. [6, Theorem 3.2]).

All rings considered below are commutative, with 1. Data consisting of a quasi-local ring R with maximal ideal M and residue-class field $k = R/M$ will be summarized as either (R, M) or (R, M, k), with k denoted by either $\boldsymbol{k}(R)$ or $\boldsymbol{k}(M)$. More generally, if P is a prime ideal of a ring R, then $\boldsymbol{k}(P)$ denotes R_P/PR_P. If R is a ring then the Krull dimension of R, the Jacobson radical of R, the complete integral closure of R and the integral closure of R are denoted by $\dim(R)$, $J(R)$, $C(R)$ and R' respectively. We assume familiarity with the literature on PVD's and with pullback techniques, as in [9]. Any unexplained terminology is standard, as in [15] and [19].

2. – Locally pseudo-valuation domains.

Let R be a domain, with quotient field K. As in [17], we say that a prime ideal P of R is *strongly prime* in case, whenever elements x and y of K satisfy $xy \in P$, then either $x \in P$ or $y \in P$; and that R is a *pseudo-valuation domain* (or, in short, a PVD) in case (R, M) is quasi-local and M is strongly prime. In order to globalize the PVD concept, we need the following definition and preliminary result. We shall say that a prime P of R is *locally strongly prime* (*in* R) if PR_P is strongly prime; equivalently, if R_P is a PVD.

LEMMA 2.1. – If $Q \subset P$ are distinct primes of a domain R and P is locally strongly prime in R, then Q is also locally strongly prime in R.

PROOF. – Since QR_P is a nonmaximal prime of the pseudo-valuation domain R_P [17, Proposition 2.6] guarantees that $(R_P)_{QR_P} \cong R_Q$ is a valuation domain and, hence, a PVD, as required.

PROPOSITION 2.2. – For a domain R, the following conditions are equivalent:

(1) R_M is a PVD, for each maximal ideal M of R;

(2) Each prime of R is locally strongly prime in R.

PROOF. – By the above comments, (1) is equivalent to requiring each maximal ideal of R to be locally strongly prime. However the latter condition is equivalent to (2) by virtue of Lemma 2.1, since each prime is contained in a suitable maximal ideal.

A domain R satisfying the equivalent conditions in Proposition 2.2 will be called a *locally pseudo-valuation domain* (or, in short, an LPVD). It is clear that any Prüfer domain is an LPVD, and so is any PVD. A more interesting family of examples is given in Example 2.5. First, we pause to relate the LPVD concept to the studies in [5], [7].

COROLLARY 2.3. – Any LPVD is a locally divided domain.

PROOF. – It suffices to recall that any PVD is a divided domain [6, p. 560].

REMARKS 2.4. – (*a*) As in the proof of Corollary 2.3, we may use earlier studies of PVD's to show that LPVD's satisfy additional local properties. For instance, any LPVD must be seminormal, since it is known (cf. [2, Proposition 3.1 (*a*)]) that any PVD is seminormal.

(*b*) The converse of Corollary 2.3 fails, even in the quasi-local integrally closed case [6, Remark 4.10 (*b*)].

(*c*) By combining Corollary 3 with [7, p. 124], we see that any LPVD is treed, in the sense that its prime spectrum, as a poset under inclusion, forms a tree. A similar appeal to [7, Theorem 2.4] reveals that if R is an LPVD, then $R + PR_P$ is an R-flat overring, for each prime P of R.

(*d*) The LPVD concept has appeared recently in [2, Corollary 3.6], which produced a natural class of one-dimensional domains satisfying condition (2) in Theorem 2.9 below.

(*e*) In view of what is known about PVD's, Proposition 2.2 readily yields that if R is an LPVD, then R/P and R_S are also LPVD's, for each prime ideal P and multiplicatively closed subset S of R.

EXAMPLE 2.5. – Let $n \geqslant 2$ be a positive integer. Then there exists a locally pseudo-valuation domain T with precisely n maximal ideals, such that T is neither a Prüfer domain nor a PVD.

To indicate such a construction, consider a field k with the following two properties: (1) there exist n pairwise incomparable valuation domains $V_i = k + M_i$ having (maximal ideal M_i, residue class field k and) a common quotient field; (2) there exist n distinct proper subfields k_i of k. Then $T = \bigcap (k_i + M_i)$ has the asserted properties.

For a proof, first set $Q_i = M_i \cap T$ for each i. By appealing to [14, Theorem 3.1] (cf. also [10, Proposizione 5.6.1 (*f*)]), we have that $Q_1, \dots, Q_n$ are distinct and form the maximal spectrum of T. Since $n \geqslant 2$, T is not quasi-local and, hence, is not a PVD. Moreover, for each i, we claim that $T_{Q_i} \cong k_i + M_i$. As each $k_i + M_i$ is a PVD but not a valuation domain (by, for instance, [17, Example 2.1]), it will follow that T is an LPVD but not a Prüfer domain.

To complete the proof, we proceed to establish the above claim. First, recall that $P = \bigcap V_i$ is a Prüfer domain with precisely n distinct maximal ideals, given by $N_i = M_i \cap P$, such that $P_{N_i} = V_i$ for each i (cf. [19, Theorem 107]). Next, it will be convenient to regard T as being constructed in n steps. The first of these steps produces a domain, say T_1, as the pullback of the diagram

$$\begin{array}{ccc} & & k_1 \\ & & \downarrow \\ P & \longrightarrow & k \end{array}$$

in which the vertical map is the inclusion and the horizontal map is the composite of the canonical surjection $P \to P/N_1$ with the isomorphism $P/N_1 \to k$. By [9, Theorem 1.4], $T_1 \subset P$ is a unibranched extension; in particular, the primes of T_1 are N_1 and ideals of the form $I \cap T_1$, for primes $I \neq N_1$ of P; and $(T_1)_{N_j \cap T_1} = P_{N_j}$ $(= V_j)$ for each $j \geqslant 2$. We shall next describe the localization of T_1 at its remaining maximal ideal, N_1.

Since [9, Theorem 1.4] yields that Spec (T_1) and Spec (P) are homeomorphic, it follows readily that $P \setminus N_1$ is the saturation in P of the multiplicatively closed set $T_1 \setminus N_1$. Thus $P_{T_1 \setminus N_1} = P_{N_1}$ $(= V_1)$, so that [9, Proposition 1.9] supplies the pullback description $(T_1)_{N_1} \cong k_1 \times_k V_1$, whence $(T_1)_{N_1} = k_1 + M_1$. The next step of the construction produces T_2 as the pullback of the diagram

$$\begin{array}{ccc} & & k_2 \\ & & \downarrow \\ T_1 & \longrightarrow & k \end{array}$$

in which the vertical map is the inclusion and the horizontal map is the composite of the canonical surjection $T_1 \to T_1/N_2 \cap T_1$ and the isomorphisms $T_1/N_2 \cap T_1 \to$ $\to V_2/M_2 \to k$. As above, [9, Theorem 1.4] may be applied to show that $T_2 \subset T_1$ is a unibranched extension; and, besides $N_2 \cap T_1$, the other primes of T_2 take the form $J \cap T_2$, for primes $J \neq N_2 \cap T_1$ of T_1. We shall next describe the localizations of T_2 at each of its maximal ideals.

To this end, note first that $N_2 \cap T_1 \not\subset N_1$. (Otherwise, $k_1 + M_1 = (T_1)_{N_1} \subset (T_1)_{N_2 \cap T_1} =$ $= k + M_2$, whence $V_1 \subset V_2$, contrary to hypothesis.) Accordingly, [9, Theorem 1.4(c)] applies, giving $(T_2)_{N_1 \cap T_2} = (T_1)_{N_1}$ $(= k_1 + M_1)$. Moreover, if $j \geqslant 3$, then $N_2 \cap T_1 \not\subset N_j$. (The point is that the homeomorphism Spec $(T_2) \to$ Spec (T_1), given by [9, Theorem 1.4], induces an isomorphism of the underlying posets.) Accordingly, [9, Theorem 1.4 (c)] applies again, yielding $(T_2)_{N_j \cap T_2} = (T_1)_{N_j \cap T_1}$ $(= V_j)$ for $j \geqslant 3$. For the localization of T_2 at its remaining maximal ideal, $N_2 \cap T_1$, one may argue as above invoking the pullback description $k_2 + M_2 \cong P_{N_2} \times_k k_2$ to show that $(T_2)_{N_2 \cap T_1} = k_2 + M_2$.

By iteration of the above process, we arrive at a domain T_n, with precisely n distinct maximal ideals, at which the respective localizations are the domains $k_i + M_i$. Since T_n is the intersection of its localizations at maximal ideals, $T_n = T$, and the proof is complete.

Propositions 2.6 and 2.7 offer additional ways to construct LPVD's. First, in order to ease the statement of Lemma 2.8, it is convenient next to recall the following terminology and facts. Let (R, M) be a quasi-local domain with quotient field K. R is a PVD if and only if M is also a maximal ideal of some valuation overring V of R [17, Theorem 2.7]. In this case, V is uniquely determined as the conductor $V = (M:M) = \{x \in K : xM \subset M\}$, by [1, Proposition 2.5], and is called the *valuation domain associated to* R; Spec (R) = Spec (V) as sets; if $R \neq V$, than $V = (R:M)$, by [17, Theorem 2.10]; and R may be recovered as the pullback $R = V \times_{\boldsymbol{k}(V)} \boldsymbol{k}(R)$, by [1, Proposition 2.6].

PROPOSITION 2.6. – If S is an overring of a locally pseudo-valuation domain R such that the extension $R \subset S$ satisfies INC, then S is also an LPVD. In particular, each integral overring of an LPVD must be an LPVD.

PROOF. – It is enough to show that S_N is a PVD, for each maximal ideal N of S. Note that $R_{N\cap R}$ is a PVD by virtue of Proposition 2.2. Since the overring extension $R_{N\cap R}\subset S_N$ inherits INC from $R\subset S$, an application of [17, Theorem 1.7] completes the proof.

PROPOSITION 2.7. – Let $R\subset S$ be an integral extension of domains, such that R is integrally closed. Then:

(1) If S_N is a PVD for some prime ideal N of S, then $R_{N\cap R}$ is also a PVD.

(2) If S is an LPVD, then R is also an LPVD.

PROOF. – Since integral extensions satisfy the lying-over property (cf. [19, Theorem 44]), (2) follows immediately from (1). As for (1), note that [15, Proposition 12.7] guarantees that $S_N\cap K = R_{N\cap R}$, where K denotes the quotient field of R, so that (1) is a consequence of part (1) of the following result.

PROPOSITION 2.8. – Let S be a PVD with maximal ideal N, quotient field F, and associated valuation domain V. Let K be a subfield of F. Set $R = S\cap K$, $W = V\cap K$ and $M = N\cap R$. Then:

(1) R is a PVD with maximal ideal $M = N\cap K$ and associated valuation domain W.

(2) If F is algebraic over K, then the contraction map $\text{Spec}\,(S)\to\text{Spec}\,(R)$ is an inclusion-preserving bijection and hence $\dim\,(R) = \dim\,(S)$.

(3) Assume that $[F:K]<\infty$. Then R is Noetherian if and only if S is Noetherian such that $[\boldsymbol{k}(S):\boldsymbol{k}(R)]<\infty$.

PROOF. – (1) Since V is a valuation domain of F with maximal ideal N, [15, Theorem 19.16 (*a*), (*b*)] implies that W is a valuation domain of K with maximal ideal $N\cap K = M$. Moreover, M is also the unique maximal ideal of R. To see this, observe for any $r\in R\setminus M$ that $r\in S\setminus N$, so that $r^{-1}\in S\cap K = R$, as desired. Since $R = W\times_{\boldsymbol{k}(W)}\boldsymbol{k}(R)$, the assertions in (1) now follow directly from [1, Proposition 2.6].

(2) As F/K is algebraic, [15, Theorem 19.16 (*b*), (*c*)] assures that the contraction map $\text{Spec}\,(V)\to\text{Spec}\,(W)$ is an inclusion-preserving bijection and $\dim\,(W) = \dim\,(V)$. Since $\text{Spec}\,(S) = \text{Spec}\,(V)$ and $\text{Spec}\,(R) = \text{Spec}\,(W)$, (2) readily follows.

(3) Suppose first that (the pseudo-valuation domain) R is Noetherian. By [17, Proposition 3.2], $\dim\,(R)\leqslant 1$. If $\dim\,(R) = 0$, then (2) shows that S is a field, trivially Noetherian. In the remaining case, $\dim\,(R) = 1$, and the Krull-Akizuki theorem may be applied, to conclude that S is Noetherian. Moreover, applying [11, Corollaire 1.6] to the pullback description of R reveals that $[\boldsymbol{k}(W):\boldsymbol{k}(R)]<\infty$. How-

ever, finiteness of $[F:K]$ guarantees finiteness of $[\boldsymbol{k}(V):\boldsymbol{k}(W)]$ (cf. [4, Lemma 2, p. 417]). Then $[\boldsymbol{k}(S):\boldsymbol{k}(R)]$, being a divisor of $[\boldsymbol{k}(V):\boldsymbol{k}(R)] = [\boldsymbol{k}(V):\boldsymbol{k}(W)][\boldsymbol{k}(W):\boldsymbol{k}(R)]$, is necessarily finite.

Conversely, to show that R is Noetherian, [11, Corollaire 1.6] reduces the task to proving $[\boldsymbol{k}(W):\boldsymbol{k}(R)] < \infty$ and W is a DVR. Since S is assumed Noetherian, applying [11, Corollaire 1.6] to the pullback description of S yields that $[\boldsymbol{k}(V):\boldsymbol{k}(S)] < \infty$ and V is a DVR. As $[\boldsymbol{k}(S):\boldsymbol{k}(R)] < \infty$ by hypothesis, we argue as above that $[\boldsymbol{k}(V):\boldsymbol{k}(R)] < \infty$, whence $[\boldsymbol{k}(W):\boldsymbol{k}(R)] < \infty$. Finally, W inherits the DVR property from V, by [4, Corollary 3, p. 418], to complete the proof.

Let R be a domain, with integral closure R'. If R is a coherent LPVD, then [6, Proposition 4.2] readily implies that R' is a Prüfer domain and [18, Theorem 1.9] readily implies that each overring of R is an LPVD. The relation between these conditions is given next in Theorem 2.9, this section's main result. First, recall from [20] that R is said to be an *i-domain* if the contraction map Spec $(S) \to$ Spec (R) is an injection for each overring S of R; equivalently (cf. [20, Corollary 2.15]), if the integral closure of R_M is a valuation domain for each maximal ideal M of R.

THEOREM 2.9. – Let R be a domain, with integral closure R'. Then the following conditions are equivalent:

(1) Each overring of R is an LPVD;

(2) R is an LPVD and each overring of R is seminormal;

(3) R is an LPVD and R' is a Prüfer domain;

(4) R is an LPVD and an i-domain.

PROOF. – (1) $\Rightarrow$ (2): Use the fact that any LPVD is seminormal (cf. Remarks 2.4 (*a*)).

(2) $\Rightarrow$ (3): Use the fact [2, Theorem 2.3] that if each overring of R is seminormal, then R' is a Prüfer domain.

(3) $\Rightarrow$ (4): Assume (3). By [20, Proposition 2.14], proving (4) reduces to showing that the (surjective) contraction map Spec $(R') \to$ Spec (R) is an injection. To this end, one need only show for each prime ideal P of R, that $S = (R')_{R\setminus P}$ is of the form $(R')_Q$ for some (uniquely determined) prime ideal Q of R'. As R' is a Prüfer domain, [19, Theorem 65] reduces the task to showing that S is a valuation domain. However this follows from [17, Theorem 1.7] (cf. also [6, Proposition 4.1]) since the Prüfer domain S, being an integral extension of the pseudo-valuation domain R_P, must be quasi-local.

(4) $\Rightarrow$ (1): Apply the first assertion in Proposition 2.6. The proof is complete.

COROLLARY 2.10. – Let R be a PVD, with associated valuation domain V and integral closure R'. Then the following conditions are equivalent:

(1) Each overring of R is seminormal;

(2) Each overring of R is an LPVD;

(3) Each overring of R is a PVD;

(4) $R' = V$.

PROOF. – The assertion follows directly from Theorem 2.9 in view of the following observations. If the pseudo-valuation domain R is an i-domain, then $R' = V$ (cf. [6, Remark 4.8 (*a*)]) and each overring of R is quasi-local (by [20, Proposition 2.34]).

REMARKS 2.11. – (*a*) The following result gives a sufficient condition for an LPVD to be a Prüfer domain. Let R be a domain, but not a field. Then R is a Prüfer domain each of whose maximal ideals is finitely generated if (and only if) R is an LPVD each of whose maximal ideals is invertible. For the proof, note for each maximal ideal M of R, that R_M is a PVD whose maximal ideal is invertible, and hence principal (cf, [19, Theorem 59]); thus, by [17, Corollary 2.9], R_M is a valuation domain, as desired.

(*b*) Let T be the LPVD constructed in Example 2.5. If each field extension $k_i \subset k$ figuring in the construction of T is taken algebraic, then [14, Proposition 3.4] shows that the integral closure of T is $\bigcap (k + M_i)$, which is well-known to be a Prüfer domain. Thus, in this case, each overring of T is an LPVD, by virtue of Theorem 2.9. The same conclusion follows (for algebraic $k_i \subset k$) from [2, Corollary 3.6] in (the less general) case $\dim (k + M_i) = 1$ for each i, for T is then (a seminormal i-domain and) one-dimensional (cf. [14, Lemma 3.1 and Theorem 3.1]).

Of course, the integral closure of T need not be a Prüfer domain in general. For instance, if each k_i is algebraically closed in k, then T is integrally closed.

(*c*) Let $1 \leqslant d \leqslant \infty$. Then there exists a d-dimensional LPVD, S, which is neither a Prüfer domain nor a PVD, such that each overring of S is an LPVD.

For the construction, note that the ring T in the first paragraph of (*b*) certainly takes care of the case $d = 1$. Let F be the quotient field of T. If $d \geqslant 2$, consider a $(d-1)$-dimensional valuation domain (V, M) of the form $V = F + M$. Then $S = T + M$ has the asserted properties. To see this, it is easiest to verify condition (3) in Theorem 2.9, using the lore of the $D + M$-construction (cf. [6, Proposition 4.9 (*i*)]).

(*d*) Let R satisfy the equivalent conditions in Theorem 2.9. For each maximal ideal M of R, let $V(M)$ be the valuation domain associated to R_M. It follows readily from the proof of (3) $\Rightarrow$ (4) in Theorem 2.9 that $R' = \bigcap V(M)$.

(*e*) The equivalence (3) $\Leftrightarrow$ (4) in Corollary 2.10 was also established in [18, Proposition 2.7] (cf. [11], Corollaire 1.4 (*c*)]).

3. – Globalized pseudo-valuation domains.

One reason that the theory of PVD's is so tractable is the presence of a valuation overring sharing the spectrum of a given PVD. As [1, Proposition 3.3] shows, a non-quasi-local LPVD (which is not a Prüfer domain) cannot admit a (Prüfer) overring with the same spectrum. However, an analogue of the « same spectrum » phenomenon is available for the non-quasi-local case. Indeed, we now proceed to introduce and characterize this section's object of study, a well-behaved class of LPVD's admitting unibranched Prüfer overrings.

THEOREM 3.1. – Let R be a subring of a Prüfer domain T. Then the following two conditions are equivalent:

(1) (*a*) $R \subset T$ is a unibranched extension;

(*b*) There exists a nonzero radical ideal A common to T and R such that each prime ideal of T (resp., R) which contains A is a maximal ideal of T (resp., R).

(2) There exist a nonzero radical ideal B common to T and R such that $\bar{R} = R/B$ and $\bar{T} = T/B$ satisfy the following:

(i) $\bar{R} \subset \bar{T}$ is a unibranched extension;

(ii) $\dim(\bar{R}) = \dim(\bar{T}) = 0$.

Next, suppose that (1) and (2) hold. Then, if A is as in (1) and N is a maximal ideal of T containing A, the square

$$\begin{array}{ccc} R_{N\cap R} & \longrightarrow & \boldsymbol{k}(N\cap R) \\ \downarrow & & \downarrow \\ T_N & \longrightarrow & \boldsymbol{k}(N) \end{array}$$

is a pullback diagram. Moreover, $J(R) = J(T)$; $T = \bigcap V(M)$, where the index M runs over the maximal ideals of R and for each M, $V(M)$ denotes the valuation domain associated to the pseudo-valuation domain R_M; and $R = T \times_{\prod \boldsymbol{k}(N)} \prod \boldsymbol{k}(N \cap R)$, where the index N runs over the maximal ideals of T.

Each domain R for which there exists a Prüfer domain T satisfying the equivalent conditions in Theorem 3.1 will be called a *globalized pseudo-valuation domain* (or, in short, a GPVD); and T will be called the *Prüfer domain associated to* R.

It will be shown, in the course of proving Theorem 3.1, that if R is a GPVD with associated Prüfer domain T, then the contraction map $\operatorname{Spec}(T) \to \operatorname{Spec}(R)$ is a homeomorphism. The induced order-theoretic isomorphism guarantees, in particular, that a prime ideal of T is maximal if and only if its contraction is a maximal ideal of R.

Granting Theorem 3.1 for the moment, we shall show next that any GPVD, R, must be an LPVD. Indeed, let T be the Prüfer domain associated to R, consider any nonzero maximal ideal M of R, and let N be the maximal ideal of T contracting to M. Let A be as in condition (1) of Theorem 3.1. If $A \not\subset N$ then, *a fortiori*, the conductor $(R:T)$ is not contained in N, and so R_M coincides with the (pseudo-) valuation domain T_N (cf. [19, Exercise 41 (*b*), p. 46]). On the other hand, if $A \subset N$, then [1, Proposition 2.6] translates the first pullback assertion in the statement of Theorem 3.1 into the statement that R_M is a PVD with associated valuation domain T_N.

It is important to observe that if R is a GPVD, then its associated Prüfer domain T is uniquely determined by conditions (1) and (2) in Theorem 3.1. Indeed $T = \bigcap V(M)$ where, as above, $V(M)$ denotes the valuation domain associated to R_M. To see this, without loss of generality, R is not a field. Then it suffices to note that the preceding paragraph established that $V(M) = T_N$.

Any Prüfer domain R is a GPVD and coincides with its associated Prüfer domain. To see this, take $A = R = T$ in condition (1) of Theorem 3.1; or take $B = R = T$ in (2), invoking the convention that the zero ring has Krull dimension 0. If the Prüfer domain R is not a field, one may verify (1) and (2) somewhat less artificially by setting $T = R$ and choosing A or B to be any (necessarily nonzero) maximal ideal of R.

If (R, M) is a PVD with associated valuation domain V, then R is a GPVD with associated Prüfer domain V. To see this, the preceding example allows us to assume $R \neq V$. Then (1) and (2) hold with $A = M = B$.

Additional examples of GPVD's will be given in Examples 3.2 and 3.4.

Proof of Theorem 3.1. – (1) $\Rightarrow$ (2): Given (1), set $B = A$. Then (i) follows from (*a*) and the observation that $(Q/A) \cap (R/A) = (Q \cap R)/A$ for each prime Q of T which contains A. In addition, (ii) follows readily from (*b*) and the assumptions about A.

(2) $\Rightarrow$ (1): Given (2), set $A = B$. Since A is an ideal of both R and T, it follows readily that $R \cong \bar{R} \times_{\bar{T}} T$. Applying [9, Theorem 1.4] to this pullback description reveals that $\operatorname{Spec}(R)$ is canonically homeomorphic to a certain quotient space which, by (i), may be identified with $\operatorname{Spec}(T)$. In particular, (*a*) holds. In addition, (*b*) follows readily from (ii) and the assumptions about B. This completes the proof of the equivalence of (1) and (2).

Henceforth, suppose that (1) and (2) hold. Let N be a maximal ideal of T containing B and set $M = N \cap R$. Since Spec (R) and Spec (T) are order-isomorphic, the saturation in T of the multiplicatively closed set $R \setminus M$ is easily seen to be $T \setminus N$, whence $T_{R \setminus M} = T_N$. Thus, applying [9, Proposition 1.9] to the pullback description $R = \bar{R} \times_{\bar{T}} T$ yields

$$R_M \cong (R_M / BR_M) \times_{T_N/BT_N} T_N .$$

However, (*b*) guarantees that MR_M is the only prime ideal of R_M which contains BR_M. Since BR_M inherits from B the property of being a radical ideal, $BR_M = MR_M$, whence $R_M/BR_M \cong \boldsymbol{k}(M)$. Similarly, $T_N/BT_N \cong \boldsymbol{k}(N)$, and so the above pullback description of R_M simplifies to $\boldsymbol{k}(M) \times_{\boldsymbol{k}(N)} T_N$, as asserted.

Let N be a maximal ideal of T and set $M = N \cap R$. We claim that $MR_M = NT_N$. This follows from the observation that $V(M) = T_N$ (whose proof was indicated earlier) since a PVD and its associated valuation domain have the same maximal ideal. Hence, $J(R) = \cap\, M = \cap\, MR_M = \cap\, NT_N = \cap\, N = J(T)$, and so the square

$$\begin{array}{ccc} R & \longrightarrow & R/J(R) \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/J(T) \end{array}$$

is a pullback diagram. Moreover, the above claim also implies that

$$\begin{array}{ccc} R/J(R) & \longrightarrow & \prod \boldsymbol{k}(M) \\ \downarrow & & \downarrow \\ T/J(T) & \longrightarrow & \prod \boldsymbol{k}(N) \end{array}$$

is a pullback diagram. Juxtaposition of the preceding two diagrams yields the asserted pullback description of R. As the proof that $T = \cap\, V(M)$ was given in the comments following the statement of the theorem, the proof is complete.

The reader may have noticed that the proof of above equivalence (1) $\Leftrightarrow$ (2) carries over if one deletes the conditions $A \neq 0$ and $B \neq 0$. However, the only additional situation covered by such generality is the case in which $R \subset T$ is a proper extension of fields.

EXAMPLES 3.2. – (*a*) The domain T constructed in Example 2.5 is a GPVD with, to use the earlier notation, associated Prüfer domain $P = \bigcap (k + M_i)$. Indeed, one verifies condition (2) of Theorem 3.1 as follows. Set $B = \bigcap N_i$, the intersection of the n maximal ideals N_i of P; evidently, B is a nonzero radical ideal of P. However, B is also $\bigcap Q_i$, the intersection of the n maximal ideals of T (cf. [14, p. 156, last line]); in particular, B is an ideal of T as well. Set $\bar{T} = T/B$ and

$\overline{P} = P/B$. Then $\overline{T} \subset \overline{P}$ is readily shown to satisfy conditions (i) and (ii) by virtue of the observations that Spec $(\overline{T}) = \{Q_i/B\}$, Spec $(\overline{P}) = \{N_i/B\}$ and $(N_i/B) \cap \overline{T} = Q_i/B$ for each i.

(*b*) Let A be the domain of all algebraic integers, $p\mathbb{Z}$ a maximal ideal of $\mathbb{Z}$, $\{M_i\}$ the uncountable set of maximal ideals of A which contract to $p\mathbb{Z}$, $S = A \setminus \bigcup \{M_i\}$, $T = A_S$ and $N_i = M_i A_S$: Then, as noted by GILMER [13, Example 1], T is a one-dimensional Prüfer (in fact, Bézout) G-domain with $\{N_i\}$ as its set of maximal ideals. Then there exists a non-Noetherian one-dimensional GPVD, R, with associated Prüfer domain T $(\neq R)$. (Thus, in contrast with the example in (*a*), R is not semi-quasi–local.)

Observe that $J(T) = pT \neq 0$, so that $\overline{T} = T/pT$ is absolutely flat. Then $I = $ Spec $(\overline{T})$, endowed with the Zariski topology, is a Boolean space. Setting $K = (I, \{\boldsymbol{k}(N_i)\})$, we thus have that $\overline{T}$ is a K-ring, in the sense of [21, p. 369]. (Note that condition (1) in [21, Théorème 3.1] is easily verified.) In particular, view $\overline{T}$ as embedded in $\prod \boldsymbol{k}(N_i)$. Set

$$\overline{R} = \{x \in \overline{T}\colon \text{ for each } i, \text{ the coordinate } x_i \in \mathbb{Z}/p\mathbb{Z}\} = \overline{T} \times_{\prod \boldsymbol{k}(N_i)} \prod \{\mathbb{Z}/p\mathbb{Z}\colon i \in I\}\,.$$

Then $\overline{R}$ is absolutely flat and one may check (cf. [21, Corollaire 3.6]) that $\overline{R}$ is the smallest K-ring. In particular, $\overline{R} \subset \overline{T}$ is a unibranched extension.

Define $R = T \times_{\prod \boldsymbol{k}(N_i)} \prod \{\mathbb{Z}/p\mathbb{Z}\colon i \in I\}$. Evidently, $\overline{R} \simeq R/pT$. The foregoing information assures that $\overline{R} \subset \overline{T}$ satisfies condition (2) of Theorem 3.1 (with $B = pT$) and so R is indeed a GPVD with associated Prüfer domain T. Moreover, the homeomorphism Spec $(T) \to$ Spec (R) yields $\dim (R) = \dim (T) = 1$. Finally, that R is non-Noetherian follows by, for example, appeal to condition (2) in Proposition 3.6 below: it suffices to observe that T, being a one-dimensional domain with infinitely many maximal ideals and nonzero pseudo-radical, cannot be Noetherian (cf. [19, Theorem 88]) and, hence, is not a Dedekind domain.

Despite the abundance of GPVD's supplied by Examples 3.2, not every LPVD is a GPVD. An example to this effect, such that its integral closure is actually a Prüfer domain, is given in Example 3.4. First, we isolate some needed facts concerning locally finite intersections of PVD's. Lemma 3.3 is a PVD-theoretic analogue of some results on locally finite intersections of valuation domains in [19, Theorems 111-113]. Its proof is a rather straightforward adaptation of the approach of KAPLANSKY [19] and, for that reason, has been placed in the appendix.

LEMMA 3.3. – Let the domain R be a locally finite intersection $\bigcap \{W_i\colon i \in I\}$ of one-dimensional pseudo-valuation overrings W_i. For each $i \in I$, let V_i be the valuation domain canonically associated to W_i. Assume that the V_i's are pairwise incomparable. Then:

(1) If S is any multiplicatively closed subset of R not containing 0, then R_S is a locally finite intersection of those W_i which contain R_S.

(2) If R is one-dimensional and quasi-local, then $R = W_i$ for some i.

(3) If P is a height 1 prime ideal of R, then $R_P = W_i$ for some i.

EXAMPLE 3.4. – There exists a one-dimensional LPVD, D, which is not a GPVD. Moreover, D can be arranged Noetherian (in which case, D is not integrally closed and D' is a Dedekind domain); alternatively, D may be chosen integrally closed (in which case, D is not Noetherian).

We next assemble the data for the construction of such D. Let k be an infinite field, I an infinite index set with cardinality at most that of k, $\{X_i: i \in I\}$ a family of algebraically independent indeterminates over k, $K = k(\{X_i: i \in I\})$, Y an indeterminate over K, $F = K(Y)$, n an integer exceeding 1, and $\{\alpha_i: i \in I\}$ a subset of k (such that $\alpha_i \neq \alpha_j$ whenever $i \neq j$ in I). For each $i \in I$, set $\check{K}_i = k(\{X_j: j \in I \setminus \{i\}\})$, $K_i = \check{K}_i(X_i^n)$, V_i the valuation domain $K[Y]_{(Y-\alpha_i)}$ expressed as usual as $K + M_i$ where $M_i = (Y - \alpha_i)V_i$, $\check{W}_i = \check{K}_i + M_i$, and $W_i = K_i + M_i$. Finally, set $\check{R} = \bigcap \check{W}_i$, $R = \bigcap W_i$, and $T = \bigcap V_i$. Then $\check{R}$ and R are each one-dimensional LPVD's with quotient field F, neither $\check{R}$ nor R is a GPVD, $\check{R}$ is integrally closed (but not Noetherian), and R is Noetherian (but not integrally closed) with T its integral closure.

The proof of the above assertions will follow from a series of observations. First, we shall show that F is the common quotient field of $\check{R}$, R and T. Since $\check{R} \subset R \subset T \subset F$, it suffices to verify the statement for $\check{R}$. To this end, note first that $k \subset \check{R}$ since $k \subset \check{K}_i$ for each i; and that $Y \in \check{R}$ since $Y = \alpha_i + (Y - \alpha_i) \in k + M_i \subset \check{W}_i$ for each i. Accordingly, it remains only to show that each X_i is in the quotient field of $\check{R}$. For this, write $X_i = uv^{-1}$ where $v = Y - \alpha_i$, and note that $v \in \check{R} + k = \check{R}$, so that we need only prove that $u \in \check{R}$. Now if $j \neq i$ in I, X_i is a unit of $\check{W}_j$ since $X_i \in \check{K}_j$, whence $u = vX_i \in \check{R}\check{W}_j = \check{W}_j$. On the other hand,

$$u = vX_i \in M_i V_i = M_i \subset \check{W}_i ,$$

whence $u \in \check{R}$, as desired.

Consider the multiplicatively closed set $S = K[Y] \setminus \bigcup (Y - \alpha_i)$ of $K[Y]$. It is straightforward to check that S satisfies the conditions in [15, (4.7)] (essentially because $K[Y]$ is a principal ideal domain) and so, by [15, Lemma 5.4],

$$K[Y]_S = \bigcap K[Y]_{(Y-\alpha_i)} = \bigcap V_i = T$$

It follows that T is a principal ideal (hence Dedekind, hence Prüfer) domain. By *abus de langage*, we shall let M_i denote $(Y - \alpha_i)T$, the typical nonzero prime ideal of T. Of course, $T_{M_i} = V_i$.

As T is a Dedekind (hence Krull) domain, the expression of T as $\bigcap V_i$ is locally finite. Therefore both the expressions $\check{R} = \bigcap \check{W}_i$ and $R = \bigcap W_i$ are also locally finite since, for each i, $\check{W}_i$ and W_i both have the same unique maximal ideal as V_i.

Moreover, each $\check{W}_i$ (resp., W_i) is a one-dimensional quasilocal (indeed, pseudo-valuation) overring of $\check{R}$ (resp., R), and so [19, Theorem 110] may be applied to the above locally finite intersections. Setting $\check{N}_i = M_i \cap \check{R}$ and $N_i = M_i \cap R$ for each $i \in I$, we deduce the following. For each nonzero prime ideal P of $\check{R}$ (resp., R), there exists $i \in I$ such that $\check{N}_i \subset P$ (resp., $N_i \subset P$).

It will be useful to observe next that for $i \neq j$ in I, $\check{N}_i$ and $\check{N}_j$ (resp., N_i and N_j) are comaximal ideals of $\check{R}$ (resp., R). This follows since

$$\alpha_i - \alpha_j = (Y - \alpha_j) - (Y - \alpha_i) \in \check{N}_j + \check{N}_i \subset N_j + N_i$$

and $\alpha_i - \alpha_j$ is a unit of both $\check{R}$ and R. The final comment of the preceding paragraph now yields that each $\check{N}_i$ (resp., N_i) has height 1 in $\check{R}$ (resp., R).

We shall show next, for each $i \in I$, that $R_{N_i} = W_i$ and $\check{R}_{\check{N}_i} = \check{W}_i$. The proofs being similar, we shall tend only to the first of these. As N_i has height 1, Lemma 3.3 (3) applies to the above noted locally finite expression for R, with the result that $R_{N_i} = W_j$ for some $j \in I$ depending on i. The above observation concerning comaximality leads easily to $i = j$, as desired.

We are now in a position to verify that R and $\check{R}$ are one-dimensional domains, with maximal spectra $\{N_i\}$ and $\{\check{N}_i\}$, respectively. (By virtue of the preceding paragraph, it will then follow that both R and $\check{R}$ are LPVD's.) As before, we shall give only the argument for R, by proving that each non-zero prime ideal P of R must coincide with some N_i. To this end, apply Lemma 3.3 (1) to the above-noted locally finite expression for R, with the result that $R_P = \bigcap \{W_j : j \in J\}$, for some subset J of I depending on P. If $j \in J$, then $R_P \subset W_j = R_{N_j}$, whence $N_j \subset P$. The earlier observation about comaximality therefore guarantees that J is a singleton set, say $\{i\}$. The above expression for R_P simplifies to $R_P = W_i = R_{N_i}$, whence $P = N_i$, as asserted.

We shall show next that neither R nor $\check{R}$ is a GPVD. As above, we argue only for R. Note, for each i, that the associated valuation domain of R_{N_i} ($= W_i$) is $V(N_i) = V_i$. Thus if one supposes that R is a GPVD, Theorem 3.1 implies that the associated Prüfer domain of R is $\bigcap V_i = T$. The desired contradiction will arise by showing that the conductor $C = (R : T)$ is zero, although C contains the nonzero ideal A satisfying condition (1) in the statement of Theorem 3.1. To this end, note first via [15, Corollary 5.2] that $T_{R \setminus N_i} = T_{M_i}$ ($= V_i$) since M_i is the only prime ideal of T which is disjoint from $R \setminus N_i$; consequently,

$$C \subset \bigcap (R_{N_i} : T_{R \setminus N_i}) = \bigcap (W_i : V_i) = \bigcap M_i .$$

However, the locally finite property of the expression $T = \bigcap V_i$ guarantees (since I is infinite) that $\bigcap M_i = 0$, whence $C = 0$, as desired.

To check that $\check{R}$ is integrally closed, it is enough to verify that each $\check{R}_{\check{N}_i}$ ($= \check{W}_i$) is integrally closed. However, this is a well-known consequence of the fact that $\check{K}_i$

is algebraically closed in K. (Moreover, $\check{R}$ is not Noetherian. Indeed, [6, Corollary 3.5] reveals that $\check{R}$ is not even a locally finite-conductor domain, lest $\check{R}$ become a Prüfer domain which is not a GPVD, an absurdity.)

We turn to the remaining assertions about R. Of course, R is not integrally closed since its typical localization, $R_{N_i} = W_i$, is not integrally closed. The point is that $W_i' = \check{K}_i(X_i) + M_i = K + M_i = V_i$. In fact,

$$R' = \bigcap (R_{N_i})' = \bigcap V_i = T\,.$$

Finally, one may see in a variety of ways that R is Noetherian. For instance, combine the following two remarks. R is locally Noetherian (since, for each i, V_i is a discrete valuation domain and $[K:K_i] = n < \infty$). Moreover, each nonzero element $r \in R$ lies in only finitely many maximal ideals of R (since $r \in N_i$ entails $r \in M_i$). This completes the proof of Example 3.4.

PROPOSITION 3.5. – Any integral overring of a GPVD is also a GPVD.

PROOF. – Let R be a GPVD with associated Prüfer domain T, and let S be an integral overring of R. We shall show that S is a GPVD with associated Prüfer domain T as well.

Let A be an ideal of both R and T which satisfies condition (1) of Theorem 3.1. Since $R \subset S \subset R' \subset T' = T$, it follows that A is also an ideal of S. If N is a prime ideal of S which contains A, then condition (b) of Theorem 3.1 assures that $N \cap R$ is a maximal ideal of R and so, by integrality, N is maximal in S.

It remains only to prove that $S \subset T$ is a unibranched extension. Since $R \subset T$ is unibranched, it suffices to show that if N is a prime ideal of S, then there exists a prime of T which contracts to N. To this end, set $M = N \cap R$ and let Q be the unique prime of T satisfying $Q \cap R = M$. Recall that T_Q is the associated valuation domain of the pseudo-valuation domain R_M. (If $M \supset A$, this is the content of the first pullback assertion in the statement of Theorem 3.1; if $M \supset A$, then $T_Q = R_M$ [19. Exercise 41 (b), p. 46].) Thus, by [11, Proposition 1.3 (a)], S_N and T_Q are comparable (via inclusion). If $T_Q \subsetneq S_N$ then S_N is a localization of the valuation domain T_Q at some nonmaximal prime, whence $S_N = T_I$ for some prime $I \subsetneq Q$ of T; then $Q \cap R = M = NS_N \cap R = IT_I \cap R = I \cap R$, contradicting the unibranchedness of $R \subset T$. Therefore $S_N \subset T_Q$, whence $QT_Q \cap S_N \subset NS_N$, so that intersecting with S yields $Q \cap S \subset N$. As $Q \cap S$ and N each contract to M, it follows that $Q \cap S = N$ since the extension $R \subset S$, being integral, must satisfy INC. This completes the proof.

The next result is motivated by the following consequence of [14, Theorem 3.4]. If T is the ring constructed in Example 2.5, then T is Noetherian if and only if $[k:k_i] < \infty$ and $k + M_i$ is a DVR for each i.

PROPOSITION 3.6. – Let R be a GPVD, with associated Prüfer domain T and integral closure R'. Then the following three conditions are equivalent:

(1) R is Noetherian;

(2) T is a Dedekind domain and $[\boldsymbol{k}(T_N):\boldsymbol{k}(R_{N\cap R})] < \infty$ for each maximal ideal N of T;

(3) T is a Dedekind domain and NT_N is a finitely generated $R_{N\cap R}$-module for each maximal ideal N of T.

Moreover, if the above conditions hold, then $\dim(R) \leqslant 1$ and $R' = T$.

PROOF. – (1) $\Rightarrow$ (2): Assume (1). Then $\dim(R) \leqslant 1$ since R is a treed Noetherian domain (cf. Remark 2.4 (*c*) and [19, Theorem 144]). By the Krull-Akizuki theorem, T is Noetherian, and hence satisfies Noether's conditions for a Dedekind domain.

Next, let N be a maximal ideal of T and set $M = N \cap R$. Since R_M is a Noetherian PVD with associated valuation domain T_N, [11, Corollaire 1.6] yields $[\boldsymbol{k}(N): :\boldsymbol{k}(M)] < \infty$, as desired.

(2) $\Leftrightarrow$ (3): This follows directly from the corresponding result in the quasi-local case [11, Corollaire 1.6].

(2) $\Rightarrow$ (1): Let M be a maximal ideal of R, with N the maximal ideal of T contracting to M. Assume (2). Then T_N is a DVR and [11, Corollaire 1.6] assures that the pseudo-valuation domain R_M is Noetherian. Hence, $(R')_{R\setminus M} = (R_M)' = T_N = T_{R\setminus M}$, the last equality holding since $T\setminus N$ is the saturation in T of $R\setminus M$. By globalization, $R' = T$.

Since the maximal ideals of R are in one-to-one correspondence with the maximal ideals of T and since T is a Dedekind domain, R inherits from T the property that each nonzero element lies in only finitely many maximal ideals. As we have also shown that R is locally Noetherian, a standard argument now yields that R is Noetherian (cf. [3, Exercise 9, p. 85]). This completes the proof.

The next result, together with Corollary 3.9, indicates compatibility of behavior between the LPVD datum and the Prüfer datum of a GPVD in the context of Prüfer's ascent theorem (cf. [15, Theorem 22.3]).

THEOREM 3.7. – Let R be a PVD, with associated valuation domain V and quotient field K. Let K^* be an extension field of K. Let R^* (resp., V^*) denote the integral closure of R (resp., V) in K^*. Then R^* is a GPVD, with associated Prüfer domain V^*.

PROOF. – Without loss of generality, R is integrally closed and distinct from V.

We consider first the case in which $[K^*:K] < \infty$. Let $V_1, \ldots, V_n$ be the (finitely many, pairwise incomparable) valuation domains of K^* such that $V_i \cap K = V$; recall that $[\boldsymbol{k}(V_i):\boldsymbol{k}(V)] < \infty$ in this case (cf. [15, Corollary 20.3]). We next make two elementary observations.

(*a*) If W is a valuation ring of K such that $R \subset W \subset V$ and if $\overline{W}$ denotes the canonical image of W in $\boldsymbol{k}(V)$, then $\boldsymbol{k}(R) \subset \overline{W} \subset \boldsymbol{k}(V)$ and the canonical homomorphism $W \to \overline{W} \times_{\boldsymbol{k}(V)} V$ is an isomorphism. (Apply [11, Proposition 1.3 (*a*)].)

(*b*) Let W and $\overline{W}$ be as in (*a*). For each $i = 1, \dots, n$, let $\{\overline{W}_{ij}: 1 \leqslant j \leqslant n_i\}$ be the set of (finitely many, pairwise incomparable) valuation domains of $\boldsymbol{k}(V_i)$ such that $\overline{W}_{ij} \cap \boldsymbol{k}(V) = \overline{W}$. Set $W_{ij} = \overline{W}_{ij} \times_{\boldsymbol{k}(V_i)} V_i$. Then, using (*a*), one sees easily that $\{W_{ij}: 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n_i\}$ is the set of all valuation domains of K^* such that $W_{ij} \cap K = W$; and $W_{ij} \not\subset W_{i',j'}$ if $(i, j) \neq (i', j')$.

For each $i = 1, \dots, n$, let k_i be the algebraic closure of $\boldsymbol{k}(R)$ in $\boldsymbol{k}(V_i)$ and set $R_i = k_i \times_{\boldsymbol{k}(V_i)} V_i$. We claim that $R^* = \bigcap R_i$. To see this, note first that $R^* = \bigcap W_{ij}$, where W, i, j range as in (*a*), (*b*) above (cf. [15, Theorem 19.8] and (*b*)). Moreover the above pullback descriptions yield that $R_i \subset W_{ij}$ since $k_i \subset \overline{W}_{ij}$, whence $\bigcap R_i \subset R^*$. For the reverse inclusion, let $x \in R^*$. In view of the pullback description of R_i, it is enough to prove for each i that x_i, the canonical image of x in $\boldsymbol{k}(V_i)$, is actually in k_i. This, however, is clear since integrality of x over R assures integrality (algebraicity) of x_i over $\boldsymbol{k}(R)$, thus proving the claim.

We proceed to show that R^* is a semi-quasi-local GPVD. By using the above claim and the definition of the R_i, one readily checks that an element $x \in V^* = \bigcap V_i$ belongs to R^* if (and only if) for each i, the canonical image of x in $\boldsymbol{k}(V_i)$ actually belongs to k_i. Thus the square

$$\begin{array}{ccc} R^* & \longrightarrow & \prod k_i \\ \downarrow & & \downarrow \\ V^* & \longrightarrow & \prod \boldsymbol{k}(V_i) \end{array}$$

is a pullback diagram. Its bottom horizontal arrow is surjective by virtue of the Chinese Remainder theorem. (The point is that each $V^* \to \boldsymbol{k}(V_i)$ is surjective. Indeed for each i, [19, Theorem 107] provides a maximal ideal M_i of V^* such that $V_i = V^*_{M_i}$, whence $\boldsymbol{k}(V_i) \cong V^*/M_i$.) Therefore, the results of [9] apply to the above diagram. In particular, [9, Theorem 1.4 (*f*), (*b*)] guarantees that the contraction map $\operatorname{Spec}(V^*) \to \operatorname{Spec}(R^*)$ is a homeomorphism; and that $\ker\big(V^* \to \prod \boldsymbol{k}(V_i)\big) = \bigcap M_i = J(V^*)$ coincides with $\ker(R^* \to \prod k_i) = J(R^*)$. It is now evident that $R^* \subset V^*$ satisfies condition (2) of Theorem 3.1, with the role of B played by $J(R^*)$, since the sets of prime ideals of $R^*/J(R^*)$ and $V^*/J(V^*)$ are but $\{(M_i \cap R^*)/J(R^*)\}$ and $\{M_i/J(R^*)\}$ respectively.

GENERAL CASE. – We may assume that K^* is algebraic over K. Let M denote the maximal ideal of R. It is well-known (cf. [3, Lemma 5.14]) that $J(R^*) = \operatorname{rad}_{R^*}(MR^*) = \{x \in R^*: x \text{ is integral over } M\} = \{x \in K^*: x \text{ is integral over } M\} = \{x \in V^*: x \text{ is integral over } M\} = \operatorname{rad}_{V^*}(MV^*) = J(V^*)$. Thus $R^*/J(R^*) \subset V^*/J(V^*)$ is an extension of zero-dimensional rings, hence satisfies the lying-over property [19, Exercise 2, p. 41]. In order to verify that condition (2) of Theorem 3.1 is satisfied, it therefore suffices to show that distinct primes of $V^*/J(R^*)$ cannot contract to the same prime of $R^*/J(R^*)$. If the assertion fails, one readily produces distinct maximal ideals N_1 and N_2 of V^* such that $N_1 \cap R^* = N_2 \cap R^*$ $(= P$, say). Select

$y \in N_1 \setminus N_2$ and set $L = K(y)$. Let R_L (resp., V_L) denote the integral closure of R (resp., V) in L. By the finite-dimensional case established above, R_L is a GPVD with associated Prüfer domain V_L; in particular, $R_L \subset V_L$ is a unibranched extension. However, $N_1 \cap V_L$ and $N_2 \cap V_L$ are distinct (since y lies in the former but not the latter) prime ideals of V_L which each meet R_L in $P \cap L$. This contradiction completes the proof.

COROLLARY 3.8. – With the same notation and hypotheses as in Theorem 3.7, we have:

(1) Suppose that K^* is algebraic over K. If W is a valuation domain of K^* which contains R^*, then W is comparable with at least one valuation domain of K^* which contains V.

(2) If V^* has nonzero pseudo-radical, then $C(R^*) = C(V^*) = \bigcap W_i$, where $\{W_i\}$ is the set of all one-dimensional valuation domains of K^* which contain V^*. If each nonzero element of V^* is contained in only finitely many maximal ideals, then $\dim(C(V^*)) \leqslant 1$. If $\dim(V) = 1$, then $V^* = C(V^*) = C(R^*)$.

PROOF. – (1) By [11, Proposition 1.3 (*a*)], $W \cap K$ and V are comparable valuation overrings of R. Let T denote the integral closure of $W \cap K$ in K^*. Since W is a valuation overring of the Prüfer domain T, there exists a prime ideal N of T such that $W = T_N$.

Suppose first that $W \cap K \subset V$. One sees readily that $R^* \subset T \subset V^* \cap W$. Set $P = N \cap R^*$ and let Q be the unique prime ideal of V^* lying over P. (Note that Q is well-defined by virtue of Theorem 3.7.) As $R^*_P \subset T_N = W$ and V^*_Q is the associated valuation domain of the pseudo-valuation domain R^*_P, it follows from [11, Proposition 1.3 (*a*)] that W and V^*_Q are comparable, as desired. (In fact, one may show in this case that $W \subset V^*_Q$.)

In the remaining case, $V \subset W \cap K$, so that $V^* \subset T$. Thus $W = T_N$ must contain the valuation domain $V^*_{N \cap V^*}$, as desired.

(2) The first assertion follows from [16, Proposition 4] and Theorem 3.7 since R^* and V^*, having a common nonzero (radical) ideal, must also have a common complete integral closure (cf. [15, Lemma 26.5]). As V^* is a Prüfer domain, the second assertion is a direct consequence of [16, Corollary 9]. Finally, the third assertion follows from the equality $C(R^*) = C(V^*)$ noted above and the fact that V^*, being a one-dimensional Prüfer domain, must be completely integrally closed.

COROLLARY 3.9. – Let R be a GPVD, with associated Prüfer domain T and quotient field K. Let K^* be an extension field of K. Let R^* (resp., T^*) denote the integral closure of R (resp., T) in K^*. Then R^* is a GPVD, with associated Prüfer domain T^*.

PROOF. – If N is a prime (resp., maximal) ideal of T and $M = N \cap R$ the corresponding prime (resp., maximal) ideal of R, then Theorem 3.7 readily implies that

$(R_M)^*$ is a GPVD, with associated Prüfer domain $(T_N)^*$. (Of course, D^* generally denotes the integral closure of D in K^*.) Moreover, one has canonical isomorphisms $R^* \otimes_R R_M \cong (R_M)^*$ and $T^* \otimes_T T_N \cong (T_N)^*$ (cf. [3, Proposition 5.12).

Hence, if Q is a prime (resp., maximal) ideal of R^* and $M = Q \cap R$, then Q induces a prime (resp., maximal) ideal P in $(R_M)^*$. If N is *the* prime (resp., maximal) ideal of T lying over M, let I be *the* prime (resp., maximal) ideal of $(T_N)^*$ which lies over P. Then $J = I \cap T^*$ is the unique prime (resp., maximal) ideal of T^* lying over Q (since any such ideal D must satisfy $D(T_N)^* = I$). In particular, $R^* \subset T^*$ is a unibranched extension.

In order to complete the proof, it suffices to produce a suitable ideal common to R^* and T^*. To this end, let A be a nonzero radical ideal of T and R satisfying condition (b) of Theorem 3.1. We claim that $\mathfrak{A} = \operatorname{rad}_{T^*}(AT^*)$ has the desired properties.

Indeed, an appeal to [3, Lemma 5.14] as in the proof of Theorem 3.7 reveals that $\mathfrak{A} = \{x \in T^*: x \text{ is integral over } A\} = \{x \in K^*: x \text{ is integral over } A\} = \{x \in R^*: x \text{ is integral over } A\} = \operatorname{rad}_{R^*}(AR^*)$. In particular, $\mathfrak{A}$ is a radical ideal of both T^* and R^* which, since it contains A, must be nonzero. Finally, if $\mathfrak{S}$ is a prime ideal of T^* (resp., R^*) which contains $\mathfrak{A}$ then $\mathfrak{S} \cap T \supset \mathfrak{A} \cap T \supset A$ (resp., $\mathfrak{S} \cap R \supset \mathfrak{A} \cap R \supset A$), and so the conditions satisfied by A guarantee that $\mathfrak{S} \cap T$ (resp., $\mathfrak{S} \cap R$) is a maximal ideal of T (resp., R); hence by integrality, $\mathfrak{S}$ is a maximal ideal of T^* (resp., R^*). Thus $\mathfrak{A}$ has all the desired properties, completing the proof.

REMARK 3.10. – It is interesting to note the following analogue of Corollary 3.9. If R is an LPVD with quotient field K and if R^* is the integral closure of R in a field extension K^* of K, then R^* is an LPVD.

For a proof, let N be a maximal ideal of R^*, and set $M = N \cap R$. Since R_M is a PVD, Theorem 3.7 assures that $R^*_{R \setminus M}$ is a GPVD, and hence an LPVD. Passing to its ring of quotients with respect to $R^* \setminus N$, we see that $(R^*)_N$ is an LPVD (cf. Remark 2.4 (e)) which, being quasi-local, is then a PVD, as desired.

The ring $\check{R}$ considered in Example 3.4 illustrates the fact that if R is as in Remark 3.10, then R^* need not be a GPVD even in case $K^* = K$.

4. – Appendix.

This brief final section contains the proof of Lemma 3.3. First, we give a PVD-theoretic analogue of [19, Theorem 107].

LEMMA A. – Let R, $\{W_i\}$ and $\{V_i\}$ be as in the riding hypotheses of Lemma 3.3. Assume also that $\{i\}$ is finite, say $\{1, \ldots, n\}$. Then:

(1) For each i, there exists a uniquely determined prime ideal Q_i of R such that $W_i = R_{Q_i}$.

(2) If Q_i is as in (1), then $Q_1, \ldots, Q_n$ are precisely the maximal ideals of R.

PROOF. – For each i, let M_i denote the maximal ideal of V_i. Set $S = V_1 \cap \ldots \cap \cap V_n$ and $P_i = M_i \cap S$ for each i. By [19, Theorem 107], $\{P_1, \ldots, P_n\}$ is the set of maximal ideals of S and $V_i = S_{P_i}$ for each i.

Consider the pullback

$$\begin{array}{ccc} T & \longrightarrow & \prod \boldsymbol{k}(W_i) = \bar{T} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \prod \boldsymbol{k}(V_i) = \bar{S} \end{array}$$

where the right vertical map is induced by the inclusions $W_i \to V_i$ and the bottom horizontal surjection is induced by the canonical surjections $V_i \to \boldsymbol{k}(V_i)$. By applying [9, Theorem 1.4] to this pullback, we infer that Spec (T) and Spec (S) are canonically homeomorphic. (In computing the intervening quotient space, the point is that Spec $(\bar{T})$ and Spec $(\bar{S})$ are in one–to–one correspondence since $\{i\}$ is finite.) In particular, S is then a unibranched extension of T and the maximal spectrum of T consists of the n ideals given by $Q_i = P_i \cap T$. Note that $B = \bigcap P_i = \ker (S \to \bar{S}) = \ker (T \to \bar{T})$ is a nonzero radical ideal of both T and S. Since Spec $(\bar{T})$ and Spec $(\bar{S})$ may be identified with $\{Q_i/B\}$ and $\{P_i/B\}$ respectively, we readily verify condition (2) of Theorem 3.1; i.e., T is a GPVD with associated Prüfer domain S. Thus, by Theorem 3.1,

$$T_{Q_i} \cong S_{P_i} \times_{\boldsymbol{k}(P_i)} \boldsymbol{k}(Q_i)$$

for each i. Since $\ker (S \to \bar{S} \to \boldsymbol{k}(V_i)) = P_i$, it follows by considering the above pullback diagram that $\ker (T \to \bar{T} \to \boldsymbol{k}(W_i)) = P_i \cap T = Q_i$, so that $\boldsymbol{k}(Q_i) = T/Q_i \cong \cong \boldsymbol{k}(W_i)$. Accordingly, T_{Q_i} may be identified with $V_i \times_{\boldsymbol{k}(V_i)} \boldsymbol{k}(W_i)$ which, according to [1, Proposition 2.6], is just W_i: Thus $T = \bigcap T_{Q_i} = \bigcap W_i = R$, and the required assertions are now immediate consequences of the foregoing comments.

PROOF OF LEMMA 3.3. – (1) Define $J \subset I$ so that $\{W_j : j \in J\}$ is the set of those W_i's which contain R_S. Set $A = \bigcap \{W_j : j \in J\}$. Evidently, $R_S \subset A$. For the reverse inclusion, let $x \in A$, and write $x = rs^{-1}$ for suitable $r, s \in R$; we shall show $x \in R_S$.

As usual, let M_i denote the common maximal ideal of W_i and V_i: Then $I(s) = = \{k \in I : s \in M_k\}$ is finite since $R = \bigcap W_i$ is locally finite; moreover, if $k \in I \setminus I(s)$, then both s^{-1} and s belong to $W_k \setminus M_k$, whence $x \in W_k$. Next, set $I(x) = \{k \in I : x \notin W_k\}$, a (necessarily finite) subset of $I(s)$. Without loss of generality, $I(x)$ is non empty (lest $x \in R \subset R_S$).

Observe, for each $k \in I(x)$, that $M_k \cap S$ is nonempty. Indeed, one would otherwise have $R_S \subset W_k$, so that $k \in J$, whence $x \in A \subset W_k$, contradicting $k \in I(x)$. Select $z_k \in M_k \cap S$.

Evidently, $I(x) = I_1 \cup I_2$, where $I_1 = \{k \in I : x \in V_k \setminus W_k\}$ and $I_2 = \{k \in I : x \notin V_k\}$. If $k \in I_1$, then $z_k x \in M_k V_k = M_k \subset W_k$. If $k \in I_2$, then x^{-1} and z_k are each nonunit elements of W_k; thus, since W_k is quasi-local and one-dimensional, [19, Theorem 108] yields a positive integer n_k so that $y_k = z_k^{n_k}$ is divisible in W_k by x^{-1}, that is, $y_k x \in W_k$.

Set $z = (\prod z_k)(\prod y_k)$, where the first product is indexed by $k \in I_1$ and the second product is indexed by $k \in I_2$. Then $0 \neq z \in S$ and the results of the preceding paragraph guarantee that $zx \in B = \bigcap \{W_k\colon k \in I(x)\}$. Since it is trivial that $zx \in D = \bigcap \{W_k\colon k \in I \setminus I(x)\}$, we infer that $zx \in B \cap D = \bigcap \{W_i\colon i \in I\} = R$, whence $x = (zx)z^{-1} \in R_S$. Thus $R_S = A$, evidently a locally-finite intersection.

(2) Let M_i (resp., M) denote the maximal ideal of W_i (resp., R). The hypotheses on R guarantee that, for each $i \in I$, $M_i \cap R$ is either M or 0. Since $R_{M_i \cap R}$ embeds in its overring W_i which is not a field, $M_i \cap R \neq 0$; thus, $M_i \cap R = M$ for each i. By applying the locally finite condition to any nonzero element of M, we see that I is finite, and the assertion therefore follows from Lemma A (1).

(3) By taking $S = R \setminus P$, we infer from (1) that R_P is a locally finite intersection of some W_i's. As R_P is one-dimensional and quasi-local, (2) may be applied, to complete the proof.

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