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CONDUCIVE INTEGRAL DOMAINS AS PULLBACKS

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This article presents a characterization of conducive (integral) domains as the pullbacks in certain types of Cartesian squares in the category of commutative rings. Such squares behave well with respect to (semi)normalization, thus permitting us to recover the recent characterization by Dobbs-Fedder of seminormal conducive domains. The conducive domains satisfying various finiteness conditions (Noetherian, Archimedean, accp)are characterized by identifying suitable restrictions on the data in the corresponding Cartesian squares. Various necessary or sufficient conditions are given for Mori conducive domains. Consequently, one has examples of several (accp non-Mori; Mori non-Noetherian) conducive domains.

1. INTRODUCTION

A conducive domain is a (commutative unitary integral) domain R, with quotient field F, such that for every overring T of R, $R \subseteq T \subsetneq F$, the conductor $(R:T) = \{x \in F \mid xT \subseteq R\}$ is nonzero. This type of integral domain was explicitly introduced and intensively studied in a recent paper by D. Dobbs and R. Fedder [6], who motivated their investigation by noticing that all domains of $(D+\mathcal{M})$-type (cf. Gilmer [9; Appendix 2,

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p. 558]), all valuation domains and, more generally, all pseudo-valuation domains (for short, PVD; cf. Hedstrom-Houston [14]) satisfy the previous property. By the way, the notion of conducive domain was also implicitly considered by E. Bastida and R. Gilmer [3] in a passage of a paper, dedicated to $(D+\mathcal{M})$-domains, in which they study the relation with the domains such that every nonzero ideal is divisorial (cf. W. Heinzer [15]). As a matter of fact, they proved the following:

PROPOSITION 0. (Bastida-Gilmer [3, Th. 4.5]) *Let* R *be an integral domain with quotient field* F. *The following statements are equivalent:*

(i) R *is a conducive domain;*

(ii) *Every* R*-submodule* E *of* F *such that* $R \subseteq E \subsetneq F$ *is a fractional ideal of* R*;*

(iii) *For every valuation overring* V *of* R *such that* $R \subseteq V \subsetneq F$, *the conductor* $(R:V) \neq 0$*;*

(iv) *Either* $R = F$ *or there exists a valuation overring* V *of* R *such that* $R \subseteq V \subsetneq F$ *and* $(R:V) \neq 0$. □

We seize the opportunity to recall that the equivalences (i) ⇔ ⇔ (iii) ⇔ (iv), among other interesting characterizations of conducive domains, were also proved in [6; Lemma 2.0 and Th. 3.2].

Although the starting point for considering conducive domains was founded on several pullback-type examples, Cartesian squares' techniques did not play a leading rôle in the work by Dobbs and Fedder. The aim of the present paper is to show how the pullback point of view (cf. [7]) can be used not only to recover rather easily the main results contained in [6], but also to deepen our knowledge of several important classes of conducive domains. In addition to a constructive characterization of general conducive domains (in Theorem 1), the main new results of this paper are three theorems each of which gives a complete description of one of the following classes of conducive domains satisfying various finiteness conditions on their ideals: Noetherian conducive domains (cf. Theorem 6); Archimedean conducive domains (cf. Theorem 10) and accp conducive domains (cf. Theorem 12). For conducive Mori domains, we give some necessary or sufficient criteria (cf. Propositions 16 and 19). (The definitions of Archimedean, accp and Mori domains will be recalled later.) In particular, these results allow one to construct explicitly new examples of Archimedean non-accp, accp non-Mori, Mori non-Noetherian and Noetherian domains in the conducive (integrally closed or not) case.

Throughout, all rings are assumed commutative with unit and all ring-homomorphisms are assumed unital. Given a domain R, we denote by C(R) the complete integral closure of R, by U(R) the set of all the units of R. Any unexplained material is standard as in [4] and [10].

2. RESULTS

We start with a theorem which gives a pullback description of a general conducive domain. From this result and some general facts about pullbacks (cf., for instance, [7, Th. 1.4]) we can recover in particular Theorem 2.4 of [6]. Moreover, we notice that the following theorem provides a suitable extension (to the non-seminormal case) of Proposition 2.12 of [6].

THEOREM 1. *Let* R *be an integral domain which is not a field. Then,* R *is a conducive domain if and only if there exists*

(1) *a nontrivial valuation overring* (V,m) *of* R*;*

(2) *an* m*-primary ideal* b *of* V*;*

(3) *an injective ring homomorphism* $u: A \hookrightarrow V/b \overset{\text{not.}}{=} B$ (B *is a* 0*-dimensional quasilocal ring);*

such that the following diagram is a Cartesian square:

$$(*) \qquad \begin{array}{ccc} R & \twoheadrightarrow & A \\ \downarrow & & \downarrow u \\ V & \twoheadrightarrow & B \end{array}$$

PROOF. We identify, for the sake of simplicity, R with its image in V.

Then, the sufficiency follows immediately from the fact that $(R:V)\supseteq \mathfrak{b}\neq 0$ (cf.[7, 1.4] and Prop. 0). Conversely, take any non trivial valuation overring W of a conducive domain R. Then, by hypothesis, $(R:W) \overset{\text{not.}}{=} \mathfrak{g} \neq 0$. We consider now the prime ideal $\mathrm{rad}_W(\mathfrak{g}) \overset{\text{not.}}{=} \mathfrak{q}$. Let $V = W_{\mathfrak{q}}$. Hence:

$$0 \neq \mathfrak{g}\cdot\mathfrak{q} = (R:W)\cdot(W:V) \subseteq \mathfrak{f} \overset{\text{def.}}{=} (R:V) \subseteq (R:W) \subseteq \mathfrak{g};$$

thus $\mathfrak{q} = \mathrm{rad}_W(\mathfrak{g}) \cap \mathrm{rad}_W(\mathfrak{q}) = \mathrm{rad}_W(\mathfrak{g}\cdot\mathfrak{q}) \subseteq \mathrm{rad}_W(\mathfrak{f}) \subseteq \mathfrak{q}$. Therefore $\mathrm{rad}_W(\mathfrak{f}) = \mathfrak{q}$. Moreover, $\mathrm{rad}_V(\mathfrak{f}) = \mathfrak{q}W_{\mathfrak{q}} \equiv \mathfrak{m}\ (=\mathfrak{q})$; thus $\mathfrak{f}$ is $\mathfrak{m}$-primary. Consider, now, the following pullback diagram:

$$\begin{array}{ccccc} R & \longrightarrow\!\!\!\!\rightarrow & R/\mathfrak{f} & & \\ \downarrow & & \downarrow & & \\ V & \longrightarrow\!\!\!\!\rightarrow & V/\mathfrak{f} & \longrightarrow & (V/\mathfrak{f})_{red} = V/\mathfrak{m} \end{array}$$

and take $\mathfrak{b} = \mathfrak{f}$ and, as u, take the canonical inclusion $R/\mathfrak{f} \hookrightarrow V/\mathfrak{f}$. □

REMARK 2. We notice that a pullback representation as in (*) of a conducive domain is very far from unique, in general. For instance, if R is a (pseudo-)valuation domain of dimension $n \geq 2$, then for each nonzero non-maximal prime ideal $\mathfrak{p}$ of R

$$\begin{array}{ccc} R & \longrightarrow & R/\mathfrak{p} = A \\ \downarrow & & \downarrow \\ V = R_{\mathfrak{p}} & \longrightarrow & R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = B \end{array}$$

is a pullback representation of R as (*) in Theorem 1, with $\mathfrak{b} = \mathfrak{p}R_{\mathfrak{p}}$ Moreover, even if we fix the valuation overring $(V,\mathfrak{m})$ in (*) then, in general, there are still several $\mathfrak{m}$-primary ideals verifying conditions (2) and (3) of the previous Theorem. For instance, if $\mathfrak{b}$ leads to a Cartesian square of type (*), then so does $\mathfrak{b}^n$, for every $n \geq 1$.

The previous Theorem allows us to give easily a pullback description of the normalization and seminormalization (in the sense of Traverso [19]) of a conducive domain. For this purpose, we fix some notation. Let R be a conducive domain which is not a field, and let (*) be (one of) its pullback description(s) (as in Theorem 1). Let A' be the integral closure of A inside B and let A^+ be the seminormalization of A inside A'. Since $\mathfrak{m}$ is the radical of $\mathfrak{b}$, it is easy to see that A_{red}, $(A^+)_{red}$ and $(A')_{red}$ are integral domains contained in the field

$B_{red} = V/\mathfrak{m}$, which we will denote by K. We notice also that $(A')_{red}$ coincides with $(A_{red})'$, which denotes the integral closure of A_{red} inside K (cf. [13; Prop. 6.5.5(i), p. 146]). Moreover, $(A^+)_{red}$ coincides with $(A_{red})^+$, which is by definition the seminormalization of A_{red} inside $(A_{red})'$, as a direct calculation shows. Therefore, the notations A^+_{red} and A'_{red} are not ambiguous.

COROLLARY 3. *Let* R *be a conducive domain, but not a field, and let* (*) *be (one of) its pullback description(s). Then, with the notation as above, the normalization* R' *(resp. the seminormalization* R^+*) of* R *is a conducive domain with the following pullback description:*

$$\begin{array}{ccc} R' & \twoheadrightarrow & A'_{red} \\ \downarrow & & \downarrow \\ V & \twoheadrightarrow & K \end{array} \qquad (\textit{resp.,} \quad \begin{array}{ccc} R^+ & \twoheadrightarrow & A^+_{red} \\ \downarrow & & \downarrow \\ V & \twoheadrightarrow & K \end{array} \quad).$$

PROOF. Without loss of generality, $R \neq V$ (lest $A = B$, in which case the assertions are obvious). It is easy to see that R' (resp., R^+) is isomorphic to $V \times_B A'$ (resp., $V \times_B A^+$). Moreover, if $\mathfrak{f} = (R:V)$, then by integrality we deduce that $\mathfrak{m} = \mathrm{rad}_V(\mathfrak{f}) \subset R'$. It is also easy to see that $\mathfrak{m} = \mathrm{rad}_{R'}(\mathfrak{f}) \subset R^+$ (cf. [10, Th. 1.1]). As a consequence, we have that A' (resp., A^+) is isomorphic to $B \times_K A'_{red}$ (resp., $B \times_K A^+_{red}$). The conclusion follows immediately, since a composite of Cartesian squares is itself Cartesian. □

An easy consequence of Corollary 3 and Theorem 1 is the following pullback description of normal or seminormal conducive domains.

COROLLARY 4. *Let* R *be an integral domain, but not a field. Then,* R *is an integrally closed (resp., seminormal) conducive domain if and only if there exists:* (1) *a non trivial valuation overring* $(V, \mathfrak{m}, K)$ *of* R*;* (2) *an injective ring homomorphism* $u: D \hookrightarrow K$ *with* D *integrally closed in* K *(resp., seminormal in its integral closure in* K*); such that the following:*

$$\begin{array}{ccc} R & \longrightarrow\!\!\!\!\to & D \\ \downarrow & & \downarrow u \\ V & \longrightarrow\!\!\!\!\to & K \end{array}$$

is a Cartesian square. □

We notice that the previous Corollary, in the seminormal case, is equivalent to Proposition 2.12 of [6] (as a matter of fact, one pseudovaluation overring of R of the type discussed in Proposition 2.12 of [6] is the pullback of $V \twoheadrightarrow K$ with the inclusion of the quotient field of D inside K). Moreover, it is interesting to see how the previous description permits us to classify easily several kinds of conducive domains.

COROLLARY 5. *Let* R *be a conducive integral domain, but not a field. Then,*

(a) R *is completely integrally closed if and only if* R *is a rank* 1 *valuation domain;*

(b) R *is a Krull domain if and only if* R *is a rank* 1 *discrete valuation ring;*

(c) R *is a Prüfer domain if and only if, with notation of* Corollary 3, D *is a Prüfer domain with* K *as field of quotients.*

PROOF. (a): By [9, Lemma 26.5], in the notation of Corollary 3, it is enough to notice that now $C(R) = C(V)$, because $(R:V) \neq 0$. (b): an easy consequence of (a), because any Krull domain is completely integrally closed. (c) follows from [7, Th. 2.4]. □

Our aim is now to give a characterization of the Noetherian conducive domains, for which only a necessary condition was known earlier (cf. [6, Cor. 2.7]).

THEOREM 6. *Let* R *be an integral domain which is not a field. Then,* R *is a Noetherian conducive domain if and only if there exists a unique pullback representation of* R *as in* (*), *in which* V *is a rank* 1 *discrete valuation domain,* $\mathfrak{b} = (R:V)$ *and* u *is a finite homomorphism.*

PROOF. For the "if" part, we remark that the inclusion $R \hookrightarrow V$ is finite, because u is finite (cf. [7, Prop. 1.8]); then by Eakin-Nagata Theorem we conclude that R is Noetherian. Moreover, Theorem 1 shows that R is conducive. Conversely, for the "only if" part, by Theorem 1, we know that R can be obtained as a pullback as in (*). We only need to prove that V must be Noetherian and unique and that u must be finite. We notice that the complete integral closure (= integral closure, since R is Noetherian) C(R) of R must coincide with the complete integral closure C(V) of V (cf. also Cor. 5(a)), i.e. $C(V) = C(R) = R' \subseteq V$. Because $V \subseteq C(V)$ always holds, we have that $R' = V$, thus V is uniquely determined. Moreover, V must be a Krull domain, by Mori-Nagata theorem and, thus, a rank 1 discrete valuation domain (cf. Cor. 5(b)). As $R \hookrightarrow V$ is integral, R is 1-dimensional; by Krull-Akizuki theorem [4; Ch. 7, Prop. 5, p. 30] we deduce that $R \hookrightarrow V \twoheadrightarrow V/\mathfrak{b}$ is finite. In particular, $u: R/\mathfrak{b} \hookrightarrow V/\mathfrak{b}$ is finite. □

From the previous result (and its proof) we deduce immediately:

COROLLARY 7. *Let* R *be an integral domain. The following statements are equivalent:*

(i) R *is a Noetherian conducive domain;*

(ii) *every overring of* R *is a Noetherian conducive domain.*

Moreover, if R *is not a field and the above conditions hold, then* R *(and all its proper overrings) is* 1*-dimensional local domain.* □

The next example constructs some nontrivial Noetherian conducive domains.

EXAMPLE 8. Let $k \hookrightarrow K$ bê a finite extension of distinct fields, and let X be an analytic indeterminate over K. Let $V = K[[X]] = K + \mathcal{M}$, where $\mathcal{M} = XV$. We consider $R = k + \mathcal{M}^n = \{\sum_{i \geq 0} a_i X^i \mid a_0 \in k, a_1 = \ldots = a_{n-1} = 0,\ a_i \in K$ for $i \geq n\}$, for $n \geq 2$. Then, clearly, the integral closure R' of R coincides with V and the seminormalization R^+ is equal to $k+\mathcal{M}$. These rings are obtained by the following pullback diagrams:

$$\begin{array}{ccccc}
R & \twoheadrightarrow & R/\mathcal{M}^n = k & & \\
\downarrow & & \downarrow & & \\
R^+ & \twoheadrightarrow & R^+/\mathcal{M}^n & \twoheadrightarrow & R^+/\mathcal{M} = k \\
\downarrow & & \downarrow & & \downarrow \\
V & \twoheadrightarrow & V/\mathcal{M}^n & \twoheadrightarrow & V/\mathcal{M} = K
\end{array}$$

By the criterion in Theorem 6, we thus see that R and R^+ are Noetherian conducive domains, with $R \subsetneq R^+ \subsetneq R' = V$. Moreover, R is neither a PVD nor a domain of nontrivial $(D+\mathcal{M})$-type, although R^+ is a PVD. The latter fact is not surprising because:

REMARK 9. A seminormal Noetherian conducive domain R is always a PVD and its integral closure is a discrete valuation domain, by Corollary 4 and Theorem 6. We notice also that the above mentioned results give back, in particular, Corollary 2.9 of [6].

As announced, our goal now is to give some characterization theorems, in the same spirit as in the Noetherian case, for other distinguished classes of conducive domains verifying finite type conditions on their ideals.

We recall that an integral domain R is <u>Archimedean</u> if $\bigcap_{n \geq 0} r^n R = 0$, for every non unit $r \in R$ (cf.[18]). Completely integrally closed or 1-dimensional or Noetherian integral domains are the most natural examples of Archimedean domains. The following result gives a constructive characterization of Archimedean conducive domains, in the spirit of Theorem 5.

THEOREM 10. *Let* R *be an integral domain, but not a field. Then,* R *is an Archimedean conducive domain if and only if there exists a unique pullback representation of* R *as in* (*) *with* $\dim(V)=1$, $\mathfrak{b} = (R:V)$ *and* A *a* 0-*dimensional quasilocal ring.*

PROOF. "Only if" part: Let (*) be a pullback representation of R (cf.

Theorem 1). By [2, Th. 2.2], we know that R is quasilocal with $\dim(R)=1$, hence A is quasilocal and $\dim(A)=0$. Since $\dim(R) = \dim(V) + \dim(A)$, [7, Th. 1.4(e)], we deduce that $\dim(V) = 1$. Moreover, since $C(R)=C(V)$ coincides with V because $\dim(V)=1$ (cf. [9, Th. 23.4]), it is clear that V is uniquely determined. "If" part: By general pullback-theoretical considerations [7, Th. 1.4(e)], R is a 1-dimensional quasilocal domain and it is conducive by Theorem 1. Finally, we notice that $U(C(R)) \cap R = U(V) \cap R = (V \setminus \mathfrak{m}) \cap R = R \setminus (\mathfrak{m} \cap R) = U(R)$. By the criterion in [2, Prop. 2.1], we conclude that R is Archimedean. □

COROLLARY 11. *The integral closure and the seminormalization of an Archimedean conducive domain is an Archimedean (conducive)* PVD.

PROOF. The statement follows from Theorem 10 and Corollary 3, after noticing that the ring A in the (unique) pullback representation (*) of the Archimedean conducive domain R is such that A_{red} is a field and, hence, A'_{red} and A^{+}_{red} are fields too. □

We recall that an <u>accp domain</u> (or 1-<u>acc domain</u>) is an integral domain for which the ascending chain condition holds for its integral principal ideals.

THEOREM 12. *Let* R *be an integral domain but not a field. Then,* R *is an accp conducive domain if and only if there exists a unique pullback representation of* R *as in* (*)*, in which* V *is a rank* 1 *discrete valuation domain,* $\mathfrak{b} = (R:V)$ *and* A *a* 0*-dimensional (quasilocal) ring.*

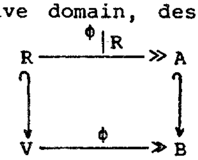
PROOF. "Only if" part: Let (*) be a pullback representation of R (cf. Theorem 1). It is easy to see that an accp domain is Archimedean; so we know, by Theorem 10, that V is unique with $\dim(V) = 1$ and A is a 0-dimensional quasilocal ring. It is enough to show that accp holds in V. Let $\{x_n V\}_{n \in \mathbb{N}}$ be an ascending chain of principal ideals of V with $x_{n+1} v_n = x_n$, for each $n \in \mathbb{N}$. We can suppose that, for each $n \in \mathbb{N}$, $x_n \in R^{\cdot}$ (otherwise, take $0 \neq b \in (R:V) = \mathfrak{b}$ and consider the chain $\{bx_n V\}_{n \in \mathbb{N}}$). If the chain $\{x_n V\}_{n \in \mathbb{N}}$ is not stationary, we can suppose it strictly ascending; hence $v_n \in \mathfrak{m}$, for $n = 1,2,\ldots$, where $\mathfrak{m}$ is the maximal ideal of V. Let ν be the ideal of V generated by the elements $v_1, v_2, \ldots, v_n, \ldots$ We have $\nu \subseteq \mathfrak{m} = \sqrt{\mathfrak{b}}$. By a well known property of valuation domains (cf. [10, Th. 17.1.(5)]), there exists an integer $k > 0$ such that $\nu^k \subset \mathfrak{b}$. Hence $\nu^k \subseteq R$, that is $v_n^k \in R$, for each n ($n = 1,2,\ldots$). Now, consider the chain of principal ideals of R, $\{x_n^k R\}_{n \in \mathbb{N}}$; since $x_{n+1}^k v_n^k = x_n^k$ and $v_n^k \in R$ ($n = 1,2,\ldots$), this is an ascending chain of principal ideals of R. Thus, by hypothesis, it is stationary, i.e. there exists $n_o \geq 1$ such that, for $n \geq n_o$, we have $v_n^k \in U(R) = R \setminus (\mathfrak{m} \cap R)$. But, we know that $v_n \in \mathfrak{m}$ and $v_n^k \in R$, hence $v_n^k \in \mathfrak{m} \cap R$: a contradiction.

"If" part: By construction, R is a conducive quasilocal domain such that $U(V) \cap R = U(R)$. Hence, we can easily transfer the accp property from V to R. □

COROLLARY 13. *The integral closure and the seminormalization of an accp conducive domain are accp (conducive) PVD's.*

PROOF. *Mutatis mutandis*, the proof is analogous to that of Corollary 11. □

EXAMPLE 14. As an application of Theorem 12, we can easily give an example of a pair of domains $R \subset S$, with the same quotient field, such that $U(S) \cap R = U(R)$ and accp holds in R, but accp does not hold in S (cf. [12, p. 324-325]). As a matter of fact, let R be an accp (hence, Archimedean) conducive domain, described by the following diagram:



where V is a rank 1 discrete valuation domain and A (and B) is a quasilocal 0-dimensional ring. Suppose that there exists a non-0-dimensional ring C, $A \subset C \subset B$. (For instance, take $V = \mathbb{C}[[X]]$, $B = V/X^2V = \mathbb{C} + x\mathbb{C}$, where $x = X + X^2V$, $A = \mathbb{Q} + x\mathbb{C}$ and $C = \mathbb{Q}[\pi] + x\mathbb{C}$.) Let $S = \phi^{-1}(C)$. By Theorem 12 and Theorem 1, S is a conducive domain but accp does not hold in S. Moreover, R,S and V have the same quotient field and

$S \subseteq C(R) = C(S) = V$. Finally $U(S) \cap R = U(R)$ by [2, Prop. 2.1].

We recall that an integral domain R is called a <u>Mori domain</u> if the ascending chain condition holds in the set of integral divisorial ideals of R (cf. [16]). A Mori domain is clearly an accp domain; thus, in the conducive case we know, by Theorem 12, that there exists a unique pullback representation of R as in (*), in which V is a rank 1 discrete valuation domain, $\mathfrak{b} = (R:V)$, and A a 0-dimensional quasilocal ring. For a Mori conducive domain, we shall exhibit a necessary "rigidity" property concerning its overrings (cf. Proposition 16). As Remark 18 will point out, this property, in general, is not sufficient for a conducive domain to be a Mori domain. However, with Proposition 19 (and Example 20), we shall show an explicit way to construct a large class of nontrivial Mori conducive domains.

Before giving these results, we recall that an integral divisorial ideal I of a domain R is <u>strongly divisorial</u> if $I \cdot (R:I) = I$, i.e. if $(I:I) = (R:I)$ (cf. [16, p. 344]). So if I is a strongly divisorial ideal of R, $(R:I)$ is an overring of R. We recall also the following result which appeared in the unpublished thesis [17].

PROPOSITION 15 (N. Raillard). *If* R *is a Mori domain and if* I *is a strongly divisorial ideal of* R, *then* $S=(R:I)$ *is a Mori domain.*

PROOF. If $\mathcal{D}(R)$ (resp., $\mathcal{D}(S)$) is the set of integral divisorial ideals of R (resp., of S), let ϕ be the function:

$$\phi : \mathcal{D}(R) \longrightarrow \mathcal{D}(S)$$
$$J \longmapsto (J:I)$$

It is not difficult to show that ϕ is a surjective map such that, if $\phi(J) \subseteq \phi(J')$, then $J \subseteq J'$. The statement follows easily. □

PROPOSITION 16. *Let* R *be an integral domain but not a field. Then* R *is a Mori conducive domain (if and) only if there exists a unique pullback representation of* R *as in* (*) *in which* V *is a rank* 1 *discrete* *valuation domain,* b = $(R:V)$, A *is a* 0*-dimensional quasilocal ring and there exists a finite sequence of overrings of* R, $R=R_1 \subset R_2 \subset \cdots \subset R_n = V$ *such that:*

a) *for each* i, i=1,2,...,n,R_i *is a Mori, quasilocal,1-dimensional domain; and*

b) *if* N_i *is the maximal ideal of* R_i, *then* $(N_i : N_i) = (R_i : N_i) = R_{i+1}$ (i=1,...,n-1).

PROOF. "Only if" part: Since a Mori domain is an accp domain, Theorem 12 supplies a unique pullback representation of R as in (*), where V is a rank 1 discrete valuation domain, b = (R:V), and A is a 0-dimensional quasilocal ring. In particular, R is a 1-dimensional quasilocal domain. (This also follows from [2, Th. 2.2]). Let $M = xV$ be the maximal ideal of V and let N be the maximal ideal of R. First of all, notice that M is a fractional ideal of R. Moreover, if $(R:V)=x^kV$, with $k \geq 1$, we have $(R:(R:M))=(R:(R:xV))=x(R:x^kV)=x(x^{-k}(x^kV))= M$. Thus M is a (fractional) divisorial ideal of R and, hence, $N = M \cap R$ is a divisorial ideal of R, as an intersection of divisorial ideals. Now, using the uniqueness of the pullback representation, if N is principal, then R=R'=V is a rank 1 discrete valuation domain and the Proposition trivially holds. Thus, without loss of generality, we can suppose that N is not principal. For each $z \in (R:N)$, we have $zN \subsetneq R$, hence $zN \subseteq N$; so $(R:N) \subseteq (N:N)$. The other inclusion being trivial, N is a strongly divisorial ideal of R. Therefore, we can construct the ring $R_2 = (R:N) = (N:N) \subseteq C(R) = V$, which is a 1-dimensional quasilocal conducive Mori domain (cf. Proposition 15 and [2, Theor. 2.2]). Iterating this construction, we get an ascending chain of quasilocal conducive domains of dimension 1, (R_i, N_i), $i \geq 1$ (where $R = R_1$). We want to show that, for some $i \geq 1$, N_i is principal, that is, $R_i = V$. Let $(R_i:V) = x^{k_i}V$, for each $i \geq 1$ (with $k_i \geq 0$). It is enough to prove that, if N_i is not principal, then $k_{i+1} \lneq k_i$. Let $l_i \geq 1$ be the largest exponent such that $x^{l_i}V \supseteq N_i$ (i.e. $x^{l_i}V \supseteq N_i$, but $x^{l_i+1}V \not\supseteq N_i$). Remark that $(x^{k_i}V:N_i) = ((R_i:V):N_i) = (R_i:VN_i) = ((R_i:N_i):V) = (R_{i+1}:V) = x^{k_{i+1}}V$. Since $x^{k_i-l_i}VN_i \subseteq x^{k_i-l_i+l_i}V = x^{k_i}V$, we have $x^{k_i-l_i}V \subseteq (x^{k_i}V:N_i)=$

$= x^{k_{i+1}}V$. On the other hand, since $x^{k_i - l_i} V\mathcal{N}_i \not\subseteq x^{k_i - l_i} V x^{l_{i+1}} V = x^{k_i + 1} V$, we have $x^{k_i - l_i} V \not\subseteq (x^{k_i + 1} V : \mathcal{N}_i) = x(x^{k_i} V : \mathcal{N}_i) = x\, x^{k_{i+1}} V = x^{k_{i+1}+1} V$. Hence $x^{k_i - l_i} V = x^{k_{i+1}} V$ and so $k_i - l_i = k_{i+1}$. In particular $k_i \gneq k_{i+1}$.

The converse is trivial by Theorem 1, because $R = R_1$ is a Mori domain. □

EXAMPLE 17: As an application of Proposition 16 and Theorem 12, we give an explicit construction of a conducive accp , non Mori domain.

Let $R = k + XD + X^2K[[X]]$, where $k \subset K$ are two distinct fields, D is an integral domain, but not a field, such that $k \subset D \subset K$ and X is an analytic indeterminate over K (take, for example, $k = \mathbb{Q}$, $K = \mathbb{R}$ and $D = \mathbb{Q}[\pi]$). By construction, R is obtained by the following pullback diagram:

$$\begin{array}{ccc} R & \longrightarrow & A = R/X^2V \\ \downarrow & & \downarrow \\ V = K[[X]] & \longrightarrow & B = V/X^2V \end{array}$$

where A is 0-dimensional, because $A_{red} \cong k$ is a field. Hence, by Theorem 12, R is a (1-dimensional) quasilocal accp conducive domain and its maximal ideal is $\mathcal{N} = XD + X^2K[[X]]$. However, by Proposition 16, R is not a Mori domain because $R_2 = (R:\mathcal{N}) = D + XK[[X]]$ is not a 1-dimensional ring.

REMARK 18. If condition a) of Proposition 16 holds only for $i=2,\dots,n$, then the conclusion ("if" part) of Proposition 16 is not true anymore. Take for example $R = k + XL + X^2K[[X]]$, where k is a field, $K = k(t_1, t_2)$, with t_1 and t_2 two indeterminates over k, and $L = kt_2 + k[t_1]$. Let $K[[X]] = V$. Then R is a quasilocal domain with maximal ideal $\mathcal{N} = XL + X^2V$. Moreover, by Theorem 12, R is an accp (conducive) domain, with $\mathfrak{b} = (R:V) = X^2V$. Consider $(\mathcal{N}:\mathcal{N}) = (R:\mathcal{N}) = k + XV \overset{\text{not.}}{=} R_2$. Then, R_2 is a Mori (cf. [1, Cor. 3.5]) (quasilocal, 1-dimensional , conducive) PVD, with maximal ideal $\mathcal{N}_2 = XV$, such that $(\mathcal{N}_2:\mathcal{N}_2) = (R_2:\mathcal{N}_2) = V$. Hence, we have a finite sequence of overrings of R, $R = R_1 \subset R_2 \subset R_3 = V$ verifying condition b) and the weaker form of condition a) of Proposition 16 (i.e. R_i is Mori only for $i \geq 2$). We want to show that R is not a Mori domain. Notice, first of all, that XV is a fractional divisorial ideal of R, as a direct calculation shows (cf. proof of Prop. 16). Hence, $\mathcal{N} = XV \cap R$ is a divisorial ideal of R and, for each nonzero element z in the quotient field of R, $z\mathcal{N}$ is still divisorial. For each $n \in \mathbb{N}$, take $z_1^{(n)} = Xt_1^{-n}$ and $z_2^{(n)} = X\,t_1^{-(n+1)}$ and consider the divisorial ideal of R

$$\begin{aligned} I^{(n)} &= z_1^{(n)}\mathcal{N} \cap z_2^{(n)}\mathcal{N} = Xt_1^{-n}(XL+X^2V) \cap Xt_1^{-(n+1)}(XL+X^2V) = \\ &= X^2(t_1^{-n}L \cap t_1^{-(n+1)}L) + X^3V \end{aligned}$$

By definition of L we have:

$$t_1^{-n}L \cap t_1^{-(n+1)}L = t_1^{-n}k + t_1^{-(n-1)}k + \ldots + t_1^{-1}k + k[t_1] \overset{\text{not.}}{=\!=} W^{(n)}$$

The ascending chain of k-vector spaces $\{W^{(n)}\}_{n\in\mathbb{N}}$ is clearly not stationary; hence the ascending chain of integral divisorial ideals $\{I^{(n)}\}_{n\in\mathbb{N}}$ of R is not stationary. Thus R is not a Mori domain.

We end the paper with a new class of examples of Mori conducive domains. We need the following preliminary result.

PROPOSITION 19. *Let* (V, M) *be the formal power series ring* $K[[X]]$ *,where* K *is a field . Let* R *be a subring of* V *of the following type:*

$$R = L_0 + XL_1 + X^2L_2 + \ldots + X^{k-1}L_{k-1} + X^kV$$

where $k \geq 1$, $L_0 \subseteq K$ *is a field and, for* $i=1,\ldots,k-1$, $L_i \subsetneq K$ *is a* L_0 *-vector space such that* $L_iL_j \subseteq L_{i+j}$ *for* $0 \leq i,j \leq k-1$ *(where* $L_{i+j} = K$ *if* $i+j \geq k$*). If the ascending chain condition holds for the* L_0*-vector spaces of the type*

$$W = \bigcap_j (c_{1,j}L_1 + \ldots + c_{k-1,j}L_{k-1}),$$

where $c_{ij} \in K$, *for* $i=1,\ldots,k-1$ *and* j *is in any index set, then* R *is a conducive Mori domain.*

PROOF. Notice, first of all, that by construction, R is obtained by the following pullback diagram:

$$\begin{array}{ccc} R & \longrightarrow\!\!\!\!\to & A = R/X^kV \\ \cap\downarrow & & \cap\downarrow \\ V & \longrightarrow\!\!\!\!\to & B = V/X^kV \end{array}$$

where A is 0-dimensional, because $A_{red} \cong L_0$ is a field. Hence, by Theorem 12, R is a (1-dimensional, quasilocal) accp conducive domain.

Notice, moreover, that the maximal ideal of R is $\mathcal{N}=\mathcal{M}\cap R = XL_1+\dots+X^{k-1}L_{k-1}+X^kV$, where $\mathcal{M}$ is a fractional ideal of R such that $(R:(R:\mathcal{M}))=(R:(R:XV))=X(X^{-k}(X^kV))=\mathcal{M}$. So also $\mathcal{N}=\mathcal{M}\cap R$ is a divisorial ideal of R.

Let F be the quotient field of R (and V). Any divisorial non-principal ideal $\mathcal{D}$ of R has the form $\mathcal{D}=\cap zR$, with $z\in F$ and $zR\supset\mathcal{D}$, so that $\mathcal{D}=\cap z\mathcal{N}$. Hence, it is enough to show that acc holds for integral divisorial ideals of R of the type $\cap z\mathcal{N}$. For each nonzero $z\in F$, we can write $z=X^hc_h+X^{h+1}c_{h+1}+\dots$, with $c_h, c_{h+1},\dots\in K$, $h\in\mathbb{Z}$, $c_h\neq 0$ (i.e., h is the value of z under the canonical valuation v associated to V). Thus we have $z\mathcal{N}=X^{h+1}c_hL_1+X^{h+2}(c_hL_2+c_{h+1}L_1)+\dots+X^{h+k}V$.

Let $\mathcal{D}=\bigcap_{z\in Z} z\mathcal{N}$, for some subset Z of K and let $h=h(\mathcal{D})=\max\{v(z)\mid z\in Z\}$, with $h<+\infty$, because $\mathcal{D}\neq(0)$.

Then, it is easy to see that

$$\mathcal{D}=X^{h+1}E_1+\dots+X^{h+\lambda}E_\lambda+\dots+X^{h+k-1}E_{k-1}+X^{h+k}V$$

where, for $\lambda=1,\dots,k-1$

$$E_\lambda=\bigcap_j (c_{1,j}L_1+\dots+c_{k-1,j}L_{k-1})$$

and $c_{i,j}$ are elements of K depending on $z\in Z$. Let, moreover, $\lambda_0=\lambda_0(\mathcal{D})=\min\{\lambda\mid E_\lambda\neq 0\}$ and $\lambda_0(X^{h+k}V)=0$.

Take any ascending chain $\{\mathcal{D}^{(n)}\}_{n\in\mathbb{N}}$ of integral divisorial ideals of R of the previous type, i.e. $\mathcal{D}^{(n)}=\bigcap_{z\in Z^{(n)}} z\mathcal{N}$, for some subset $Z^{(n)}$ of K. Notice that if $n\leq n'$, then $h(\mathcal{D}^{(n)})+\lambda_0(\mathcal{D}^{(n)})\geq h(\mathcal{D}^{(n')})+\lambda_0(\mathcal{D}^{(n')})\geq 0$, because $\mathcal{D}^{(n)}\subseteq\mathcal{D}^{(n')}\subseteq R$. So, without loss of generality, we can suppose that, for each n, $h(\mathcal{D}^{(n)})+\lambda_0(\mathcal{D}^{(n)})=\mu$, where μ is a fixed positive integer. In this case, for each $n\in\mathbb{N}$, we have:

$$\mathcal{D}^{(n)}=X^\mu E^{(n)}_{\lambda_0(\mathcal{D}^{(n)})}+\dots+X^{h(\mathcal{D}^{(n)})+k-1}E^{(n)}_{k-1}+X^{h(\mathcal{D}^{(n)})+k}V$$

where,

$$E^{(n)}_\lambda=\bigcap_j (c^{(n)}_{1,j}L_1+\dots+c^{(n)}_{k-1,j}L_{k-1})\ \text{for}\ \lambda=\lambda_0(\mathcal{D}^{(n)}),\dots,k-1,\ \text{with}$$

$c_{i,j}\in K$.

Since, for each λ, the ascending chain of L_0-vector spaces $\{E^{(n)}_\lambda\}_{n\in\mathbb{N}}$ is stationary by hypothesis, we deduce that the chain $\{\mathcal{D}^{(n)}\}_{n\in\mathbb{N}}$ of divisorial ideals of R is stationary too. □

EXAMPLE 20: A new class of examples of Mori, non-Noetherian conducive domains. Let $R = L_o + XL_1 + \ldots + X^{k-1} L_{k-1} + X^k V$ be a ring, as in the statement of Proposition 19, where the field extension $L_o \subset K$ is non-finite and $L_1, \ldots, L_{k-1}$ are finite dimensional as L_o-vector spaces. As in the proof of Proposition 19, we see that R is obtained by the following pullback diagram:

$$\begin{array}{ccc} R & \twoheadrightarrow & A = R/X^k V \\ \downarrow & & \downarrow u \\ V & \rightarrow & B = V/X^k V \end{array}$$

where the ring extension $A \hookrightarrow B$ is non-finite, because $L_o \subset K$ is non finite. Hence, by Proposition 19 and the proof of Theorem 6, we easily conclude that R is a Mori non-Noetherian conducive domain.

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