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Amalgamation of algebras and the ultrafilter topology on the Zariski space of valuation overrings of an integral domain

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Introduction

This thesis consists of three chapters: the first one covers essentially my first year of research activity in Rome, last two chapters contain the results that I have obtained during the remaining part of my PhD studies.

Chapter 1 is devoted to the study of the following "new" ring construction. Let $f : A \longrightarrow B$ be a ring homomorphism and let $\mathfrak{b}$ be an ideal of B. Consider the following subring of $A \times B$:

$$A \bowtie^f \mathfrak{b} := \{(a, f(a) + b) : a \in A,\ b \in \mathfrak{b}\},$$

called *the amalgamation of A with B along $\mathfrak{b}$ with respect to f*. This construction generalizes the amalgamated duplication of a ring along an ideal (introduced and studied in [13], [12], [14] and in [55]). Moreover, several classical constructions (such as the $A+XB[X]$, the $A+XB[\![X]\!]$ and the $D+M$ constructions) can be studied as particular cases of the amalgamation (see Examples 1.11 and 1.14) and other classical constructions, such as the Nagata's idealization (cf. [57, page 2], [44, Chapter VI, Section 25]), also called Fossum's trivial extension (cf. [27], [48] and [4]), and the CPI extensions (in the sense of Boisen and Sheldon [7]) are strictly related to it (see Example 1.15).

The firt thing to remark about the ring $A \bowtie^f \mathfrak{b}$ is that it is always a ring extension of A, since the map $\iota : A \longrightarrow A \bowtie^f \mathfrak{b}$, $a \mapsto (a, f(a))$, is clearly a ring embedding. Moreover, the structure of $A \bowtie^f \mathfrak{b}$ is deeply related to that of A by the fact that A is a ring retract of $A \bowtie^f \mathfrak{b}$, via the canonical surjective map $p_{_A} : A \bowtie^f \mathfrak{b} \longrightarrow A$, $(a, f(a) + b) \mapsto a$ (i.e. $p_{_A} \circ \iota$ is the identity of A). This fact will be an useful tool in our investigation, since several algebraic properties are inherited from a ring to a subring retract of it.

The other the key tools for studying $A \bowtie^f \mathfrak{b}$ are based on the fact that the amalgamation can be studied in the frame of pullbacks constructions (see Proposition 1.17). This point of view allows us to provide an ample description of various properties of $A \bowtie^f \mathfrak{b}$, in connection with the properties of A, $\mathfrak{b}$ and f.

For example, we characterize those distinguished pullbacks that can be expressed as an amalgamation (see Propositions 1.19 and 1.23) and investigate the basic properties of this construction (e.g., we provide characterizations for $A \bowtie^f \mathfrak{b}$ to be a Noetherian ring, an integral domain, a reduced ring).

Then, we describe the topological and order structure of the prime spectrum of $A \bowtie^f \mathfrak{b}$, remarking also that $\mathrm{Spec}(A)$ is identified with a closed subspace of $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$, via the canonical surjective map $p_{_A} : A \bowtie^f \mathfrak{b} \longrightarrow A$, $(a, f(a) + b) \mapsto a$. We use the ring retraction structure of $A \bowtie^f \mathfrak{b}$ to show that (the canonical image of) $\mathrm{Spec}(A)$ is also a topological retract of $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$. More precisely, the ring embedding $\iota : A \longrightarrow A \bowtie^f \mathfrak{b}$ induces a surjective map $\iota^* : \mathrm{Spec}(A \bowtie^f \mathfrak{b}) \longrightarrow \mathrm{Spec}(A)$ and $\iota^* \circ p_{_A}^*$ is the identity of $\mathrm{Spec}(A)$. The closed embedding $p_{_A}^*$ induced by $p_{_A}$ identify each prime ideal $\mathfrak{p}$ of A with the prime ideal $\mathfrak{p}'^{_f} := \mathfrak{p} \bowtie^f \mathfrak{b} := \{(p, f(p) + b) : p \in \mathfrak{p}, b \in \mathfrak{b}\}$ and this corrispondence identifies $\mathrm{Spec}(A)$ with the closed subspace $V(\mathfrak{b}_0)$ of $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$, where $\mathfrak{b}_0 := \{0\} \times \mathfrak{b}$. The complement of $V(\mathfrak{b}_0)$ in $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$ is constructed by contracting to $A \bowtie^f \mathfrak{b}$ the prime ideals of $A \times B$ not containing $\{0\} \times \mathfrak{b}$. More precisely, the restriction of the map $\mathrm{Spec}(A \times B) \longrightarrow \mathrm{Spec}(A \bowtie^f \mathfrak{b})$, induced by the inclusion $A \bowtie^f \mathfrak{b} \subseteq A \times B$, gives rise to a homeomorphism of the subspace $\mathrm{Spec}(A \times B) \backslash V(\mathfrak{b}_0)$ onto $\mathrm{Spec}(A \bowtie^f \mathfrak{b}) \backslash V(\mathfrak{b}_0)$, sending a prime ideal of the type $A \times \mathfrak{q}$ to the prime ideal $\overline{\mathfrak{q}}^f := (A \times \mathfrak{q}) \cap A \bowtie^f \mathfrak{b}$, for $\mathfrak{q}$ varying in $\mathrm{Spec}(B) \backslash V(\mathfrak{b})$. Moreover, for each prime ideal $\mathfrak{q}$ of B containing $\mathfrak{b}$, then $(A \times \mathfrak{q}) \cap A \bowtie^f \mathfrak{b} = f^{-1}(\mathfrak{q})'^{_f}$. Thus we can describe $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$ as the union of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$, modulo the equivalence relation generated by the identification of $\mathfrak{q}$ and $f^{-1}(\mathfrak{q})$, for each $\mathfrak{q} \in V(\mathfrak{b})$ (see Proposition 1.36 and Remark 1.37).

Section 1.4 is completely devoted to the study of the Prüfer like conditions on $A \bowtie^f \mathfrak{b}$. By using the local structure of $A \bowtie^f \mathfrak{b}$ and its ring retraction structure, we provide necessary and sufficient conditions for $A \bowtie^f \mathfrak{b}$ to satisfy Prüfer like conditions. Moreover, we show, when the conductor of $A \bowtie^f \mathfrak{b}$ in $A \times B$ is a regular ideal, then $A \bowtie^f \mathfrak{b}$ may be semihereditary (resp. of weak global dimension ≤ 1, Gauss, locally Prüfer) only in the trivial case $\mathfrak{b} = B$ (Proposition 1.88).

In Proposition 1.54 and Lemma 1.55 we compute the integral closure of $A \bowtie^f \mathfrak{b}$ in $\mathrm{Tot}(A) \times \mathrm{Tot}(B)$ and we characterize when the ring embedding $\iota : A \longrightarrow A \bowtie^f \mathfrak{b}$ is integral. This means that in some relevant cases (e.g. when the ring homomorphism f is integral) the Krull dimension of $A \bowtie^f \mathfrak{b}$ is equal to the Krull dimension of A. More generally, we show that the ring extension $A \bowtie^f \mathfrak{b} \subseteq A \times (f(A) + \mathfrak{b})$ is always integral, and thus $\dim(A \bowtie^f \mathfrak{b}) = \sup\{\dim(A), \dim(f(A)+\mathfrak{b})\}$. In general, it is not easy to compute $\dim(f(A)+\mathfrak{b})$ in terms of the given objects $A, B, \mathfrak{b}$ and f. Thus, in Proposition 1.95 and Theorem 1.101, we provide lower and upper bounds for $\dim(A \bowtie^f \mathfrak{b})$ and we show that these bounds, obtained in a such general setting, are so sharp that generalize, and possibly improve, analogous bounds established for the very particular cases of integral domains of the form $A + XB[X]$ (see [21]) or $A + XB[\![X]\!]$ (see [17]). Most of the results of this chapter are contained in three papers in collaboration with M. D'Anna and M. Fontana.

Chapter 2 is devoted to the study of a new topology defined in the space, denoted by $\mathrm{Zar}(K|A)$, of all the valuation domains of a field K contained a fixed subring A of K. The motivations for studying spaces of valuation domains come from various directions and, historically, mainly from Zariski's work for building up Algebraic Geometry by algebraic means. From another point of view, the crucial role played by valuation domains in Commutative Algebra is due to the fact that each integrally closed domain A is the intersection of all its valuation overrings, i.e. $\mathrm{Zar}(K|A)$ is *a representation of* A, as it is said usually. The first topological approach to the study of collections of valuation domains is due to Zariski, that put on $\mathrm{Zar}(K|A)$ the topology whose subbasic open sets are the sets of the form $B_x := \{V \in \mathrm{Zar}(K|A) : x \in V\}$, for x varying in K, and showed the compactness of this space (see [62] and [63]), endowed with what is know called *the Zariski topology on* $\mathrm{Zar}(K|A)$. Later, it was proven, and rediscovered by several authors with a variety of different techniques, that, if K is the quotient field of A, then $\mathrm{Zar}(K|A)$, endowed with Zariski's topology, is a spectral space in the sense of Hochster [40] (see [15], [16], [43] and the appendix of [50]). As it is possible to see at a glance, the Zariski topology on $\mathrm{Zar}(K|A)$ is almost never Hausdorff. In the case of the prime spectrum of a ring, there is a natural way to construct a topology finer or equal than the Zariski topology and making the prime spectrum a compact topological space: the point is to consider the coarsest topology for which the principal open subsets (of the Zariski topology) are clopen. This topology is known as the constructible topology (see [2, Chapter 3, Exercises 27,28,30] or [28]). Recently, M. Fontana and K. A. Loper, by using ultrafilters, have constructed a "new" topology on the prime spectrum of a ring, called *the ultrafilter topology*, and proved that this topology is identical to the constructible topology (see [26]). In Chapter 2, we extend the argument given in [26] to define a topology on $\mathrm{Zar}(K|A)$ that is finer than the

Zariski topology. In fact, we use ultrafilters to construct an operator that is a closure operator and we call the topology we obtain *the ultrafilter topology on* $\mathrm{Zar}(K|A)$. Moreover, it is possible to endow $\mathrm{Zar}(K|A)$ with the smallest topology for which the the sets $B_F := \{V \in \mathrm{Zar}(K|A) : V \supseteq F\}$ (for F varying in the collection of all the finite subsets of K). It is clear that the role played by this topology is exactly the same role played by the constructible topology on the prime spectrum of a ring. Thus, we call this topology *the constructible topology on* $\mathrm{Zar}(K|A)$ With different approach from [26], we prove that the constructible topology and the ultrafilter topology on $\mathrm{Zar}(K|A)$ are identical (cite Theorem 2.14(6)). The first key point for providing applications of the ultrafilter topology on $\mathrm{Zar}(K|A)$ is to study the properties of the canonical map $\gamma : \mathrm{Zar}(K|A) \longrightarrow \mathrm{Spec}(A)$, sending a valuation domain to its center on A. This map (clearly surjective, by Zorn's Lemma) is continuous and closed, if $\mathrm{Zar}(K|A)$ and $\mathrm{Spec}(A)$ are both endowed with the Zarisky topology, by [15, Lemma (2.1) and Theorem (2.5)]. We show that these properties are preserved also if $\mathrm{Zar}(K|A)$ and $\mathrm{Spec}(A)$ are equipped with the ultrafilter topology (Theorem 2.17). It follows, in particular that, when the map γ is injective, then it is an homeomorphism. The other step toward applications of ultrafilter topological techniques to integrally closed domains is the fact that the space $\mathrm{Zar}(K|A)$ can be identified to the space of the valuation overrings of some Prüfer domain, both equipped with the Zariski topology or with the ultrafilter topology. To see this, we use the following abstract approach to the Kronecker function rings, due to F. Halter–Koch (see [34]): if K is a field and T is an indeterminate over K, then a subring S of $K(T)$ is called *a* $K-$*function ring* if $T, T^{-1} \in S$ and $f(0)/f(T) \in S$ for each polynomial $f(T) \in K[T]\backslash\{0\}$. F. Halter–Koch proved, in particular, that a $K-$function ring is a Bézout domain (in particular, a Prüfer domain) whose quotient field is $K(T)$. To provide an easy description of the space of the valuation overrings of a $K-$function ring, it is necessary to recall the following relevant notion: if V is a valuation domain having K as the quotient field and $\mathfrak{m}$ is its maximal ideal, there is a very particular way to extend V to a valuation domain of $K(T)$. This extension is obtained by localizing the polynomial ring $V[T]$ to the prime ideal $\mathfrak{m}[T]$. This ring, usually denoted by $V(T)$, is a valuation domain of $K(T)$ extending V and it is called *the trivial extension of* V *in* $K(T)$. In this setting, it is easily proved that the space of the valuation overrings of a $K-$function ring S is exactly

$$\mathrm{Zar}_0(K(T)|S) := \{(W \cap K)(T) : W \in \mathrm{Zar}(K(T)|S)\},$$

that is a valuation overring W of S is obtained by contracting itself over K and taking the trivial extension of this contraction (see Proposition 2.24). Moreover, it is proved that, if S is a $K-$function ring, then the contraction map $\varphi : \mathrm{Zar}(K(T)|S) \longrightarrow \mathrm{Zar}(K|S \cap K)$, $W \mapsto W \cap K$, is an homeomorphism, both with respect to the Zariski topology and the ultrafilter topology. In particular, this allows easily to conclude that $\mathrm{Zar}(K|A)$ is a spectral space for any field K and any subring A of K (note that this result generalizes [16, Theorem 2]). In fact, we can consider the $K-$function ring $\mathrm{Kr}(K|A) := \bigcap\{V(T) : V \in \mathrm{Zar}(K|A)\}$ and, keeping in mind the facts explained above, it follows that $\mathrm{Zar}(K|A)$ is homeomorphic to $\mathrm{Spec}(\mathrm{Kr}(K|A))$. Section 4 of Chapter 2 provides applications of the topological material introduced above to representations of integrally closed domains. The first fact to notice is that, if Y_1 and Y_2 are spaces of valuation domains having the same closure, with respect to the ultrafilter topology, then Y_1 and Y_2 are representations of the same integrally closed domain (i.e. $\bigcap\{V : V \in Y_1\} = \bigcap\{V : V \in Y_2\}$) (see Proposition 2.31). Moreover, we provide a class of integral domains for which the converse of the previous statement is not true. To do this, we show that if K is a field, A is a subring of K and $\Sigma \subseteq \mathrm{Zar}(K|A)$ is an infinite and locally finite collection of valuation domains, then the closure of Σ, with respect to the ultrafilter topology, is obtained just by adding the field K to Σ (Proposition 2.33). Thus, each integral domain that admits at least 2 distinct infinite and locally finite representation (examples of such a domain are given) does not satisfy the converse of Proposition 2.31. The point is that the fact that two collection of valuation domains have the same closure, with respect to the ultrafilter topology, imply a statement stronger than that given in Proposition 2.31. To explain this, we will use the following distinguished class of semistar operations, that plays a central role in multiplicative ideal theory. If A is an integral domain and K is its quotient field, we say that a semistar operation $\star$ on A is ***e.a.b.*** if, for each finitely generated $A-$submodules F, G, H of K, the condition $(FG)^\star \subseteq (FH)^\star$ implies $G^\star \subseteq H^\star$. Among the collection of all the `e.a.b.` semistar operations, we can consider $\wedge_Y$, with $Y \subseteq \mathrm{Zar}(K|A)$, defined by $F^{\wedge_Y} := \bigcap\{FV : V \in Y\}$, for each nonzero $A-$submodule F of K (see [25, Proposition 7]). In this setting, we show, in particular, that, if Y_1 and Y_2 are subsets of $\mathrm{Zar}(K|A)$ with the same closure, with respect to the ultrafilter topology, then the semistar operations $(\wedge_{Y_1})_f$ and $(\wedge_{Y_2})_f$ are identical (and, as a particular case, we rediscover that $A^{\wedge_{Y_1}} = \bigcap\{V : V \in Y_1\} = \bigcap\{V : V \in Y_2\} = A^{\wedge_{Y_2}}$). We note that the equality $(\wedge_{Y_1})_f = (\wedge_{Y_2})_f$ does not imply, in general, the equality of the closure of

Y_1 and Y_2, with respect to the ultrafilter topology, as we see in Example 2.43. We note also that, if we consider the Zariski–generic closure $Y^{\uparrow}$ of a subset Y of $\mathrm{Zar}(K|A)$, that is $Y^{\uparrow} := \{W \in \mathrm{Zar}(K|A) : W \supseteq V, \text{ for some } V \in Y\}$, then the operation to take the Zariski–generic closure of a set does not change the semistar operations, i.e. $\wedge_Y = \wedge_{Y^{\uparrow}}$. We show that the equality $(\wedge_{Y_1})_f = (\wedge_{Y_2})_f$ is equivalent to the fact that the sets $\mathrm{Ad}^{\mathrm{ultra}}(Y_1), \mathrm{Ad}^{\mathrm{ultra}}(Y_2)$ have the same Zariski–generic closure (see Theorem 2.40). We show also that the operation to take the Zariski–generic closure of the closure of a subset Y, with respect to the ultrafilter topology, gives rise to closure operation. We show that the topology determined by this closure is the so called *inverse topology* (or *dual topology*), with respect to the Zariski topology (for example, see [40, Proposition 8]). Thus, we can formulate the previous result by saying that $(\wedge_{Y_1})_f = (\wedge_{Y_2})_f$ if and only if Y_1 and Y_2 have the same closure, with respect to the inverse topology. From this result and the natural relation between `e.a.b.` semistar operations and Kronecker function rings, we prove that an integrally closed domain A (with quotient field K) is *vacant* (that is, it has a unique Kronecker function ring [19]) if and only if each representation of A is a dense subspace of $\mathrm{Zar}(K|A)$, with respect to the inverse topology. Moreover, in Theorem 2.45, we characterize the `e.a.b.` semistar operations that are of finite type (i.e. the *complete* semistar operations). In particular, we show that a semistar operation $\star$ on A is complete if and only if it is there exists a compact subspace Y (resp. Y') of $\mathrm{Zar}(K|A)$, with respect to the ultrafilter topology (resp. the Zariski topology) such that $\star = \wedge_Y$ (resp. $\star = \wedge_{Y'}$). By using the results given before, it is not hard to deduce that, given a `e.a.b.` semistar operation of the type $\wedge_Y$, for some subspace Y of $\mathrm{Zar}(K|A)$, then $(\wedge_Y)_f = \wedge_{\mathrm{Ad}^{\mathrm{ultra}}(Y)}$. Thus the algebraic operation to take the completion of a semistar operation corresponds topologically to a compactification. The results of this chapter are the subject of a preprint in collaboration with M. Fontana and A. Loper.

Finally, in the Appendix the notion of ultrafilter closure and ultrafilter limit point, given in the prime spectrum of a ring and in the Zariski–Riemann surfaces are vastly generalized. In fact, given a set X and a nonempty collection $\mathcal{F}$ of subsets of X, we define a new closure operator, depending of $\mathcal{F}$. This operator defines new ultrafilter topologies for which $\mathcal{F}$ is a collection of clopen sets. In this way we get, as very particular cases, the ultrafilter topology of the prime spectrum of a ring and of an abstract Zariski–Riemann surface. We describe several properties of these topologies and provide applications.

Chapter 1

Amalgamated algebras along an ideal

1.0 Preliminaries

1.0.1 Notation

If $\mathcal{A}$ is a nonempty collection of subsets of a set X, we will denote by $\bigcap \mathcal{A}$ the intersection of all the members of $\mathcal{A}$. In the following, unless otherwise specified, with the term *ring* we will mean always a commutative unitary ring. Moreover, if A and B are rings and $f : A \longrightarrow B$ is a ring homomorphism, we will assume that f maps the identity of A into the identity of B. Every prime ideal of a ring is, in particular, a proper ideal. If A is a ring, we set

$$\mathrm{Spec}(A) := \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ is a prime ideal of } A\}$$

$$\mathrm{Max}(A) := \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ is a maximal ideal of } A\}$$

$$\mathrm{Min}(A) := \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ is a minimal prime ideal of } A\}$$

$$\mathrm{Nilp}(A) := \{a \in A : a \text{ is a nilpotent element of } A\} =: \text{ the nilradical of } A$$

$$\mathrm{Jac}(A) := \bigcap \mathrm{Max}(A) =: \text{ the Jacobson radical of } A$$

$$\mathrm{Reg}(A) := \{a \in A : a \text{ is a regular (non-zerodivisor) element of } A\}$$

$$\mathrm{Tot}(A) := \text{ the total ring of fractions of } A =: \mathrm{Reg}(A)^{-1}A$$

Unless otherwise specified, we shall consider the set $\mathrm{Spec}(A)$ endowed with the Zariski topology, i.e., the topology whose closed sets are the subsets of

Spec(A) of the form $V_A(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\}$, for each ideal $\mathfrak{a}$ of A. When there is no danger of confusion, we denote $V_A(\mathfrak{a})$ simply by $V(\mathfrak{a})$. Let $f : A \longrightarrow B$ be a ring homomorphism. We will denote by $f^* : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ the canonical continuous function associated to f, that is, the function defined by $f^*(\mathfrak{p}) := f^{-1}(\mathfrak{p})$, for each $\mathfrak{p} \in \operatorname{Spec}(B)$.

1.0.2 Fiber products

In this section, we collect some facts about fiber products of rings and their prime spectra.

1.1 Definition. *Let $\alpha : A \longrightarrow C$, $\beta : B \longrightarrow C$ be ring homomorphisms. Then, the subring $D := \alpha \times_C \beta := \{(a,b) \in A \times B : \alpha(a) = \beta(b)\}$ of $A \times B$ is called the* fiber product *(or* pullback*)* of α and β.

We begin to collect some properties for a fiber product to be a reduced ring.

1.2 Proposition. *With the notation of Definition 1.1, we have:*

(1) *If D $(= \alpha \times_C \beta)$ is reduced, then*

$$\operatorname{Nilp}(A) \cap \operatorname{Ker}(\alpha) = \{0\} \quad \text{and} \quad \operatorname{Nilp}(B) \cap \operatorname{Ker}(\beta) = \{0\}.$$

(2) *If at least one of the following conditions holds*

(a) *A is reduced and* $\operatorname{Nilp}(B) \cap \operatorname{Ker}(\beta) = \{0\}$,

(b) *B is reduced and* $\operatorname{Nilp}(A) \cap \operatorname{Ker}(\alpha) = \{0\}$,

then D is reduced.

Proof. (1) Assume D reduced. By simmetry, it sufficies to show that $\operatorname{Nilp}(A) \cap \operatorname{Ker}(\alpha) = \{0\}$. If $a \in \operatorname{Nilp}(A) \cap \operatorname{Ker}(\alpha)$, then $(a, 0)$ is a nilpotent element of D, and thus $a = 0$.

(2) By the simmetry of conditions (a) and (b), it is enough to show that, if condition (a) holds, then D is reduced. Let (a, b) be a nilpotent element of D. Then $a = 0$, since $a \in \operatorname{Nilp}(A)$ and A is reduced. Thus, we have $(a, b) = (0, b) \in \operatorname{Nilp}(D)$, hence $b \in \operatorname{Nilp}(B) \cap \operatorname{Ker}(\beta) = \{0\}$. □

1.3 Remark. We preserve the notation of Definition 1.1 and endow $p_A(D)$ with the natural $D-$module structure given by p_A. Obviously $\mathrm{Ker}(p_A) = \{0\}\times\mathrm{Ker}(\beta)$, and thus we have the following short exact sequence of $D-$modules

$$0 \longrightarrow \mathrm{Ker}(\beta) \xrightarrow{i} D \xrightarrow{p_A} p_A(D) \longrightarrow 0,$$

where i is the natural D–module embedding (defined by $x \mapsto (0, x)$ for all $x \in \mathrm{Ker}(\beta)$). Moreover, it is clear that the $D-$submodules of $p_A(D)$ are exactly the ideals of the ring $p_A(D)$.

By using [2, Proposition 6.3] and the previous Remark 1.3, the following results follow easily.

1.4 Proposition. *With the notation of Definition 1.1, the following conditions are equivalent.*

(i) *D $(= \alpha \times_C \beta)$ is a Noetherian ring.*

(ii) *$\mathrm{Ker}(\beta)$ is a Noetherian D–module (with the D–module structure naturally induced by p_B) and $p_A(D)$ is a Noetherian ring.*

1.5 Proposition. *With the notation of Definition 1.1, the following conditions are equivalent.*

(i) *D $(= \alpha \times_C \beta)$ is a Artin ring.*

(ii) *$\mathrm{Ker}(\beta)$ is a Artin D–module (with the D–module structure naturally induced by p_B) and $p_A(D)$ is a Artin ring.*

The following result, due to M. Fontana, gives a complete description of the prime spectrum of a fiber product.

1.6 Theorem. ([20, Theorem 1.4]) *With the notation of Definition 1.1, set $X := \mathrm{Spec}(A)$, $Y := \mathrm{Spec}(B)$, $Z := \mathrm{Spec}(C)$, and $W := \mathrm{Spec}(D)$. Assume that β is surjective. Then, the following statements hold.*

(1) *If $\mathfrak{h} \in W\backslash V(\mathrm{Ker}(p_A))$, then there is a unique prime ideal $\mathfrak{q}$ of B such that $p_B^{-1}(\mathfrak{q}) = \mathfrak{h}$. Moreover, $\mathfrak{q} \in Y\backslash V(\mathrm{Ker}(\beta))$ and $D_{\mathfrak{h}} \cong B_{\mathfrak{q}}$, under the canonical homomorphism induced by p_B.*

(2) *The continuous map p_A^* is a closed embedding of X into W. Thus X is homeomorphic to its image, $V(\mathrm{Ker}(p_A))$, under p_A^*.*

(3) *The restriction of the continuous map p_B^* to $Y\backslash V(\mathrm{Ker}(\beta))$ is an homeomorphism of $Y\backslash V(\mathrm{Ker}(\beta))$ with $W\backslash V(\mathrm{Ker}(p_A))$ (hence, in particular, it is an isomorphism of partially ordered sets).*

In particular, the prime ideals of D are of the type $p_A^{-1}(\mathfrak{p})$ or $p_B^{-1}(\mathfrak{q})$, where $\mathfrak{p}$ is any prime ideal of A and $\mathfrak{q}$ is a prime ideal of B, with $\mathfrak{q} \not\supseteq \mathrm{Ker}(\beta)$.

The next result is straightforward, but we will give the proof in order to illustrate some general machineries concering fiber products.

1.7 Corollary. *With the notation of Definition 1.1, assume that β is surjective. Let $\mathfrak{h}$ be a prime ideal of D $(= \alpha \times_C \beta)$. The following properties hold.*

(1) *Assume that $\mathfrak{h}$ contains $\mathrm{Ker}(p_A)$. Let $\mathfrak{p}$ be the only prime ideal of A such that $\mathfrak{h} = p_A^*(\mathfrak{p})$ (Theorem 1.6(2)). Then, $\mathfrak{h}$ is a maximal ideal of D if and only if $\mathfrak{p}$ is a maximal ideal of A.*

(2) *Assume that $\mathfrak{h}$ does not contain $\mathrm{Ker}(p_A)$. Let $\mathfrak{q}$ be the only prime ideal of B ($\mathfrak{q} \notin V(\mathrm{Ker}(\beta))$) such that $p_B^*(\mathfrak{q}) = \mathfrak{h}$ (Theorem 1.6(1)). Then, $\mathfrak{h}$ is a maximal ideal of D if and only if $\mathfrak{q}$ is a maximal ideal of B.*

Proof. Statement (1) is an easy consequence of the fact that p_A^* is a closed embedding.

(2) Assume that $\mathfrak{q}$ is a maximal ideal of B not containing $\mathrm{Ker}(\beta)$, and let $\mathfrak{d}$ be an ideal of D such that $\mathfrak{h} = p_B^*(\mathfrak{q}) \subsetneq \mathfrak{d}$. Thus, we can find an element $(a,b) \in \mathfrak{d}\backslash\mathfrak{h}$, where $a \in A$, $b \in B$, and $\alpha(a) = \beta(b)$. By the choice of (a,b), we have $b \notin \mathfrak{q}$; hence there exist $k_1 \in B$ and $q_1 \in \mathfrak{q}$ such that $k_1 b + q_1 = 1$, by maximality of $\mathfrak{q}$. Moreover, since $\mathfrak{q} \not\supseteq \mathrm{Ker}(\beta)$, we can pick an element $x \in \mathrm{Ker}(\beta)\backslash\mathfrak{q}$ and, again by maximality of $\mathfrak{q}$, we can find $k_2 \in B$ and $q_2 \in \mathfrak{q}$ such that $k_2 x + q_2 = 1$. Therefore, we have $kbx + q = 1$, for some $k \in B, q \in \mathfrak{q}$, and thus $(1,q) \in \mathfrak{h} \subset \mathfrak{d}$. Moreover, since $(0,kx) \in D$, we have $(0,kbx) = (0,kx)(a,b) \in \mathfrak{d}$, and and finally $(0,kbx)+(1,q) = (1,1) \in \mathfrak{d}$. This prove that $\mathfrak{h}$ is a maximal ideal of D.

Conversely, assume that $\mathfrak{h}$ is a maximal ideal of D not containing $\mathrm{Ker}(p_A)$, and let $\mathfrak{q}$ be the unique prime ideal of B such that $p_B^*(\mathfrak{q}) = \mathfrak{h}$ (Theorem 1.6(1)). If $\mathfrak{q}$ is not a maximal ideal of B, we can find a prime ideal $\mathfrak{q}'$ of B such that $\mathfrak{q} \subsetneq \mathfrak{q}'$. Since, in particular, $p_B^*(\mathfrak{q}')$ is a proper ideal of D containing the maximal ideal $\mathfrak{h}$, we have $\mathfrak{h} = p_B^*(\mathfrak{q}')$, a contradiction, since $\mathfrak{q} \neq \mathfrak{q}'$. $\square$

1.8 Corollary. *With the notation of Definition 1.1, assume β surjective. Then, D $(= \alpha \times_{C} \beta)$ is a local ring if and only if A is a local ring and* $\mathrm{Ker}(\beta) \subseteq \mathrm{Jac}(B)$. *In particular, if A and B are local rings, then D is a local ring. Moreover, if D is a local ring and $\mathfrak{m}$ is the only maximal ideal of A, then* $\{p_{A}^{-1}(\mathfrak{m})\} = \mathrm{Max}(D)$.

Proof. It is enough to apply Corollary 1.7. □

1.1 The construction $A \bowtie^f \mathfrak{b}$: basic algebraic properties

Let $f : A \longrightarrow B$ be a ring homomorphism and let $\mathfrak{b}$ be an ideal of B. With this notation, consider the following subring

$$A \bowtie^f \mathfrak{b} := \{(a, f(a) + b) : a \in A, b \in \mathfrak{b}\}$$

of $A \times B$, and call it *the amalgamation of the ring A with B along the ideal $\mathfrak{b}$, with respect to f*. The aim of this chapter is to study several algebraic properties of the ring $A \bowtie^f \mathfrak{b}$, and the conditions on A, B, f and $\mathfrak{b}$ to transfer them from A and B to $A \bowtie^f \mathfrak{b}$. We will discover that several constructions studied in the recent years are particular cases of the amalgamation and we will see how our results often generalize well known results. Before giving examples and beginning the systematic study of the ring $A \bowtie^f \mathfrak{b}$, we give some elementary properties of the amalgamation, whose proof is straightforward and follows by definitions.

1.9 Proposition. *Let $f : A \longrightarrow B$ be a ring homomorphism, $\mathfrak{b}$ an ideal of B and let* $A \bowtie^f \mathfrak{b} := \{(a, f(a) + b) : a \in A,\ b \in \mathfrak{b}\}$

(1) *Let $\iota := \iota_{A,f,\mathfrak{b}} : A \longrightarrow A \bowtie^f \mathfrak{b}$ be the natural the ring homomorphism defined by $\iota(a) := (a, f(a))$, for all $a \in A$. Then, ι is ring embedding, making $A \bowtie^f \mathfrak{b}$ a ring extension of A (with $\iota(A) = \Gamma(f)$ $(:= \{(a, f(a)) : a \in A\}$ subring of $A \bowtie^f \mathfrak{b}$).*

(2) *Let $\mathfrak{a}$ be an ideal of A and set $\mathfrak{a} \bowtie^f \mathfrak{b} := \{(a, f(a) + b) : a \in \mathfrak{a}, b \in \mathfrak{b}\}$. Then $\mathfrak{a} \bowtie^f \mathfrak{b}$ is an ideal of $A \bowtie^f \mathfrak{b}$, the composition of canonical homomorphisms $A \stackrel{\iota}{\hookrightarrow} A \bowtie^f \mathfrak{b} \twoheadrightarrow A \bowtie^f \mathfrak{b}/\mathfrak{a} \bowtie^f \mathfrak{b}$ is a surjective ring homomorphism and its kernel coincides with $\mathfrak{a}$.*

Hence, we have the following canonical isomorphism:

$$\frac{A \bowtie^f \mathfrak{b}}{\mathfrak{a} \bowtie^f \mathfrak{b}} \cong \frac{A}{\mathfrak{a}}\,.$$

(3) *Let* $p_{_A} : A \bowtie^f \mathfrak{b} \longrightarrow A$ *and* $p_{_B} : A \bowtie^f \mathfrak{b} \longrightarrow B$ *be the natural projections of* $A \bowtie^f \mathfrak{b} \subseteq A \times B$ *into* A *and* B*, respectively. Then,* $p_{_A}$ *is surjective and* $\mathrm{Ker}(p_{_A}) = \{0\} \times \mathfrak{b}$*.*
Moreover, $p_{_B}(A \bowtie^f \mathfrak{b}) = f(A) + \mathfrak{b}$ *and* $\mathrm{Ker}(p_{_B}) = f^{-1}(\mathfrak{b}) \times \{0\}$*. Hence, the following canonical isomorphisms hold:*

$$\frac{A \bowtie^f \mathfrak{b}}{(\{0\} \times \mathfrak{b})} \cong A \qquad \textit{and} \qquad \frac{A \bowtie^f \mathfrak{b}}{f^{-1}(\mathfrak{b}) \times \{0\}} \cong f(A) + \mathfrak{b}\,.$$

(4) *Let* $\gamma : A \bowtie^f \mathfrak{b} \longrightarrow (f(A) + \mathfrak{b})/\mathfrak{b}$ *be the natural ring homomorphism, defined by* $(a, f(a) + b) \mapsto f(a) + \mathfrak{b}$*. Then* γ *is surjective and* $\mathrm{Ker}(\gamma) = f^{-1}(\mathfrak{b}) \times \mathfrak{b}$*. Thus, we have the following natural isomorphisms*

$$\frac{A \bowtie^f \mathfrak{b}}{f^{-1}(\mathfrak{b}) \times \mathfrak{b}} \cong \frac{f(A) + \mathfrak{b}}{\mathfrak{b}} \cong \frac{A}{f^{-1}(\mathfrak{b})}\,.$$

In particular, when f *is surjective we have*

$$\frac{A \bowtie^f \mathfrak{b}}{f^{-1}(\mathfrak{b}) \times \mathfrak{b}} \cong \frac{B}{\mathfrak{b}}\,.$$

1.10 Example. A particular case of the construction introduced above is the amalgamated duplication of a ring [13]. Let A be a commutative ring with unity, and let E be an A–submodule of the total ring of fractions $\mathrm{Tot}(A)$ of A such that $E \cdot E \subseteq E$. In this case, E is an ideal in the subring $B := (E : E)$ $(:= \{z \in \mathrm{Tot}(A) : zE \subseteq E\})$ of $\mathrm{Tot}(A)$. If $\iota : A \longrightarrow B$ is the natural embedding, then $A \bowtie^{\iota} E$ coincides with $A \bowtie E$, the amalgamated duplication of A along E, as defined in [13]. A particular and relevant case is when $E := I$ is an ideal in A. In this case, we can take $B := A$, we can consider the identity map $id := id_A : A \longrightarrow A$ and we have that $A \bowtie I$, the amalgamated duplication of A along the ideal I (as denoted in [13]), coincides with $A \bowtie^{id} I$, that we will call also *the simple amalgamation of* A *along* I (instead of the amalgamation of A along I, with respect to id_A).

1.11 Example. Let $A \subset B$ be a ring extension and $\boldsymbol{X} := \{X_1, X_2, \ldots, X_n\}$ a finite set of indeterminates over B. In the polynomial ring $B[\boldsymbol{X}]$, we can consider the following subring

$$A + \boldsymbol{X}B[\boldsymbol{X}] := \{h \in B[\boldsymbol{X}] : h(\mathbf{0}) \in A\},$$

where $\mathbf{0}$ is the $n-$tuple whose components are 0. This is a particular case of the general construction introduced above. In fact, if $\sigma' : A \hookrightarrow B[\boldsymbol{X}]$ is the natural embedding and $\mathfrak{I}' := \boldsymbol{X}B[\boldsymbol{X}]$, then $A \bowtie^{\sigma'} \mathfrak{I}'$ is isomorphic to $A + \boldsymbol{X}B[\boldsymbol{X}]$, by Proposition 1.9(3)).

Similarly, the subring $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!] := \{h \in B[\![\boldsymbol{X}]\!] : h(\mathbf{0}) \in A\}$ of the ring of power series $B[\![\boldsymbol{X}]\!]$ is isomorphic to $A \bowtie^{\sigma''} \mathfrak{I}''$, where $\sigma'' : A \hookrightarrow B[\![\boldsymbol{X}]\!]$ is the natural embedding and $\mathfrak{I}'' := \boldsymbol{X}B[\![\boldsymbol{X}]\!]$.

1.12 Example. The following variant of the construction $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$ is presented and studied by S. Hizem and A. Benhissi in [38] and by S. Hizem in [37]. Let A be a ring and $\mathfrak{a}$ be an ideal of A. If X is an indeterminate over A, consider the following subring

$$A + X\mathfrak{a}[\![X]\!] := \{f(X) \in A[\![X]\!] : f(X) - f(0) \in \mathfrak{a}[\![X]\!]\}$$

of the ring of formal power series $A[\![X]\!]$. Hizem and Benhissi study the topological structure of the prime spectrum of $A + X\mathfrak{a}[\![X]\!]$, providing bounds for Krull dimension this ring. This ring can be study as a particular case of the amalgamation, as we will see below.

1.13 Example. Let $A \subseteq B$ be a ring extension, $\mathfrak{b}$ be an ideal of B and $\boldsymbol{X}$ a finite collection of indeterminates over B. Then, consider the following subrings

$$A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}] := \{f(\boldsymbol{X}) \in B[\boldsymbol{X}] : f(\mathbf{0}) \in A \text{ and } f(\boldsymbol{X}) - f(\mathbf{0}) \in \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]\}$$

$$A + \boldsymbol{X}\mathfrak{b}[\![\boldsymbol{X}]\!] := \{f(\boldsymbol{X}) \in B[\![\boldsymbol{X}]\!] : f(\mathbf{0}) \in A \text{ and } f(\boldsymbol{X}) - f(\mathbf{0}) \in \boldsymbol{X}\mathfrak{b}[\![\boldsymbol{X}]\!]\}$$

of $B[\boldsymbol{X}]$ and $B[\![\boldsymbol{X}]\!]$, respectively. It is immediately seen that the present example generalizes both the constructions of Examples 1.11 and 1.12 (in the first case, it is enough to choose $\mathfrak{b} := B$, in the second case $B := A$ and $\mathfrak{b} := \mathfrak{a}$). Moreover, we can see the ring $A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$ (resp., $A + \boldsymbol{X}\mathfrak{b}[\![\boldsymbol{X}]\!]$) as an amalgamation. Indeed, if $\sigma' : A \hookrightarrow B[\boldsymbol{X}]$ (resp., $\sigma'' : A \hookrightarrow B[\![\boldsymbol{X}]\!]$) is the natural embedding, then, by Proposition 1.9(3) we deduce that $A \bowtie^{\sigma'} \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$ (resp., $A \bowtie^{\sigma''} \boldsymbol{X}\mathfrak{b}[\![\boldsymbol{X}]\!]$) is isomorphic to $A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$ (resp., $A + \boldsymbol{X}\mathfrak{b}[\![\boldsymbol{X}]\!]$).

1.14 Example. *The $D+\mathfrak{m}$ construction.*
Let $\mathfrak{m}$ be a maximal ideal of a ring (usually, an integral domain) T and let D be a subring of T such that $\mathfrak{m} \cap D = (0)$. The ring $D+\mathfrak{m} := \{x+m : x \in D, m \in \mathfrak{m}\}$ is canonically isomorphic to $D \bowtie^{\iota} \mathfrak{m}$, where $\iota : D \hookrightarrow T$ is the natural embedding (Proposition 1.9(3)).

More generally, let $\mathcal{S}$ be a subset of $\mathrm{Max}(T)$, such that $\mathfrak{m} \cap D = (0)$ for all $\mathfrak{m} \in \mathcal{S}$, and set $\mathfrak{J} := \bigcap \mathcal{S}$. The ring $D+\mathfrak{J} := \{x+j : x \in D, j \in \mathfrak{J}\}$ is canonically isomorphic to $D \bowtie^{\iota} \mathfrak{J}$. In particular, if $D := K$ is a field contained in T and $\mathfrak{J} := \mathrm{Jac}(T)$, then $K + \mathrm{Jac}(T)$ is canonically isomorphic to $K \bowtie^{\iota} \mathrm{Jac}(T)$, where $\iota : K \hookrightarrow T$ is the natural embedding.

1.15 Example. *The CPI–extensions (in the sense of Boisen-Sheldon [7]).*
Let A be a ring and $\mathfrak{p}$ be a prime ideal of A. Let $\boldsymbol{k}(\mathfrak{p})$ be the residue field of the localization $A_{\mathfrak{p}}$ and denote by $\psi_{\mathfrak{p}}$ (or simply, by ψ) the canonical surjective ring homomorphism $A_{\mathfrak{p}} \longrightarrow \boldsymbol{k}(\mathfrak{p})$. It is wellknown that $\boldsymbol{k}(\mathfrak{p})$ is canonically isomorphic to the quotient field of $A/\mathfrak{p}$, so we can identify $A/\mathfrak{p}$ with its canonical image into $\boldsymbol{k}(\mathfrak{p})$. Then, the subring $\boldsymbol{C}(A,\mathfrak{p}) := \psi^{-1}(A/\mathfrak{p})$ of $A_{\mathfrak{p}}$ is called the *CPI–extension of A with respect to* $\mathfrak{p}$. It is immediately seen that, if we denote by $\lambda_{\mathfrak{p}}$ (or, simply, by λ) the localization homomorphism $A \longrightarrow A_{\mathfrak{p}}$, then $\boldsymbol{C}(A,\mathfrak{p})$ coincides with the ring $\lambda(A) + \mathfrak{p}A_{\mathfrak{p}}$. On the other hand, if $\mathfrak{m} := \mathfrak{p}A_{\mathfrak{p}}$, we can consider $A \bowtie^{\lambda} \mathfrak{m}$ and we have the canonical projection $A \bowtie^{\lambda} \mathfrak{m} \longrightarrow \lambda(A) + \mathfrak{m}$, defined by $(a, \lambda(a)+m) \mapsto \lambda(a)+m$, where $a \in A$ and $m \in \mathfrak{m}$. It follows that $\boldsymbol{C}(A,\mathfrak{p})$ is canonically isomorphic to $(A \bowtie^{\lambda} \mathfrak{m})/(\mathfrak{p} \times \{0\})$ (Proposition 1.9(3)).

More generally, let $\mathfrak{a}$ be an ideal of A and let $S_{\mathfrak{a}}$ be the set of the elements $s \in A$ such that $s+\mathfrak{a}$ is a regular element of $A/\mathfrak{a}$. Obviously, $S_{\mathfrak{a}}$ is a multiplicative subset of A and if $\overline{S_{\mathfrak{a}}}$ is its canonical projection onto $A/\mathfrak{a}$, then $\mathrm{Tot}(A/\mathfrak{a}) = (\overline{S_{\mathfrak{a}}})^{-1}(A/\mathfrak{a})$. Let $\varphi_{\mathfrak{a}} : S^{-1}A \longrightarrow \mathrm{Tot}(A/\mathfrak{a})$ be the canonical surjective ring homomorphism defined by $\varphi_{\mathfrak{a}}(as^{-1}) := (a+I)(s+I)^{-1}$, for all $a \in A$ and $s \in S$. Then, the subring $\boldsymbol{C}(A,\mathfrak{a}) := \varphi_{\mathfrak{a}}^{-1}(A/\mathfrak{a})$ of $S_{\mathfrak{a}}^{-1}A$ is called the *CPI–extension of A with respect to* $\mathfrak{a}$. If $\lambda_{\mathfrak{a}} : A \longrightarrow S_{\mathfrak{a}}^{-1}A$ is the localization homomorphism, then it is easy to see that $\boldsymbol{C}(A,\mathfrak{a})$ coincides with the ring $\lambda_{\mathfrak{a}}(A) + S_{\mathfrak{a}}^{-1}\mathfrak{a}$. It follows by Proposition 1.9(3) that, if we consider the ideal $\mathfrak{J} := S_{\mathfrak{a}}^{-1}\mathfrak{a}$ of $S_{\mathfrak{a}}^{-1}A$, then $\boldsymbol{C}(A,\mathfrak{a})$ is canonically isomorphic to $(A \bowtie^{\lambda_{\mathfrak{a}}} \mathfrak{J})/(\lambda_{\mathfrak{a}}^{-1}(\mathfrak{J}) \times \{0\})$.

1.16 Remark. *Nagata's idealization.*
Let A be a commutative ring and $\mathcal{M}$ a A–module. We recall that, in 1955,

Nagata introduced the ring extension of A called *the idealization of* $\mathcal{M}$ *in* A*,* denoted here by $A \ltimes \mathcal{M}$, as the A–module $A \oplus \mathcal{M}$ endowed with a multiplicative structure defined by:

$$(a, x)(a', x') := (aa', ax' + a'x)\,, \quad \text{for all } a, a' \in A \text{ and } x, x' \in \mathcal{M}$$

(cf. [58], Nagata's book [59, page 2], and Huckaba's book [44, Chapter VI, Section 25]). The idealization $A \ltimes \mathcal{M}$ is a ring, such that the canonical embedding $\iota_A : A \hookrightarrow A \ltimes \mathcal{M}$ (defined by $a \mapsto (a, 0)$, for all $a \in A$) induces a subring $A^{\ltimes}$ $(:= \iota_A(A))$ of $A \ltimes \mathcal{M}$ isomorphic to A and the embedding $\iota_{\mathcal{M}} : \mathcal{M} \hookrightarrow A \ltimes \mathcal{M}$ (defined by $x \mapsto (0, x)$, for all $x \in \mathcal{M}$) determines an ideal $\mathcal{M}^{\ltimes}$ $(:= \iota_{\mathcal{M}}(\mathcal{M}))$ in $A \ltimes \mathcal{M}$ (isomorphic, as an A–module, to $\mathcal{M}$), which is nilpotent of index 2 (i.e. $\mathcal{M}^{\ltimes} \cdot \mathcal{M}^{\ltimes} = 0$).

For the sake of simplicity, we will identify $\mathcal{M}$ with $\mathcal{M}^{\ltimes}$ and A with $A^{\ltimes}$. If $p_A : A \ltimes \mathcal{M} \longrightarrow A$ is the canonical projection (defined by $(a, x) \mapsto a$, for all $a \in A$ and $x \in \mathcal{M}$), then

$$0 \longrightarrow \mathcal{M} \xrightarrow{\iota_{\mathcal{M}}} A \ltimes \mathcal{M} \xrightarrow{p_A} A \longrightarrow 0$$

is a spitting exact sequence of A–modules. (Note that the idealization $A \ltimes \mathcal{M}$ is also called in [27] *the trivial extension of* A *by* $\mathcal{M}$.)

Now we note the Nagata's idealization can be interpreted as a particular case of the general amalgamation construction. Let $B := A \ltimes \mathcal{M}$ and ι $(= \iota_A) : A \hookrightarrow B$ be the canonical embedding. After identifying $\mathcal{M}$ with $\mathcal{M}^{\ltimes}$, $\mathcal{M}$ becomes an ideal of B. It is now straighforward that $A \ltimes \mathcal{M}$ coincides with the amalgamation $A \bowtie^{\iota} \mathcal{M}$.

Although this, the Nagata's idealization and the constructions of the type $A \bowtie^f \mathfrak{b}$ can be very different from an algebraic point of view. In fact, for example, if $\mathcal{M}$ is a nonzero A–module, the ring $A \ltimes \mathcal{M}$ is always non reduced (the element $(0, x)$ is nilpotent, for all $x \in \mathcal{M}$), but the amalgamation $A \bowtie^f \mathfrak{b}$ can be an integral domain (see Example 1.14 and next Proposition 1.25).

Now, we begin the study of the ring $A \bowtie^f \mathfrak{b}$, by explaining its fiber product structure.

1.17 Proposition. *Let* $f : A \longrightarrow B$ *be a ring homomorphism and* $\mathfrak{b}$ *be an ideal of* B*. If* $\pi : B \longrightarrow B/\mathfrak{b}$ *is the canonical projection and* $\breve{f} := \pi \circ f$*, then* $A \bowtie^f \mathfrak{b} = \breve{f} \times_{B/J} \pi$*.*

Proof. It is an immediate consequence of definitions. □

1.18 Remark. Notice that we have many other ways to describe the ring $A \bowtie^f \mathfrak{b}$ as a pullback. In fact, if $C := A \times B/\mathfrak{b}$ and $u : A \longrightarrow C$, $v : A \times B \longrightarrow C$ are the canonical ring homomorphisms defined by $u(a) := (a, f(a) + \mathfrak{b})$, $v((a,b)) := (a, b + \mathfrak{b})$, for every $(a,b) \in A \times B$, it is straightforward to show that $A \bowtie^f \mathfrak{b}$ is canonically isomorphic to $u \times_C v$. On the other hand, if $\mathfrak{a} := f^{-1}(\mathfrak{b})$, $\breve{u} : A/I \longrightarrow A/\mathfrak{a} \times B/\mathfrak{b}$ and $\breve{v} : A \times B \longrightarrow A/\mathfrak{a} \times B/\mathfrak{b}$ are the natural ring homomorphisms induced by u and v, respectively, then $A \bowtie^f \mathfrak{b}$ is also canonically isomorphic to the fiber product of $\breve{u}$ and $\breve{v}$.

The next goal is to show that the rings of the form $A \bowtie^f \mathfrak{b}$, for some ring homomorphism $f : A \longrightarrow B$ and some ideal $\mathfrak{b}$ of B, determine a distinguished subclass of the class of all fiber products.

1.19 Proposition. *Let A, B, C, α, β as in Definition 1.1, and let $f : A \longrightarrow B$ be a ring homomorphism. Then the following conditions are equivalent.*

(i) *There exist an ideal $\mathfrak{b}$ of B such that $A \bowtie^f \mathfrak{b}$ is the fiber product of α and β.*

(ii) *α is the composition $\beta \circ f$.*

If the previous conditions hold, then $\mathfrak{b} = \mathrm{Ker}(\beta)$.

Proof. Assume condition (i) holds, and let a be an element of A. Then $(a, f(a)) \in A \bowtie^f \mathfrak{b}$ and, by assumption, we have $\alpha(a) = \beta(f(a))$. This prove condition (ii).

Conversely, assume that $\alpha = \beta \circ f$. We want to show that the ring $A \bowtie^f \mathrm{Ker}(\beta)$ is the fiber product of α and β. The inclusion $A \bowtie^f \mathrm{Ker}(\beta) \subseteq \alpha \times_C \beta$ is clear. On the other hand, let $(a,b) \in \alpha \times_C \beta$. By assumption, we have $\beta(b) = \alpha(a) = \beta(f(a))$. This shows that $b - f(a) \in \mathrm{Ker}(\beta)$, and thus $(a,b) = (a, f(a) + k)$, for some $k \in \mathrm{Ker}(\beta)$. Then $A \bowtie^f \mathrm{Ker}(\beta) = \alpha \times_C \beta$ and condition (i) is true.

The last statement of the proposition is straightforward. $\square$

In the previous proposition we assume the existence of the ring homomorphism f. The next step is to give a condition for the existence of f. We start by recalling the following definition.

1.20 Definition. ([32, Pag. 111]) *Let $r : B \longrightarrow A$ be a ring homomorphism. We say that r is* ring retraction*, and A is* a retract of B (via r)*, if there exists a ring homomorphism $\iota : A \longrightarrow B$, such that $r \circ \iota = \mathrm{id}_A$.*

1.21 Remark. Let $r : B \longrightarrow A$ be a ring retraction and let ι be as in Definition 1.20. The following properties are easy to be verified.

(1) r is surjective and ι is a ring embedding. Thus, the ring retract A of B can be identified with a subring of B.

(2) We have $B = \iota(A) + \mathrm{Ker}(r)$ and that $\iota^{-1}(\mathrm{Ker}(r)) = \{0\}$. Thus A can be identified with a direct summand of the $A-$module B (with respect to the $A-$module structure given by ι).

1.22 Example. (1) If $r : B \longrightarrow A$ is a ring retraction and $\iota : A \hookrightarrow B$ is a ring embedding such that $r \circ \iota = \mathrm{id}_A$, then B is naturally isomorphic to $A \bowtie^{\iota} \mathrm{Ker}(r)$. Indeed, it is enough to apply Proposition 1.9(3) and Remark 1.21(2).

(2) Let $f : A \longrightarrow B$ be a ring homomorphism and $\mathfrak{b}$ be an ideal of B. Then, A is a retract of $A \bowtie^f \mathfrak{b}$. More precisely, $\pi_{_A} : A \bowtie^f \mathfrak{b} \longrightarrow A$, $(a, f(a), b) \mapsto a$, is a retraction, since the map $\iota : A \longrightarrow A \bowtie^f \mathfrak{b}$, $a \mapsto (a, f(a))$, is a ring embedding such that $\pi_{_A} \circ \iota = \mathrm{id}_A$.

1.23 Proposition. *Let $A, B, C, \alpha, \beta, p_{_A}, p_{_B}$ be as in Definition 1.1. Then, the following conditions are equivalent.*

(i) $p_{_A} : \alpha \times_{_C} \beta \longrightarrow A$ *is a ring retraction.*

(ii) *There exist an ideal $\mathfrak{b}$ of B and a ring homomorphism $f : A \longrightarrow B$ such that $\alpha \times_{_C} \beta = A \bowtie^f \mathfrak{b}$.*

Proof. Set $D := \alpha \times_{_C} \beta$. Assume that condition (i) holds and let $\iota : A \hookrightarrow D$ be a ring embedding such that $p_{_A} \circ \iota = \mathrm{id}_A$. If we consider the ring homomorphism $f := p_{_B} \circ \iota : A \longrightarrow B$, then, by using the definition of a fiber product, we have $\beta \circ f = \beta \circ p_{_B} \circ \iota = \alpha \circ p_{_A} \circ \iota = \alpha \circ \mathrm{id}_A = \alpha$. Then, condition (ii) follows by applying Proposition 1.19. Conversely, let $f : A \longrightarrow B$ be a ring homomorphism such that $D = A \bowtie^f \mathfrak{b}$, for some ideal $\mathfrak{b}$ of B. From Example 1.22(2), the projection of $A \bowtie^f \mathfrak{b}$ onto A is a ring retraction. □

The following easy example shows that, given A, B and $\mathfrak{b}$, the ring $A \bowtie^f \mathfrak{b}$ does not determine uniquely the ring homomorphism f.

1.24 Example. Let $f, g : A \longrightarrow B$ be ring homomorphisms and $\mathfrak{b}$ be an ideal of B.

(1) It is immediately verified that $A \bowtie^f \mathfrak{b} = A \bowtie^g \mathfrak{b}$ if, and only if, $f(a) - g(a) \in \mathfrak{b}$, for each $a \in A$.

(2) Now, let A be a ring and X be an indeterminate over A. Let f be the identity map of $A[X]$ and let $g : A[X] \longrightarrow A[X]$ be the ring homomorphism such that $g(p(X)) = p(X^2)$, for each $p(X) \in A[X]$. If $\mathfrak{b} := XA[X]$, then $p(X) - p(X^2) \in \mathfrak{b}$, for each $p(X) \in A[X]$. Hence, by the previous statement (1), we have that $A[X] \bowtie^f \mathfrak{b} = A[X] \bowtie^g \mathfrak{b}$, and trivially g is not the identity.

Let A, B, f and $\mathfrak{b}$ be as in Proposition 1.9. The subring $B_\diamond := f(A) + \mathfrak{b}$ of B has an important role in the construction $A \bowtie^f \mathfrak{b}$. For instance, if $f^{-1}(\mathfrak{b}) = \{0\}$, we have $A \bowtie^f \mathfrak{b} \cong B_\diamond$ (Proposition 1.9(3)). Moreover, in general, $\mathfrak{b}$ is an ideal also in $B_\diamond$ and, if we denote by $f_\diamond : A \longrightarrow B_\diamond$ the ring homomorphism induced from f, then $A \bowtie^{f_\diamond} \mathfrak{b} = A \bowtie^f \mathfrak{b}$. The next result shows one more aspect of the essential role of the ring $B_\diamond$ for the construction $A \bowtie^f \mathfrak{b}$.

1.25 Proposition. *With the notation of Proposition 1.9, assume $\mathfrak{b} \neq \{0\}$. Then, the following conditions are equivalent.*

(i) *$A \bowtie^f \mathfrak{b}$ is an integral domain.*

(ii) *$f(A) + \mathfrak{b}$ is an integral domain and $f^{-1}(\mathfrak{b}) = \{0\}$.*

In particular, if B is an integral domain and $f^{-1}(\mathfrak{b}) = \{0\}$, then $A \bowtie^f \mathfrak{b}$ is an integral domain.

Proof. (ii)$\longrightarrow$(i) is obvious, since $f^{-1}(\mathfrak{b}) = \{0\}$ implies that $A \bowtie^f \mathfrak{b} \cong f(A) + \mathfrak{b}$ (Proposition 1.9(3)).

Assume that condition (i) holds. If there exists an element $a \in A \setminus \{0\}$ such that $f(a) \in \mathfrak{b}$, then $(a, 0) \in (A \bowtie^f \mathfrak{b}) \setminus \{(0, 0)\}$. Hence, if b is a nonzero element of $\mathfrak{b}$, we have $(a, 0)(0, b) = (0, 0)$, a contradiction. Thus $f^{-1}(\mathfrak{b}) = \{0\}$. In this case, as observed above, $A \bowtie^f \mathfrak{b} \cong f(A) + \mathfrak{b}$ (Proposition 1.9(3)), so $f(A) + \mathfrak{b}$ is an integral domain. $\square$

1.26 Remark. (1) Note that, if $A \bowtie^f \mathfrak{b}$ is an integral domain, then A is also an integral domain, by Proposition 1.9(1).

(2) Let $B = A$, $f = id_A$ and $\mathfrak{b} = \mathfrak{a}$ be an ideal of A. In this situation, $A \bowtie^{id_A} \mathfrak{a}$ (the simple amalgamation of A along $\mathfrak{a}$) coincides with the amalgamated duplication of A along $\mathfrak{a}$ (Example 1.10) and it is never an integral domain, unless $\mathfrak{a} = \{0\}$ and A is an integral domain.

Now, we characterize when the amalgamated algebra $A \bowtie^f \mathfrak{b}$ is a reduced ring.

1.27 PROPOSITION. *We preserve the notation of Proposition 1.9. The following conditions are equivalent.*

(i) $A \bowtie^f \mathfrak{b}$ *is a reduced ring.*

(ii) A *is a reduced ring and* $\mathrm{Nilp}(B) \cap \mathfrak{b} = \{0\}$.

In particular, if A and B are reduced, then $A \bowtie^f \mathfrak{b}$ is reduced; conversely, if J is a radical ideal of B and $A \bowtie^f \mathfrak{b}$ is reduced, then B (and A) is reduced.

Proof. From Proposition 1.2(2, a) we deduce easily that (ii)$\longrightarrow$(i), after noting that, with the notation of Proposition 1.17, we have $\mathrm{Ker}(\pi) = \mathfrak{b}$.

(i)$\longrightarrow$(ii) By Proposition 1.2(1) and the previous equality, it is enough to show that if $A \bowtie^f \mathfrak{b}$ is reduced, then A is reduced. This is trivial because, if $a \in \mathrm{Nilp}(A)$, then $(a, f(a)) \in \mathrm{Nilp}(A \bowtie^f \mathfrak{b})$.

Finally, the first part of the last statement is straightforward. As for the second part, we have $\{0\} = \mathrm{Nilp}(B) \cap \mathfrak{b} = \mathrm{Nilp}(B)$ (since $\mathfrak{b}$ is radical, and so $\mathfrak{b} \supseteq \mathrm{Nilp}(B)$). Hence B is reduced. $\square$

1.28 REMARK. (1) Note that, from the previous result, when $B = A$, $f = id_A$ ($= id$) and $\mathfrak{b} = \mathfrak{a}$ is an ideal of A, we reobtain easily that $A \bowtie \mathfrak{a}$ ($= A \bowtie^{id} \mathfrak{a}$) is a reduced ring if and only if A is a reduced ring [14, Proposition 2.1].

(2) The previous proposition implies that the property of being reduced for $A \bowtie^f \mathfrak{b}$ is independent of the nature of f.

(3) If A and $f(A) + \mathfrak{b}$ are reduced rings, then $A \bowtie^f \mathfrak{b}$ is a reduced ring, by Proposition 1.27. But the converse is not true in general. As a matter of fact, let $A := \mathbb{Z}$, $B := \mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})$, $f : A \longrightarrow B$ be the ring homomorphism such that $f(n) = (n, [n]_4)$, for every $n \in \mathbb{Z}$ (where $[n]_4$ denotes the class of n modulo 4). If we set $\mathfrak{b} := \mathbb{Z} \times \{[0]_4\}$, then $\mathfrak{b} \cap \mathrm{Nilp}(B) = \{0\}$, and thus $A \bowtie^f \mathfrak{b}$ is a reduced ring, but $(0, [2]_4) = (2, [2]_4) + (-2, [0]_4)$ is a nonzero nilpotent element of $f(\mathbb{Z}) + \mathfrak{b}$.

We continue to study of transferring algebraic properties from A and B into $A \bowtie^f \mathfrak{b}$ by giving a characterization of Noetherianity of $A \bowtie^f \mathfrak{b}$.

1.29 PROPOSITION. *With the notation of Proposition 1.9, the following conditions are equivalent.*

(i) $A \bowtie^f \mathfrak{b}$ *is a Noetherian ring.*

(ii) A *and* $f(A) + \mathfrak{b}$ *are Noetherian rings.*

Proof. (ii)$\longrightarrow$(i). Recall that $A \bowtie^f \mathfrak{b}$ is the fiber product of the ring homomorphism $\breve{f} : A \longrightarrow B/\mathfrak{b}$ (defined by $a \mapsto f(a) + \mathfrak{b}$) and of the canonical projection $\pi : B \longrightarrow B/\mathfrak{b}$. Since the projection $p_{_A} : A \bowtie^f \mathfrak{b} \longrightarrow A$ is surjective (Proposition 1.9(3)) and A is a Noetherian ring, by Proposition 1.4, it sufficies to show that $\mathfrak{b}(= \mathrm{Ker}(\pi))$, with the structure of $A \bowtie^f \mathfrak{b}-$module induced by $p_{_B}$, is Noetherian. But this fact is easy, since every $A \bowtie^f \mathfrak{b}-$submodule of $\mathfrak{b}$ is an ideal of the Noetherian ring $f(A) + \mathfrak{b}$.

(i)$\longrightarrow$(ii) is a trivial consequence of Proposition 1.9(3). $\square$

Note that, from the previous result, when $B = A$, $f = id_A$ $(= id)$ and $\mathfrak{a}$ is an ideal of A, we reobtain easily that $A \bowtie \mathfrak{a}$ $(= A \bowtie^{id} \mathfrak{a})$ is a Noetherian ring if and only if A is a Noetherian ring [13, Corollary 2.11].

However, the previous proposition has a moderate interest because the Noetherianity of $A \bowtie^f \mathfrak{b}$ is not directly related to the data (i.e., A, B, f and $\mathfrak{b}$), but to the ring $B_\diamond = f(A) + \mathfrak{b}$ which is canonically isomorphic $A \bowtie^f \mathfrak{b}$, if $f^{-1}(\mathfrak{b}) = \{0\}$ (Proposition 1.9(3)). Therefore, in order to obtain more useful criteria for the Noetherianity of $A \bowtie^f \mathfrak{b}$, we specialize Proposition 1.29 in some relevant cases.

1.30 Proposition. *With the notation of Proposition 1.9, assume that at least one of the following conditions holds:*

(a) $\mathfrak{b}$ *is a finitely generated A–module (with the structure induced by* f*).*

(b) $\mathfrak{b}$ *is a Noetherian A–module (with the structure induced by* f*).*

(c) $f(A) + \mathfrak{b}$ *is Noetherian as A–module (with the structure induced by* f*).*

(d) f *is a finite homomorphism.*

Then $A \bowtie^f \mathfrak{b}$ *is Noetherian if and only if* A *is Noetherian. In particular, if* A *is a Noetherian ring and* B *is a Noetherian A–module (e.g., if* f *is a finite homomorphism [2, Proposition 6.5]), then* $A \bowtie^f \mathfrak{b}$ *is a Noetherian ring for all ideal* $\mathfrak{b}$ *of* B*.*

Proof. Clearly, without any extra assumption, if $A \bowtie^f \mathfrak{b}$ is a Noetherian ring, then A is a Noetherian ring, since it is isomorphic to $A \bowtie^f \mathfrak{b}/(\{0\} \times \mathfrak{b})$ (Proposition 1.9(3)).

Conversely, assume that A is a Noetherian ring. In this case, it is straighforward to verify that conditions (a), (b), and (c) are equivalent [2, Propositions 6.2, 6.3, and 6.5]. Moreover (d) implies (a), since $\mathfrak{b}$ is an A–submodule of B, and B is a Noetherian A–module under condition (d) [2, Proposition 6.5].

Using the previous observations, it is enough to show that $A \bowtie^f \mathfrak{b}$ is Noetherian if A is Noetherian and condition (c) holds. If $f(A) + \mathfrak{b}$ is Noetherian as an A–module, then $f(A) + \mathfrak{b}$ is a Noetherian ring (every ideal of $f(A) + \mathfrak{b}$ is an A–submodule of $f(A) + \mathfrak{b}$). The conclusion follows from Proposition 1.29((ii)$\longrightarrow$(i)).

The last statement is a consequence of the first part and of the fact that, if B is a Noetherian A–module, then (a) holds [2, Proposition 6.2]. □

1.31 PROPOSITION. *We preserve the notation of Propositions 1.9 and 1.17. If B is a Noetherian ring and the ring homomorphism $\breve{f} : A \longrightarrow B/\mathfrak{b}$ is finite, then $A \bowtie^f \mathfrak{b}$ is a Noetherian ring if and only if A is a Noetherian ring.*

Proof. If $A \bowtie^f \mathfrak{b}$ is Noetherian we already know that A is Noetherian. Hence, we only need to show that if A and B are Noetherian rings and $\breve{f}$ is finite then $A \bowtie^f \mathfrak{b}$ is Noetherian. But this fact follows immediately from [20, Proposition 1.8]. □

As a consequence of the previous proposition, we can characterize when rings of the form $A + XB[X]$ and $A + XB[\![X]\!]$ are Noetherian. Note that S. Hizem and A. Benhissi [39] have already given a characterization of the Noetherianity of the power series rings of the form $A + XB[\![X]\!]$. The next corollary provides a simple proof of Hizem and Benhissi's Theorem and shows that a similar characterization holds for the polynomial case (in several indeterminates). At the Fez Conference in June 2008, S. Hizem has announced to have proven a similar result in the polynomial ring case with a totally different approach.

1.32 COROLLARY. *Let $A \subseteq B$ be a ring extension and $\boldsymbol{X} := \{X_1, \ldots, X_n\}$ a finite set of indeterminates over B. Then the following conditions are equivalent.*

(i) *$A + \boldsymbol{X}B[\boldsymbol{X}]$ is a Noetherian ring.*

(ii) *$A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$ is a Noetherian ring.*

(iii) *A is a Noetherian ring and $A \subseteq B$ is a finite ring extension.*

Proof. (iii)$\longrightarrow$(i, ii). With the notations of Example 1.11, recall that $A + \boldsymbol{X}B[\boldsymbol{X}]$ is isomorphic to $A \bowtie^{\sigma'} \boldsymbol{X}B[\boldsymbol{X}]$ (and $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$ is isomorphic to $A \bowtie^{\sigma''} \boldsymbol{X}B[\![\boldsymbol{X}]\!]$). Since we have the following canonical isomorphisms

$$\frac{B[\boldsymbol{X}]}{\boldsymbol{X}B[\boldsymbol{X}]} \cong B \cong \frac{B[\![\boldsymbol{X}]\!]}{\boldsymbol{X}B[\![\boldsymbol{X}]\!]},$$

in the present situation, the homomophism $\breve{\sigma}' : A \hookrightarrow B[\boldsymbol{X}]/\boldsymbol{X}B[\boldsymbol{X}]$ (or, $\breve{\sigma}'' : A \hookrightarrow B[\![\boldsymbol{X}]\!]/\boldsymbol{X}B[\![\boldsymbol{X}]\!]$) is finite. Hence, statements (i) and (ii) follow easily from Proposition 1.74.

(i) (or, (ii)) $\longrightarrow$ (iii). Assume that $A + \boldsymbol{X}B[\boldsymbol{X}]$ (or, $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$) is a Noetherian ring. By Proposition 1.29 we deduce that A is also a Noetherian ring. Moreover, by assumption, the ideal $\mathfrak{I}$ of $A + \boldsymbol{X}B[\boldsymbol{X}]$ (respectively, of $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$) generated by the set $\{bX_k : b \in B, 1 \leq k \leq n\}$ is finitely generated. Hence $\mathfrak{I} = (f_1, f_2, \ldots, f_m)$, for some $f_1, f_2, \ldots, f_m \in \mathfrak{I}$. Let $\{b_{jk} : 1 \leq k \leq n\}$ be the set of coefficients of linear monomials of the polynomial (respectively, power series) f_j, $1 \leq j \leq m$. It is easy to verify that the set $\{b_{jk} : 1 \leq j \leq m, 1 \leq k \leq n\}$ generates B as A–module; thus $A \subseteq B$ is a finite ring extension. $\square$

1.33 Remark. Let $A \subseteq B$ be a ring extension, and let X be an indeterminate over B. Note that the ideal $\mathfrak{I}' = XB[X]$ of $B[X]$ is never finitely generated as an A–module (with the structure induced by the inclusion $\sigma' : A \hookrightarrow B[X]$). As a matter of fact, assume that $\{g_1, g_2, \ldots, g_r\}$ ($\subset B[X]$) is a set of generators of $\mathfrak{I}'$ as $A-$module and set $N := \max\{\deg(g_i) : i = 1, 2, \ldots, r\}$. Clearly, we have $X^{N+1} \in J' \setminus \sum_{i=1}^{r} Ag_i$, which is a contradiction. Therefore, the previous observation shows that the Noetherianity of the ring $A \bowtie^f \mathfrak{b}$ does not imply that $\mathfrak{b}$ is finitely generated as an A–module (with the structure induced by f); for instance $\mathbb{R} + X\mathbb{C}[X]$ ($\cong \mathbb{R} \bowtie^{\sigma'} X\mathbb{C}[X]$, where $\sigma' : \mathbb{R} \hookrightarrow \mathbb{C}[X]$ is the natural embedding) is a Noetherian ring (Proposition 1.32), but $X\mathbb{C}[X]$ is not finitely generated as an $\mathbb{R}$–vector space. This fact shows that condition (a) (or, equivalently, (b) or (c)) of Proposition 1.30 is not necessary for the Noetherianity of $A \bowtie^f \mathfrak{b}$.

Let $A \subseteq B$ be a ring extension, $\mathfrak{b}$ an ideal of B and $\boldsymbol{X} := \{X_1, \ldots, X_r\}$ a finite set of intederminates over B. As we saw in Example 1.13, we can see the ring $A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$ as an amalgamation. Indeed, we have $A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}] \cong$

$A \bowtie^{\sigma'} \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$, where $\sigma' : A \longrightarrow B[\boldsymbol{X}]$ is the natural embedding. Now, we will characterize the Noetherianity of the ring $A+\boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$, without assuming a finiteness condition on the inclusion $A \subseteq B$ (as in Corollary 1.32 (iii)) or on the inclusion $A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}] \subseteq B[\boldsymbol{X}]$.

1.34 THEOREM. *Let $A \subseteq B$ be a ring extension, $\mathfrak{b}$ be an ideal of B and $\boldsymbol{X} := \{X_1, \ldots, X_r\}$ a finite set of intederminates over B. Then, the following conditions are equivalent.*

(i) *$A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$ is a Noetherian ring.*

(ii) *A is a Noetherian ring, $\mathfrak{b}$ is an idempotent ideal of B and it is finitely generated as an A–module.*

Proof. (i)$\longrightarrow$(ii). Assume that $R := A+\boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$ is a Noetherian ring. Then, clearly, A is Noetherian, in view of Proposition 1.9(3). Now, consider the ideal $\mathfrak{L}$ of R generated by the set of linear monomials $\{bX_i : 1 \leq i \leq r,\ b \in \mathfrak{b}\}$. By assumption, we can find $\ell_1, \ell_2, \ldots, \ell_t \in \mathfrak{L}$ such that $\mathfrak{L} = \sum_{k=1}^{t} \ell_k R$. Note that $\ell_k(0, 0, \ldots, 0) = 0$, for all k, $1 \leq k \leq t$. If we denote by b_k the coefficient of the monomial X_1 in the polynomial ℓ_k, then it is easy to see that $\{b_1, b_2, \ldots, b_t\}$ is a set of generators of $\mathfrak{b}$ as an A–module.

The next step is to show that $\mathfrak{b}$ is an idempotent ideal of B. By assumption, $\mathfrak{J}' := \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$ is a finitely generated ideal of R. Let

$$g_h := \sum_{j_1+\ldots+j_r=1}^{m_h} c_{h,j_1\ldots j_r} X_1^{j_1} \cdots X_r^{j_r}, \text{ with } h = 1, 2, \ldots, s,$$

be a finite set of generators of $\mathfrak{J}'$ in R. Set $\overline{j_1} := \max\{j_1 : c_{h,j_1 0\ldots 0} \neq 0, \text{ for } 1 \leq h \leq s\}$. Take now an arbitrary element $b \in \mathfrak{b}$ and consider the monomial $bX_1^{\overline{j_1}+1} \in \mathfrak{J}'$. Clearly, we have

$$bX_1^{\overline{j_1}+1} = \sum_{h=1}^{s} f_h g_h, \text{ with } f_h := \sum_{e_1+\ldots+e_r=0}^{n_h} d_{h,e_1\ldots e_r} X_1^{e_1} \cdots X_r^{e_r} \in R\,.$$

Therefore,

$$b = \sum_{h=1}^{s} \sum_{j_1+e_1=\overline{j_1}+1} c_{h,j_1 0\ldots 0} d_{h,e_1 0\ldots 0}\,.$$

Since $j_1 < \overline{j_1} + 1$, we have necessarily that $e_1 \geq 1$. Henceforth f_h belongs to $\mathfrak{J}'$ and so $d_{h,e_1 0\ldots 0} \in \mathfrak{b}$, for all h, $1 \leq h \leq s$. This proves that $b \in \mathfrak{b}^2$.

(ii)$\longrightarrow$(i). In this situation, by Nakayama's lemma, we easily deduce that $\mathfrak{b} = eB$, for some idempotent element $e \in \mathfrak{b}$. Let $\{b_1, \ldots, b_s\}$ be a set of generators of $\mathfrak{b}$ as an A–module, i.e., $\mathfrak{b} = eB = \sum_{1\leq h\leq s} b_h A$. We consider a new set of indeterminates over B (and A) and precisely $\boldsymbol{Y} := \{Y_{ih} : 1 \leq i \leq r, 1 \leq h \leq s\}$. We can define a map $\varphi : A[\boldsymbol{X}, \boldsymbol{Y}] \longrightarrow B[\boldsymbol{X}]$ by setting $\varphi(X_i) := eX_i$, and $\varphi(Y_{ih}) := b_h X_i$, for all $i = 1, \ldots, r,\ h = 1, \ldots, s$. It is easy to see that φ is a ring homomorphism and $\mathrm{Im}(\varphi) \subseteq R$ $(= A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}])$. Conversely, let

$$f := a+\sum_{i=1}^{r}\left(\sum_{e_{i_1}+\ldots+e_{i_r}=0}^{n_i} c_{i,e_{i_1}\ldots e_{i_r}} X_1^{e_{i_1}}\cdots X_r^{e_{i_r}}\right)X_i \in R \text{ (and so } c_{i,e_{i_1}\ldots e_{i_r}} \in \mathfrak{b}) \,.$$

Since $\mathfrak{b} = \sum_{1\leq h\leq s} b_h A$, then for all $i = 1, \ldots, r$ and $e_{i_1}, \ldots, e_{i_r}$, with $e_{i_1} + \ldots + e_{i_r} \in \{0, \ldots, n_i\}$, we can find elements $a_{i,e_{i_1}\ldots e_{i_r},h} \in A$, with $1 \leq h \leq s$, such that $c_{i,e_{i_1}\ldots e_{i_r}} = \sum_{h=1}^{s} a_{i,e_{i_1}\ldots e_{i_r},h} b_h$. Consider the polynomial

$$g := a + \sum_{i=1}^{r}\sum_{h=1}^{s}\sum_{e_{i_1}+\ldots+e_{i_r}=0}^{n_i} a_{i,e_{i_1}\ldots e_{i_r},h} X_1^{e_{i_1}}\cdots X_r^{e_{i_r}} Y_{ih} \in A[\boldsymbol{X}, \boldsymbol{Y}].$$

It is straightforward to see that $\varphi(g) = f$ and so $\mathrm{Im}(\varphi) = R$. By Hilbert Basis Theorem, we conclude easily that R is Noetherian. $\square$

1.35 Remark. We preserve notation of the previous Theorem 1.34.

(1) We reobtain Corollary 1.32 ((i)$\Leftrightarrow$(iii)), by taking $\mathfrak{b} = B$.

(2) If $B = A$ and $\mathfrak{a}$ is an ideal of A, then we simply have that $A + \boldsymbol{X}\mathfrak{a}[\boldsymbol{X}]$ *is a Noetherian ring if and only if A is a Noetherian ring and $\mathfrak{a}$ is an idempotent ideal of A.* Note the previous two cases were studied as separate cases by S. Hizem, who announced similar results in her talk at the Fez Conference in June 2008, presenting an ample and systematic study of the transfer of various finiteness conditions in the constructions $A + \boldsymbol{X}\mathfrak{a}[\boldsymbol{X}]$ and $A + \boldsymbol{X}B[\boldsymbol{X}]$.

(3) The Noetherianity of B it is not a necessary condition for the Noetherianity of the ring $A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$. For instance, take A any field, B the product of infinitely many copies of A, so that we can consider A as a subring of B, via the diagonal ring embedding $a \mapsto (a, a, \ldots)$, $a \in A$.

Set $\mathfrak{b} := (1, 0, \ldots)B$. Then, $\mathfrak{b}$ is an idempotent ideal of B and, at the same time, a cyclic A-module. Thus, by Theorem 1.34, $A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$ is a Noetherian ring. Obviously, B is not Noetherian.

(4) Note that, if $A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}]$ is Noetherian and B is not Noetherian, then $A \subseteq B$ and $A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}] \subseteq B[\boldsymbol{X}]$ are necessarily not finite. Moreover, it is easy to see that $A + \boldsymbol{X}\mathfrak{b}[\boldsymbol{X}] \subseteq B[\boldsymbol{X}]$ is a finite extension if and only if the canonical homomorphism $A \hookrightarrow B[\boldsymbol{X}]/(\boldsymbol{X}\mathfrak{b}[\boldsymbol{X}])$ is finite. Finally, it can be shown that last condition holds if and only if $\mathfrak{b} = B$ and $A \subseteq B$ is finite.

1.2 $A \bowtie^f \mathfrak{b}$: the prime spectrum

The following proposition, whose proof is a straightforward consequence of Theorem 1.6, Corollaries 1.7 and 1.8, and Proposition 1.9, gives the description of the prime and maximal spectrum of the ring $A \bowtie^f \mathfrak{b}$.

1.36 Proposition. *With the notation of Proposition 1.9, set* $X := \mathrm{Spec}(A)$, $Y := \mathrm{Spec}(B)$, *and* $W := \mathrm{Spec}(A \bowtie^f \mathfrak{b})$, *and* $\mathfrak{b}_0 := \{0\} \times \mathfrak{b} (\subseteq A \bowtie^f \mathfrak{b})$. *For all* $\mathfrak{p} \in X$ *and* $\mathfrak{q} \in Y$, *set:*

$$\mathfrak{p}'^f := \mathfrak{p} \bowtie^f \mathfrak{b} := \{(p, f(p) + b) : p \in \mathfrak{p},\ b \in \mathfrak{b}\}$$

$$\overline{\mathfrak{q}}^f := \{(a, f(a) + b) : a \in A, b \in \mathfrak{b}, f(a) + b \in \mathfrak{q}\}.$$

Then, the following statements hold.

(1) *The map* $\mathfrak{p} \mapsto \mathfrak{p}'^f$ *establishes a closed embedding of* X *into* W, *so its image, which coincides with* $V(\mathfrak{b}_0)$, *is homeomorphic to* X.

(2) *The map* $\mathfrak{q} \mapsto \overline{\mathfrak{q}}^f$ *is a homeomorphism of* $Y \backslash V(\mathfrak{b})$ *onto* $W \backslash V(\mathfrak{b}_0)$.

(3) *The prime ideals of* $A \bowtie^f \mathfrak{b}$ *are of the type* $\mathfrak{p}'^f$ *or* $\overline{\mathfrak{q}}^f$, *for* $\mathfrak{p}$ *varying in* X *and* $\mathfrak{q}$ *in* $Y \backslash V(\mathfrak{b})$.

(4) *Let* $\mathfrak{p} \in \mathrm{Spec}(A)$. *Then,* $\mathfrak{p}'^f$ *is a maximal ideal of* $A \bowtie^f \mathfrak{b}$ *if and only if* $\mathfrak{p}$ *is a maximal ideal of* A.

(5) *Let* $\mathfrak{q}$ *be a prime ideal of* B *not containing* $\mathfrak{b}$. *Then,* $\overline{\mathfrak{q}}^f$ *is a maximal ideal of* $A \bowtie^f \mathfrak{b}$ *if and only if* $\mathfrak{q}$ *is a maximal ideal of* B.
In particular:

$$\mathrm{Max}(A \bowtie^f \mathfrak{b}) = \{\mathfrak{p}'^f : \mathfrak{p} \in \mathrm{Max}(A)\} \cup \{\overline{\mathfrak{q}}^f : \mathfrak{q} \in \mathrm{Max}(B) \backslash V(\mathfrak{b})\}.$$

(6) *The ring* $A \bowtie^f \mathfrak{b}$ *is local if, and only if,* A *is local and* $\mathfrak{b} \subseteq \mathrm{Jac}(B)$*. In this case, if* $\mathfrak{m}$ *is the maximal ideal of* A*, then the maximal ideal of* $A \bowtie^f \mathfrak{b}$ *is* $\mathfrak{m}'^f$*. In particular, if* A *and* B *are local rings and* $\mathfrak{b}$ *is a proper ideal of* B*, then* $A \bowtie^f \mathfrak{b}$ *is a local ring.*

1.37 Remark. We preserve the notation of Proposition 1.9. The prime ideals of $A \bowtie^f \mathfrak{b}$ described in Proposition 1.36 can be also represented as contractions of prime ideals of $A \times B$ under the ring extension $A \bowtie^f \mathfrak{b} \subseteq A \times B$. In particular, for each prime ideal $\mathfrak{q}$ of B (even when $\mathfrak{q} \supseteq \mathfrak{b}$), we can consider the prime ideal

$$(A \times \mathfrak{q}) \cap (A \bowtie^f \mathfrak{b}) = \{(a, f(a) + b) : a \in A,\ b \in \mathfrak{b},\ f(a) + b \in \mathfrak{q}\}$$

which coincides with $\overline{\mathfrak{q}}^f$ (notation of Proposition 1.36) when $\mathfrak{q} \not\supseteq \mathfrak{b}$. If $\mathfrak{q} \supseteq \mathfrak{b}$, then we can consider the prime ideal $\mathcal{P} := f^{-1}(\mathfrak{q})$ of A and it is not difficult to see that, in this case, $(A \times \mathfrak{q}) \cap (A \bowtie^f \mathfrak{b})$ coincides with $\mathcal{P}'^f$).

On the other hand, notice also that, for every prime ideal $\mathfrak{p}$ of A, the prime ideal $\mathfrak{p}'^f$ (i.e., $\pi_A^*(\mathfrak{p})$) of $A \bowtie^f \mathfrak{b}$ coincides also with $(\mathfrak{p} \times B) \cap (A \bowtie^f \mathfrak{b})$.

The next result provides a description of the minimal prime ideals of $A \bowtie^f \mathfrak{b}$.

1.38 Proposition. *With the notations of Proposition 1.36, set*

$$\mathcal{X} := \mathcal{X}_{(f,\mathfrak{b})} := \bigcup_{\mathfrak{q} \in \mathrm{Spec}(B) \setminus V(\mathfrak{b})} V(f^{-1}(\mathfrak{q} + \mathfrak{b})).$$

Then the following properties hold.

(1) *The map defined by* $\mathfrak{q} \mapsto \overline{\mathfrak{q}}^f$ *establishes a homeomorphism of* $\mathrm{Min}(B) \setminus V(\mathfrak{b})$ *with* $\mathrm{Min}(A \bowtie^f \mathfrak{b}) \setminus V(\mathfrak{b}_0)$*.*

(2) *The map defined by* $\mathfrak{p} \mapsto \mathfrak{p}'^f$ *establishes a homeomorphism of* $\mathrm{Min}(A) \setminus \mathcal{X}$ *with* $\mathrm{Min}(A \bowtie^f \mathfrak{b}) \cap V(\mathfrak{b}_0)$*.*

Therefore, we have:

$$\mathrm{Min}(A \bowtie^f \mathfrak{b}) = \{\mathfrak{p}'^f : \mathfrak{p} \in \mathrm{Min}(A) \setminus \mathcal{X}\} \cup \{\overline{\mathfrak{q}}^f : \mathfrak{q} \in \mathrm{Min}(B) \setminus V(\mathfrak{b})\}.$$

Proof. The statement (1) follows easily from the fact that the continuous map $\mathrm{Spec}(B) \setminus V(\mathfrak{b}) \longrightarrow \mathrm{Spec}(A \bowtie^f \mathfrak{b}) \setminus V(\mathfrak{b}_0)$, defined by $\mathfrak{q} \mapsto \overline{\mathfrak{q}}^f$, establishes an isomorphism of partially ordered sets (Proposition 1.36(2) or Theorem 1.6(3)).

By Proposition 1.36(1) or Theorem 1.6(2), note that statement (2) is equivalent to:

(2′) $\mathfrak{p} \in \mathrm{Min}(A)\backslash\mathcal{X}$ if and only if $\mathfrak{p}'^{f} \in \mathrm{Min}(A \bowtie^f \mathfrak{b}) \cap V(\mathfrak{b}_0)$.

Let $\mathfrak{p}$ be a minimal prime ideal of A such that $\mathfrak{p} \notin \mathcal{X}$. If $\mathfrak{p}'^{f}$ is not a minimal prime ideal of $A \bowtie^f \mathfrak{b}$, then $\mathfrak{p}'^{f}$ necessarily contains a prime ideal of the type $\overline{\mathfrak{q}}^{f}$, for some $\mathfrak{q} \in \mathrm{Spec}(B)\backslash V(\mathfrak{b})$ (Proposition 1.36(3)). From the definitions of $\mathfrak{p}'^{f}$ and $\overline{\mathfrak{q}}^{f}$ it is easy to verify that $\mathfrak{p} \supseteq f^{-1}(\mathfrak{q}+\mathfrak{b})$, a contradiction.

Conversely, let $\mathfrak{p}'^{f} \in \mathrm{Min}(A \bowtie^f \mathfrak{b}) \cap V(\mathfrak{b}_0)$. It is easy to see that $\mathfrak{p} \in \mathrm{Min}(A)$. Assume that $\mathfrak{p}$ is in $\mathcal{X}$. Hence, $\mathfrak{p}$ contains an ideal of the type $f^{-1}(\mathfrak{q} + \mathfrak{b})$, for some $\mathfrak{q} \in \mathrm{Spec}(B)\backslash V(\mathfrak{b})$, then we have immediately $\overline{\mathfrak{q}}^{f} \subseteq \mathfrak{p}'^{f}$, and $\overline{\mathfrak{q}}^{f} \neq \mathfrak{p}'^{f}$, since $\overline{\mathfrak{q}}^{f} \notin V(\mathfrak{b}_0)$ and $\mathfrak{p}'^{f} \in V(\mathfrak{b}_0)$. This leads to a contradiction.

The last statement is an easy consequence of (1) and (2) and of Proposition 1.36. □

1.39 Remark. Let $A, B, \mathfrak{b}$ and f be as in Proposition 1.9.

(1) Let $B = A$, $f = id_A$ and $\mathfrak{a}$ be an ideal of A. In this situation, we know that $A \bowtie^{id_A} \mathfrak{a}$ coincides with the amalgamated duplication of A along $\mathfrak{a}$. If we assume that A is an integral domain then, from the previous proposition, we recover that $A \bowtie \mathfrak{a}$ has two minimal prime ideals [13, Corollary 2.5 and Remark 2.8], that is $(A \times (0)) \cap (A \bowtie \mathfrak{a}) = \mathfrak{a} \times (0)$ and $((0) \times A) \cap (A \bowtie \mathfrak{a}) = (0) \times \mathfrak{a}$ since, in this case, $\mathrm{Min}(A)\backslash V(\mathfrak{a}) = \mathrm{Min}(A)\backslash\mathcal{X} = \{(0)\}$.

(2) With the notation of Proposition 1.38, note that $\mathcal{X} \subseteq V(f^{-1}(\mathfrak{b}))$ and that, in general, $\mathcal{X} \neq V(f^{-1}(\mathfrak{b}))$. For instance, let A be a zero-dimensional ring and let f be the identity map id_A of A. If $\mathfrak{b}$ is equal to a prime ideal $\mathfrak{p}$ of A, we have immediately $\mathcal{X} = \emptyset$. But $V(id_A^{-1}(\mathfrak{p})) = \{\mathfrak{p}\}$. Moreover, this shows that the closure of $\mathcal{X}$ is different from $V(f^{-1}(\mathfrak{b}))$, in general.

According to the notation of Proposition 1.36, the prime spectrum W of $A \bowtie^f \mathfrak{b}$ is decomposed into two disjoint set, $V(\mathfrak{b}_0)$ and $W\backslash V(\mathfrak{b}_0)$. In the next result, we describe a different decomposition of $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$.

1.40 Proposition. *We preserve the notation of Propositions 1.9 and 1.36, and set* $\mathfrak{b}_1 := f^{-1}(\mathfrak{b}) \times \{0\}$. *Then* $W = V(\mathfrak{b}_0) \cup V(\mathfrak{b}_1)$, *and the set* $V(\mathfrak{b}_0) \cap V(\mathfrak{b}_1)$ *is homeomorphic to* $\mathrm{Spec}((f(A) + \mathfrak{b})/\mathfrak{b})$, *via the continuous map associated to the natural ring homomorphism* $\gamma : A \bowtie^f \mathfrak{b} \longrightarrow (f(A) + \mathfrak{b})/\mathfrak{b}$,

$(a, f(a)+b) \mapsto f(a)+\mathfrak{b}$. *In particular, we have that the closed subspace* $V(\mathfrak{b}_0) \cap V(\mathfrak{b}_1)$ *of* W *is homeomorphic to the closed subspace* $V(\mathfrak{b})$ *of* $Y(=\mathrm{Spec}(B))$*, when* f *is surjective.*

Proof. We have $\mathfrak{b}_0 \cap \mathfrak{b}_1 = \{0\}$; thus the equality $W = V(\mathfrak{b}_0) \cup V(\mathfrak{b}_1)$ is obvious. Moreover, $V(\mathfrak{b}_0) \cap V(\mathfrak{b}_1) = V(\mathfrak{b}_0 + \mathfrak{b}_1) = V(f^{-1}(\mathfrak{b}) \times \mathfrak{b}) = V(\mathrm{Ker}(\gamma))$, by Proposition 1.9(4). Since γ is surjective, the continuous map canonically associated to γ establishes a homeomorphism of $V(\mathfrak{b}_0) \cap V(\mathfrak{b}_1)$ with $\mathrm{Spec}((f(A)+\mathfrak{b})/\mathfrak{b})$. The last claim is clear, since $\mathrm{Spec}((f(A)+\mathfrak{b})/\mathfrak{b}) \cong V(\mathfrak{b})$, when f is surjective. $\square$

1.41 Example. Let K be an algebraically closed field and X, Y two indeterminates over K. Set $A := K[X,Y]$, $B := K[X]$ and $f : K[X,Y] \longrightarrow K[X]$ defined by $Y \mapsto 0$ and $X \mapsto X$. Let $\mathfrak{b} := XK[X]$. We want to study the ring $K[X,Y] \bowtie^f \mathfrak{b}$, and to investigate whether it is the coordinate ring of some affine variety. (Note that, from a geometrical point of view, f^* determines the inclusion of the line defined by the equation $Y = 0$ into the affine space $\mathbb{A}^2_K$.)
According to the notation of Proposition 1.40, we have $V(\mathfrak{b}_1) \cong \mathrm{Spec}(K[Y])$. Moreover, the projection π_B of $A \bowtie^f \mathfrak{b}$ into B is surjective, since f is surjective, and its kernel is $\mathfrak{b}_1$ (see Proposition 1.9). Thus $\mathrm{Spec}(A \bowtie^f \mathfrak{b}/\mathfrak{b}_1) \cong V\mathfrak{b}_1) \cong \mathrm{Spec}(B) = \mathrm{Spec}(K[X])$. We have also $V(\mathfrak{b}_1) \cap V(\mathfrak{b}_2) \cong \mathrm{Spec}(B/\mathfrak{b}) = \mathrm{Spec}(K)$, by Proposition 1.40. Then, $A \bowtie^f \mathfrak{b}$ is the coordinate ring of the union of a plane (i.e., $\mathrm{Spec}(K[X,Y])$) and a line (i.e., $\mathrm{Spec}(K[X])$) with one common point (i.e., $\mathrm{Spec}(K)$). Note that, in this case, the ring $A \bowtie^f \mathfrak{b}$ can be also presented by a quotient of a polynomial ring. In fact, consider the ring homomorphism $\varphi : K[X,Y,Z] \to A \times B = K[X,Y] \times K[X]$ defined by $X \mapsto (X, X)$, $Y \mapsto (Y, 0)$, and $Z \mapsto (0, X)$. It is easy to check that $\mathrm{Im}(\varphi)$ coincides with $K[X,Y] \bowtie^f J$. In fact, since $\varphi(X), \varphi(Y), \varphi(Z) \in A \bowtie^f \mathfrak{b}$ and $\varphi(K) \subseteq A \bowtie^f \mathfrak{b}$, it follows that $\mathrm{Im}(\varphi) \subseteq A \bowtie^f \mathfrak{b}$. Conversely, take an element $\psi \in A \bowtie^f \mathfrak{b}$. Then ψ is of the form $\psi = (f(X,Y), f(X,0) + Xg(X))$, for some polynomials $f(X,Y) \in K[X,Y], g(X) \in K[X]$. If $h(X,Y,Z) := f(X,Y) + Zg(Z)$, then it follows immediately that $\varphi(h(X,Y,Z)) = \psi$, that is $\mathrm{Im}(\varphi) = A \bowtie^f \mathfrak{b}$. Moreover, $\mathrm{Ker}(\varphi) = (Z^2 - ZX, YZ)$. In fact, the inclusion $\supseteq$ is trivial. For the converse, let $k(X,Y,Z) \in \mathrm{Ker}(\varphi)$. Since the polynomial $Z^2 - XZ$ is monic, with respect to the indeterminate Z, there exist polyno-

mials $q(X,Y,Z) \in K[X,Y,Z], r_0(X,Y), r_1(X,Y) \in K[X,Y]$ such that

$$k(X,Y,Z) = (Z^2 - XZ)q(X,Y,Z) + r_0(X,Y) + r_1(X,Y)Z$$

Since $k(X,Y,Z) \in \mathrm{Ker}(\varphi)$, the previous equality implies $(r_0(X,Y), r_1(X,0)X) = (0,0)$. Obviously, this condition is equivalent to $r_0(X,Y) = 0$ and $r_1(X,Y) \in YK[X,Y]$. It follows immediately that $k(X,Y,Z) \in (Z^2 - ZX, YZ)$. Hence, $K[X,Y] \bowtie^f J$ is canonically isomorphic to $K[X,Y,Z]/(Z^2 - ZX, YZ)$, which is the coordinate ring of the affine variety $V(Z^2 - ZX, YZ) = V(Z) \cup V(Z - X, Y)$ in the affine space $\mathbb{A}^3_K$.

The description of the relation between A and $A \bowtie^f \mathfrak{b}$ in terms of retractions gives a nice topological relation between $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$. Firstly, recall that, if X, Y are topological spaces, a continuous map $r : Y \longrightarrow X$ is called *a topological retraction* if there exists a continuous map $i : X \longrightarrow Y$ such that $r \circ i = id_X$. In this case, we say also that X is *a retract of* Y. The following remark is straightforward.

1.42 Remark. If X and Y are topological spaces and $r : Y \longrightarrow X$ is a topological retraction, then r is surjective and every continuous map $i : X \longrightarrow Y$ such that $r \circ i = id_X$ is a topological embedding.

1.43 Proposition. *We preserve the notation of Propositions 1.9 and 1.36. Then, the continuous map* $\iota^* : \mathrm{Spec}(A \bowtie^f \mathfrak{b}) \longrightarrow \mathrm{Spec}(A)$ *associated to the ring embedding* $\iota : A \longrightarrow A \bowtie^f \mathfrak{b}$ *(Proposition 1.9(1)) is a topological retraction.*

Proof. Preserving the notation of Proposition 1.9, we have $p_A \circ \iota = id_A$ (Example 1.22(1)), and then $\iota^* \circ p_A^* = id_A^* = id_{\mathrm{Spec}(A)}$. The conclusion follows immediately. □

Now the following consequence of Proposition 1.43 is clear.

1.44 Corollary. *With the notation of Propositions 1.9 and 1.36, the canonical embedding* $\iota : A \hookrightarrow A \bowtie^f \mathfrak{b}$ *verifies the lying-over property (or, equivalently, the mapping* $\iota^* : \mathrm{Spec}(A \bowtie^f \mathfrak{b}) \longrightarrow \mathrm{Spec}(A)$ *is surjective).*

The following Proposition provides a description of the fibers of the topological retration $\iota^* : \mathrm{Spec}(A \bowtie^f \mathfrak{b}) \longrightarrow \mathrm{Spec}(A)$.

1.45 Proposition. *With the notation of Propositions 1.9 and 1.36, let* $\iota : A \hookrightarrow A \bowtie^f \mathfrak{b}$ *be the canonical embedding and let* $f^{*_\mathfrak{b}} : \mathrm{Spec}(B) \setminus V(\mathfrak{b}) \longrightarrow$

$\mathrm{Spec}(A)$ *denote the restriction of* $f^* : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ *to the open set* $\mathrm{Spec}(B)\backslash V(\mathfrak{b})$. *Then, we have*

$$\iota^{*-1}(\mathfrak{p}) = \{\overline{\mathfrak{q}}^f : \mathfrak{q} \in f^{*_\mathfrak{b}-1}(\mathfrak{p})\} \cup \{\mathfrak{p}'^f\},$$

for every $\mathfrak{p} \in \mathrm{Spec}(A)$.

In particular, $\iota^{*-1}(\mathfrak{p}) = \{\mathfrak{p}'^f\}$ *if and only if* $f^{*_\mathfrak{b}-1}(\mathfrak{p}) = \emptyset$. *Moreover, the following conditions are equivalent.*

(i) ι^* *has finite fibers.*

(ii) $f^{*_\mathfrak{b}}$ *has finite fibers.*

More precisely, if $\mathfrak{p}$ *is a prime ideal of* A *and the fiber* $f^{*_\mathfrak{b}-1}(\mathfrak{p})$ *has* n *elements, then the fiber* $\iota^{*-1}(\mathfrak{p})$ *has* $n+1$ *elements.*

Proof. Let $\pi_B : A \bowtie^f \mathfrak{b} \longrightarrow B$ be the canonical map defined by $(a, f(a)+b) \mapsto f(a)+b$ for all $a \in A$ and $b \in \mathfrak{b}$. From Proposition 1.36, we have $\iota^*(\mathfrak{p}'^f) = \iota^{-1}(\mathfrak{p}'^f) = \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$. Moreover, it is obvious that $\pi_B \circ \iota = f$. Therefore, in particular, $\iota^*(\overline{\mathfrak{q}}^f) = \iota^{-1}(\pi_B^{-1}(\mathfrak{q})) = f^{-1}(\mathfrak{q}) = f^{*_\mathfrak{b}}(\mathfrak{q})$, for every prime ideal $\mathfrak{q}$ in $\mathrm{Spec}(B)\backslash V(\mathfrak{b})$. Thus, if $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\mathfrak{q} \in f^{*_\mathfrak{b}-1}(\mathfrak{q})$, then $\iota^*(\overline{\mathfrak{q}}^f) = \mathfrak{p}$. Now, the conclusion is a straightforward consequence of Proposition 1.36(3). □

1.46 Example. Let A be a ring and $\mathfrak{a}$ be an ideal of A. Then, we can consider the ring

$$A \bowtie^n \mathfrak{a} := \{(a, a+\alpha_1, \ldots, a+\alpha_n) : a \in A, \alpha_1, \ldots, \alpha_n \in \mathfrak{a}\}.$$

If we consider the ideal $\mathfrak{b} := \mathfrak{a}^n := \mathfrak{a} \times \ldots \times \mathfrak{a}$ ($n-$times) of the ring $B := A^n := A \times \ldots \times A$ ($n-$times), then $A \bowtie^n \mathfrak{a}$ is equal to $A \bowtie^\delta \mathfrak{b}$, where $\delta : A \hookrightarrow B$ is the diagonal embedding. As in Proposition 1.45, let $\delta^{*_\mathfrak{b}} : \mathrm{Spec}(B)\backslash V(\mathfrak{b}) \longrightarrow \mathrm{Spec}(A)$ denote the restriction of $\delta^* : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ to the open set $\mathrm{Spec}(B)\backslash V(\mathfrak{b})$. If $\mathfrak{p}$ is a prime ideal of A not containing $\mathfrak{a} = \delta^{-1}(\mathfrak{b})$, then $\delta^{*_\mathfrak{b}-1}(\mathfrak{p})$ is precisely the set of all prime ideals $\mathfrak{p}(h)$ of B, with $h = 1, 2, \ldots, n$, where $\mathfrak{p}(h) := A^{h-1} \times \mathfrak{p} \times A^{n-h}$ (for $h = 1$, $\mathfrak{p}(1) := \mathfrak{p} \times A^{n-1}$ and, for $h = n$, $\mathfrak{p}(n) := A^{n-1} \times \mathfrak{p}$). Moreover, if $\mathfrak{p} \in V(\mathfrak{a})$, then $\delta^{*_\mathfrak{b}-1}(\mathfrak{p}) = \emptyset$. Thus, by Corollary 1.44 and by Proposition 1.45, the embedding $\iota : A \hookrightarrow A \bowtie^n \mathfrak{a}$ has the lying-over property and ι^* has finite fibers. Precisely, for every prime ideal $\mathfrak{p}$ of A, we have $\iota^*(\mathfrak{p}') = \mathfrak{p}$, where $\mathfrak{p}' := \mathfrak{p}'^\delta$ (and so ι^* is surjective). Moreover, if $\mathfrak{p} \in V(\mathfrak{a})$, then $\mathfrak{p}'$ is the only prime of $A \bowtie^n \mathfrak{a}$ lying-over $\mathfrak{p}$ (Proposition 1.45). Otherwise, if $\mathfrak{p} \notin V(\mathfrak{a})$ and if $\mathfrak{p}_h := \overline{\mathfrak{p}(h)}^{\delta}$ $(h = 1, 2, \ldots, n)$, then the prime ideals of $A \bowtie^f \mathfrak{b}$ lying-over $\mathfrak{p}$ are exactly $\mathfrak{p}', \mathfrak{p}_1, \ldots, \mathfrak{p}_n$, again by Proposition 1.45.

In particular, if $n = 1$, we reobtain the the lying-over property for the ring $A \bowtie \mathfrak{a}$ [12, Proposition 2.5].

The structure of the fibers of the map ι^* is more simple if f is surjective, as we show in the following result.

1.47 Corollary. *With the notation of Propositions 1.9 and 1.36, assume f surjective, set $\mathfrak{a} := f^{-1}(\mathfrak{b})$, and let $\mathfrak{p}$ be a prime ideal of A. Then, the following statements hold.*

(1) *If $\mathfrak{p} \in V(\mathfrak{a}) \cup (\mathrm{Spec}(A) \backslash V(\mathrm{Ker}(f)))$, then ${\iota^*}^{-1}(\mathfrak{p}) = \{\mathfrak{p}'^{f}\}$.*

(2) *If $\mathfrak{p} \in (\mathrm{Spec}(A) \backslash V(\mathfrak{a})) \cap V(\mathrm{Ker}(f))$ and $\mathfrak{q}$ is the only prime ideal of B such that $f^*(\mathfrak{q}) = \mathfrak{p}$ (since, in the present situation, f^* is injective), then ${\iota^*}^{-1}(\mathfrak{p}) = \{\mathfrak{p}'^{f}, \overline{\mathfrak{q}}^{f}\}$*

Proof. Since f is surjective, f^* is injective, and so, a fortiori, $f^{*_\mathfrak{b}}$ has fibers whose cardinality is at most 1. Thus, ι^* has finite fibers, by Proposition 1.45. Moreover, since $f^* : \mathrm{Spec}(B) \longrightarrow V(\mathrm{Ker}(f))$ is an isomorphism of partially ordered sets, in particular, then ${f^{*_\mathfrak{b}}}^{-1}(\mathfrak{p}) = \emptyset$, for every $\mathfrak{p} \in V(\mathfrak{a}) \cup (\mathrm{Spec}(A) \backslash V(\mathrm{Ker}(f)))$. Moreover, if $\mathfrak{p} \in (\mathrm{Spec}(A) \backslash V(\mathfrak{a})) \cap V(\mathrm{Ker}(f))$, and $\mathfrak{q}$ is the unique prime ideal of B such that $f^*(\mathfrak{q}) = \mathfrak{p}$, then ${f^{*_\mathfrak{b}}}^{-1}(\mathfrak{p}) = \{\mathfrak{q}\}$. Now, the conclusion follows immediately by Proposition 1.45. $\square$

After describing the topological and ordering properties of the prime spectrum of the ring $A \bowtie^f \mathfrak{b}$, we want to describe the localizations of $A \bowtie^f \mathfrak{b}$ at each of its prime ideals.

1.48 Remark. (1) Let A be a ring, S be a multiplicative subset of A, and $\mathfrak{a}$ be an ideal of A. Then, the set $T := S + \mathfrak{a}$ is still a multiplicative subset of A, and $0 \notin S$ if and only if $S \cap \mathfrak{a} = \emptyset$.

(2) Let $f : A \longrightarrow B$ be a ring homomorphism, $\mathfrak{b}$ be an ideal of B and $\mathfrak{p}$ be a prime ideal of A. If we set $S := f(A \backslash \mathfrak{p})$ and $T := S + \mathfrak{b}$, then by (1) it follows that $0 \in T$ if and only if $\mathfrak{p} \not\supseteq f^{-1}(\mathfrak{b})$.

1.49 PROPOSITION. *With the notation of Propositions 1.9 and 1.36, the following properties hold.*

(1) *For every prime ideal $\mathfrak{q}$ of B not containing $\mathfrak{b}$, the ring $(A \bowtie^f \mathfrak{b})_{\overline{\mathfrak{q}}^f}$ is canonically isomorphic to $B_{\mathfrak{q}}$.*

(2) *Let $\mathfrak{p}$ be a prime ideal of A. Consider the multiplicative subset $S := S_{(\mathfrak{p},f,\mathfrak{b})} := f(A \backslash \mathfrak{p}) + \mathfrak{b}$ of B and set $B_S := S^{-1}B$ and $\mathfrak{b}_S := S^{-1}\mathfrak{b}$. Let $f_{\mathfrak{p}} : A_{\mathfrak{p}} \longrightarrow B_S$ be the ring homomorphism induced by f. Then, the ring $(A \bowtie^f \mathfrak{b})_{\mathfrak{p}'^f}$ is canonically isomorphic to $A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} \mathfrak{b}_S$. In particular, for every prime ideal $\mathfrak{p}$ of A not containing $f^{-1}(\mathfrak{b})$, $(A \bowtie^f \mathfrak{b})_{\mathfrak{p}'^f}$ is isomorphic to $A_{\mathfrak{p}}$.*

Proof. Let $\breve{f}, \pi, p_A, p_B$ be as in Propositions 1.17 and 1.9, and let $\mathfrak{q}$ be a prime ideal of B not containing $\mathfrak{b} = \mathrm{Ker}(\pi)$). Since $p_B^{-1}(\mathfrak{q}) = \overline{\mathfrak{q}}^f$, then $(A \bowtie^f \mathfrak{b})_{\overline{\mathfrak{q}}^f}$ is isomorphic to $B_{\mathfrak{q}}$, by Theorem 1.6(1). This proves statement (1).

(2). With the notation of Proposition 1.17, the ring homomorphism $f_{\mathfrak{p}} : A_{\mathfrak{p}} \longrightarrow B_S$ naturally induces a ring homomorphism $\breve{f}_{\mathfrak{p}} : A_{\mathfrak{p}} \longrightarrow B_S/\mathfrak{b}_S$. If $\pi_{(\mathfrak{p})} : B_S \longrightarrow B_S/J_S$ is the canonical projection, then, by Proposition 1.17, the ring $A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} \mathfrak{b}_S$ is the fiber product of $\breve{f}_{\mathfrak{p}}$ and $\pi_{(\mathfrak{p})}$. Now, we notice that $\pi_A(A \bowtie^f \mathfrak{b} \backslash \mathfrak{p}'^f) = A \backslash \mathfrak{p}$, and moreover $\pi_B(A \bowtie^f \mathfrak{b} \backslash \mathfrak{p}'^f) = S$. Then, the fact that $(A \bowtie^f \mathfrak{b})_{\mathfrak{p}'^f}$ is canonically isomorphic to $A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} \mathfrak{b}_S$ follows from [20, Proposition 1.9]. Part (2) of Remark 1.48 proves last statement of (2). $\square$

1.3 Computing integral closure of $A \bowtie^f \mathfrak{b}$

We begin this section with the following Lemma.

1.50 LEMMA. *Let $A, B, \mathfrak{b}$ and f be as in Proposition 1.9. Then $\mathfrak{b}_1 := f^{-1}(\mathfrak{b}) \times \mathfrak{b}$ is the conductor of $A \times B$ into $A \bowtie^f \mathfrak{b}$ (i.e., the largest ideal of $A \bowtie^f \mathfrak{b}$ that is also an ideal of $A \times B$).*

Proof. It is obvious that $\mathfrak{b}_1$ is both an ideal of $A \bowtie^f \mathfrak{b}$ and $A \times B$. On the other hand, let $\mathfrak{c}$ be an ideal both of $A \bowtie^f \mathfrak{b}$ and $A \times B$. Since, in particular, $\mathfrak{c} \subseteq A \bowtie^f \mathfrak{b}$, each element of $\mathfrak{c}$ is of the form $(a, f(a)+b)$, for some $a \in A, b \in \mathfrak{b}$. Moreover, $\mathfrak{c}$ is an ideal of $A \times B$ and thus $(0, f(a)+b) = (a, f(a)+b)(0,1) \in \mathfrak{c} \subseteq A \bowtie^f \mathfrak{b}$. It follows, by definition, that $f(a)+b \in \mathfrak{b}$, and so $a \in f^{-1}(\mathfrak{b})$. This shows that $\mathfrak{c} \subseteq f^{-1}(\mathfrak{b}) \times \mathfrak{b}$. The result is now clear. $\square$

Now, we want to determine the integral closure of the ring $A \bowtie^f \mathfrak{b}$ in its total ring of fractions. It is easy to compute $\mathrm{Tot}(A \bowtie^f \mathfrak{b})$ in the following relevant case.

1.51 PROPOSITION. *Let $f : A \longrightarrow B$ be a ring homomorphism, $\mathfrak{b}$ an ideal of B, and let $A \bowtie^f \mathfrak{b}$ be as in Proposition 1.9. Assume that $\mathfrak{b}$ and $f^{-1}(\mathfrak{b})$ are regular ideals of B and A, respectively. Then $\mathrm{Tot}(A \bowtie^f \mathfrak{b})$ is canonically isomorphic to $\mathrm{Tot}(A) \times \mathrm{Tot}(B)$.*

Proof. Since both $f^{-1}(\mathfrak{b})$ and $\mathfrak{b}$ are regular ideals, then $\mathfrak{b}_1$ is a regular ideal of $A \times B$. Now, the conclusion follows immediately by applying [29, pag. 326] and keeping in mind Lemma 1.50. □

1.52 REMARK. Note that, in Proposition 1.51, the assumption that $\mathfrak{b}$ and $f^{-1}(\mathfrak{b})$ are regular ideals is essential. For example, let A be an integral domain with quotient field K, B an overring of A, and let $\mathfrak{b} = \{0\}$. Then, in this situation, $A \bowtie^f \mathfrak{b} \cong A$ (Proposition 1.17), and thus $\mathrm{Tot}(A \bowtie^f \mathfrak{b})$ is isomorphic to K, but $\mathrm{Tot}(A) \times \mathrm{Tot}(B) = K \times K$.

In the previous example, $\mathfrak{b}$ and $f^{-1}(\mathfrak{b})$ are both the zero ideal. Another example, for which $\mathfrak{b}$ is a nonzero regular ideal, is given next. Let A be an integral domain with quotient field K, set $B := A[X]$ and $\mathfrak{b} := (X)$, and let $f : A \hookrightarrow A[X]$ be the natural inclusion. In this case, from Proposition 1.17 we deduce that $A \bowtie^f \mathfrak{b} \cong A + XA[X] = A[X]$, and hence $\mathrm{Tot}(A \bowtie^f \mathfrak{b}) = K(X)$. However, $\mathrm{Tot}(A) \times \mathrm{Tot}(B) = K \times K(X)$. (Note that in this example $f^{-1}(\mathfrak{b}) = A \cap \mathfrak{b} = \{0\}$.)

Another example, for which both $\mathfrak{b}$ and $f^{-1}(\mathfrak{b})$ are nonzero and not regular ideals, is the following. Let K be a field and set $A := K^{(3)}, B := K^{(2)}$, and $J := \{0\} \times K$, where $K^{(n)}$ is the direct product ring $K \times K \times ... \times K$ (n–times). If f is the projection defined by $(a, b, c) \mapsto (a, b)$, it is immediately seen that $A \bowtie^f \mathfrak{b} \cong K^{(4)}$. Then $\mathrm{Tot}(A \bowtie^f \mathfrak{b}) \cong K^{(4)}$, but $\mathrm{Tot}(A) \times \mathrm{Tot}(B) \cong K^{(5)}$.

The next result provides further evidence to the relevant role that the subring $B_\diamond := f(A) + \mathfrak{b}$ of B plays in the construction $A \bowtie^f \mathfrak{b}$.

1.53 LEMMA. *We preserve the notation of Proposition 1.9. Then, the ring $A \times (f(A) + \mathfrak{b})$, subring of $A \times B$, which contains $A \bowtie^f \mathfrak{b}$, is integral over $A \bowtie^f \mathfrak{b}$. More precisely, every element of $A \times (f(A) + \mathfrak{b})$ has degree at most two over $A \bowtie^f \mathfrak{b}$.*

Proof. Let $(\alpha, f(a) + b) \in A \times (f(A) + \mathfrak{b})$ with $\alpha, a \in A$ and $b \in \mathfrak{b}$. It is immediately checked that $(\alpha, f(a)+b)$ is a root of the monic polynomial $g(X) := (X - (\alpha, f(\alpha)))(X - (a, f(a)+b))$. To complete the proof, it is enough to note that

$$g(X) = X^2 - (a+\alpha, f(a+\alpha)+b)X + (\alpha, f(\alpha))(a, f(a)+b) \in A \bowtie^f \mathfrak{b}[X].$$

$\square$

1.54 Proposition. *We preserve the notation of Proposition 1.9. The following statements hold.*

(1) *The integral closure of* $A \bowtie^f \mathfrak{b}$ *in* $\mathrm{Tot}(A) \times \mathrm{Tot}(B)$ *is* $\overline{A} \times \overline{f(A)+\mathfrak{b}}$.

(2) *If* $\mathfrak{b}$ *and* $f^{-1}(\mathfrak{b})$ *are regular ideals, then the integral closure of* $A \bowtie^f \mathfrak{b}$ *is* $\overline{A} \times \overline{f(A)+\mathfrak{b}}$.

Proof. Set $C := \mathrm{Tot}(A) \times \mathrm{Tot}(B)$. Then a straightforward argument shows that

$$\overline{A \bowtie^f \mathfrak{b}}^{C} \subseteq \overline{A} \times \overline{f(A)+\mathfrak{b}}^{\mathrm{Tot}(B)}.$$

Conversely, it is obvious that the ring $\overline{A} \times \overline{f(A)+\mathfrak{b}}^{\mathrm{Tot}(B)}$ is integral over $A \times (f(A)+\mathfrak{b})$ and, moreover, $A \times (f(A)+\mathfrak{b})$ is integral over $A \bowtie^f \mathfrak{b}$, by Lemma 1.53. Now (1) is clear.

To prove (2), note that if $\mathfrak{b}$ and $f^{-1}(\mathfrak{b})$ are regular ideals, then C the total ring of fractions of $A \bowtie^f \mathfrak{b}$, by Proposition 1.51. Now, it sufficies to apply (1). $\square$

Now, we want to investigate when the ring $A \bowtie^f \mathfrak{b}$ is integral over its subring $\Gamma(f) := \{(a, f(a)) : a \in A\}$.

1.55 Lemma. *We preserve the notation of Proposition 1.9. The following conditions are equivalent.*

(i) $f(A)+\mathfrak{b}$ *is integral over* $f(A)$.

(ii) $A \bowtie^f \mathfrak{b}$ *is integral over* $\Gamma(f)$.

In particular, if f *is an integral homomorphism, then* $A \bowtie^f \mathfrak{b}$ *is integral over* $\Gamma(f)$ $(\cong A)$.

Proof. (i) implies (ii). Let $(a, f(a)+b)$ be a nonzero element of $A \bowtie^f \mathfrak{b}$. Thus, by condition (i), there exist a positive integer n and $a_0, a_1, \ldots, a_{n-1} \in A$ such that $(f(a)+b)^n + \sum_{i=0}^{n-1} f(a_i)(f(a)+b)^i = 0$. Therefore, it is easy to verify that $(a, f(a)+j)$ is a root of the monic polynomial

$$[X - (a, f(a))][X^n + \sum_{i=0}^{n-1}(a_i, f(a_i))X^i] \in \Gamma(f)[X].$$

Conversely, consider an element $f(a)+b \in f(A)+\mathfrak{b}$. By condition (ii), $(a, f(a)+b)$ is integral over $\Gamma(f)$, and hence the equation of integral dependence of $(a, f(a)+b)$ over $\Gamma(f)$ gives us the equation of integral dependence of $f(a)+b$ over $f(A)$. The last statement is obvious. □

1.4 Prüfer–like conditions on the ring $A \bowtie^f \mathfrak{b}$

In 1932, H. Prüfer (see [60]) introduced the following class of integral domains, that play a crucial role in multiplicative ideal theory.

1.56 Definition. *Let A be an integral domain and K be its quotient field. We say that A is* a Prüfer domain *if each nonzero finitely generated ideal $\mathfrak{a}$ of A is invertible, that is $\mathfrak{a}(A :_K \mathfrak{a}) = A$.*

Another crucial definition in multiplicative ideal theory is the following.

1.57 Definition. *Let A be a ring and T an indeterminate over A.*

(1) *If $f(T) := \sum_{i=0}^{n} a_i T^i \in A[T]$, then we shall denote by $c_A(f)$, or simply by $c(f)$, the ideal of A generated by $a_0, \ldots, a_n$, and we shall call it* the content of f.

(2) *Let $f(T) \in A[T]$. We say that $f(T)$ is* a Gauss polynomial over A *if, for each polynomial $g(T) \in A[T]$, we have $c(f)c(g) = c(fg)$. Note that the inclusion $c(fg) \subseteq c(f)c(g)$ is always true.*

Through the years, several equivalent conditions for an integral domain to be a Prüfer domain were found. We collect some of them in the following result.

1.58 Theorem. *Let A be an integral domain. Then, the following conditions are equivalent.*

(i) *A is a Prüfer domain.*

(ii) *Every nonzero finitely generated ideal of A is projective.*

(iii) *$A_{\mathfrak{p}}$ is a valuation domain, for each $\mathfrak{p} \in \mathrm{Spec}(A)$.*

(iv) *Every finitely generated ideal $\mathfrak{a}$ of A is locally principal (i.e. $\mathfrak{a}A_{\mathfrak{m}}$ is a principal ideal of $A_{\mathfrak{m}}$, for each $\mathfrak{m} \in \mathrm{Max}(A)$.*

(v) *Every polynomial $f(T) \in A[T]$ is Gauss polynomial over A.*

In 1969, M. Griffin generalized the previous notion to rings with zero divisors, in the following natural way (see [33]).

1.59 Definition. *Let A be a ring and K be its total ring of quotient. We say that A is* a Prüfer ring *if each regular (i.e. with a regular element) and finitely generated ideal $\mathfrak{a}$ of A is invertible, that is $\mathfrak{a}(A :_K \mathfrak{a}) = A$.*

Recall the following characterization of Prüfer rings.

1.60 Theorem. (M. Griffin [33, Theorem 13]) *Let A be a ring. Then the following conditions are equivalent.*

(i) *A is a Prüfer ring.*

(ii) *If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are ideals of A and $\mathfrak{b}$ or $\mathfrak{c}$ is regular, then*

$$\mathfrak{a}(\mathfrak{b} \cap \mathfrak{c}) = \mathfrak{a}\mathfrak{b} \cap \mathfrak{a}\mathfrak{c}.$$

(iii) *If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are ideals of A, one of which is regular, then*

$$\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} \cap \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{c}).$$

In [6], S. Bazzoni and S. Glaz proved that the conditions of 1.58 are not equivalent if we take rings that are not integral domains. Thus, with rings with zero-divisors, we can define the following Prüfer–like classes of rings.

1.61 Definition. *Let A be a ring.*

(P-1) *We say that A is* semihereditary *if every finitely generated ideal of A is projective*

(P-2) *We say that A* has weak global dimension at most 1, *and write* w.gl.dim$(A) \leq 1$, *if $A_{\mathfrak{p}}$ is a valuation domain, for each $\mathfrak{p} \in \mathrm{Spec}(A)$.*

(P-3) *We say that A is* arithmetical *if every finitely generated ideal of A is locally principal.*

(P-4) *We say that A is a Gauss ring if each polynomial in $A[T]$ is a Gauss polynomial over A.*

More precisely, in [6] it is shown, by presenting appropriate examples, that there are the following strict inclusions between the classes of rings introduced above:

$$\{\text{Semihereditary rings}\} \subsetneq \{\text{w.gl.dim} \leq 1\} \subsetneq \{\text{Arithmetical rings}\} \subsetneq$$

$$\subsetneq \{\text{Gauss rings}\} \subsetneq \{\text{Prüfer rings}\}$$

The next goal is to investigate about the Prüfer like conditions in the ring $A \bowtie^f \mathfrak{b}$.
We begin by studying the tranfer of the the Prüfer like conditions from $A \bowtie^f \mathfrak{b}$ to A.

1.62 Proposition. *We preserve the notation of Proposition 1.9. Then, the following statements hold*

(1) *If $A \bowtie^f \mathfrak{b}$ is an Arithmetical ring, then A is an Arithmetical ring.*

(2) *If $A \bowtie^f \mathfrak{b}$ is a Gauss ring, then A is a Gauss ring.*

Proof. By Proposition 1.23, A is a ring retract of $A \bowtie^f \mathfrak{b}$, via the projection $p_{_A} : A \bowtie^f \mathfrak{b} \longrightarrow A$, $((a, f(a) + b) \mapsto a)$. Then, the conclusion follows by applying [5, Theorem 2.1(1) and Theorem 2.5], keeping in mind that $\mathrm{Ker}(p_{_A}) = \{0\} \times \mathfrak{b}$. □

1.63 Definition. *We say that a ring A is* a locally Prüfer ring *if $A_{\mathfrak{m}}$ is a Prüfer ring, for each $\mathfrak{m} \in \mathrm{Max}(A)$.*

1.64 Remark. Let A be a ring.

(a) By [54, Proposition 2.10], if A is a locally Prüfer ring, then A is a Prüfer ring.

(b) If A is Gauss ring, then so is $A_{\mathfrak{m}}$, for each maximal ideal $\mathfrak{m}$ of A, by [3, Theorem 2.5(1)]. It follows that A is a locally Prüfer ring.

Thus, we have the following inclusions of classes of rings

$$\{\text{Semihereditary rings}\} \subsetneq \{\text{w.gl.dim} \leq 1\} \subsetneq \{\text{Arithmetical rings}\} \subsetneq$$

$$\subsetneq \{\text{Gauss rings}\} \subsetneq \{\text{Locally Prüfer rings}\} \subsetneq \{\text{Prüfer rings}\}$$

1.65 Proposition. *We preserve the notation of Proposition 1.9 and set* $S_{\mathfrak{p}} := S_{(\mathfrak{p},f,\mathfrak{b})} := f(A \setminus \mathfrak{p}) + \mathfrak{b}$, *for each prime ideal* $\mathfrak{p} \in \mathrm{Spec}(A)$. *Assume that the ideal* $\mathfrak{b}_{S_{\mathfrak{p}}} = \{0\}$, *for each* $\mathfrak{m} \in \mathrm{Max}(A) \cap V(f^{-1}(\mathfrak{b}))$. *Then, the following statements hold.*

(1) *If* A *is a locally Prüfer ring and* $B_{\mathfrak{n}}$ *is a Prüfer ring, for each* $\mathfrak{n} \in \mathrm{Max}(B) \setminus V(\mathfrak{b})$, *then* $A \bowtie^f \mathfrak{b}$ *is a locally Prüfer ring.*

(2) *If* A *is a Gauss ring and* $B_{\mathfrak{n}}$ *is a Gauss ring, for each* $\mathfrak{n} \in \mathrm{Max}(B)\setminus V(\mathfrak{b})$, *then* $A \bowtie^f \mathfrak{b}$ *is a Gauss ring.*

(3) *If* A *has weak global dimension at most 1 and* $B_{\mathfrak{n}}$ *is a valuation domain, for each* $\mathfrak{n} \in \mathrm{Max}(B)\setminus V(\mathfrak{b})$, *then* $A \bowtie^f \mathfrak{b}$ *has weak global dimension at most 1.*

Proof. By Proposition 1.36(5), we have

$$\mathrm{Max}(A \bowtie^f \mathfrak{b}) = \{\mathfrak{m}'_f : \mathfrak{m} \in \mathrm{Max}(A)\} \cup \{\overline{\mathfrak{n}}^f : \mathfrak{n} \in \mathrm{Max}(B)\setminus V(\mathfrak{b})\}.$$

Keeping in mind the assumptions and Proposition 1.49, we have $(A \bowtie^f \mathfrak{b})_{\overline{\mathfrak{n}}^f} \cong B_{\mathfrak{n}}$, for each $\mathfrak{n} \in \mathrm{Max}(B) \setminus V(\mathfrak{b})$, and $(A \bowtie^f \mathfrak{b})_{\mathfrak{m}'_f} \cong A_{\mathfrak{m}}$, for each $\mathfrak{m} \in \mathrm{Max}(A)$. Then (1) and (3) follow by definition. (2) follows by noting that the property of being Gauss, for a ring, is local (see [3, Theorem 2.5 (1)]). $\square$

1.66 Remark. We preserve the notation of Proposition 1.9. If $\mathfrak{b} \subseteq f(A)$, then $A \bowtie^f \mathfrak{b}$ is an integral domain if and only if A is an integral domain and $\mathfrak{b} = \{0\}$. As a matter of fact, assume $A \bowtie^f \mathfrak{b}$ is an integral domain. Then, of course, A is an integral domain, being it isomorphic to a subring of $A \bowtie^f \mathfrak{b}$, via the ring embedding $\iota : A \longrightarrow A \bowtie^f \mathfrak{b}$. If $\mathfrak{b} \neq \{0\}$, take elements $b \in \mathfrak{b}\setminus\{0\}$ and $a \in f^{-1}(\mathfrak{b})$ such that $f(a) = b$. Then $(a, 0)$ is a nonzero element of $A \bowtie^f \mathfrak{b}$ and $(a, 0)(0, b) = (0, 0)$. Thus $(0, b)$ is a nonzero zerodivisor of $A \bowtie^f \mathfrak{b}$, contradiction. The other implication is trivial, by Proposition 1.9(3).

1.67 Proposition. *We preserve the notation of Proposition 1.9 and assume that the map* $f_{\mathfrak{m}} : A_{\mathfrak{m}} \longrightarrow B_{S_{\mathfrak{m}}}$ *is surjective, for each* $\mathfrak{m} \in \mathrm{Max}(A) \cap V(f^{-1}(\mathfrak{b}))$*. Then, the following conditions are equivalent.*

(i) $A \bowtie^f \mathfrak{b}$ *has weak global dimension at most 1.*

(ii) A *has weak global dimension at most 1,* $B_{\mathfrak{n}}$ *is a valuation domain, for each* $\mathfrak{n} \in \mathrm{Max}(B) \backslash V(\mathfrak{b})$ *and* $\mathfrak{b}_{S_{\mathfrak{m}}} = \{0\}$*, for each* $\mathfrak{m} \in \mathrm{Max}(A) \cap V(f^{-1}(\mathfrak{b}))$*.*

Proof. (ii)$\Longrightarrow$(i). It is the statement of Proposition 1.65(3) and does not require the extra assumption on the maps $f_{\mathfrak{m}}$ induced by f.

(i)$\Longrightarrow$(ii). By Proposition 1.49(1), we have $(A \bowtie^f \mathfrak{b})_{\overline{\mathfrak{n}}^f} \cong B_{\mathfrak{n}}$, for any maximal ideal $\mathfrak{n}$ of B not containing $\mathfrak{b}$. Then, it follows, by definition, that $B_{\mathfrak{n}}$ is a valuation domain for each $\mathfrak{n} \in \mathrm{Max}(B) \backslash V(\mathfrak{b})$. Now, let $\mathfrak{m}$ be a maximal ideal of A containing $f^{-1}(\mathfrak{b})$. By Proposition 1.49(2), the localization $(A \bowtie^f \mathfrak{b})_{\mathfrak{m}'^f}$ of $A \bowtie^f \mathfrak{b}$ is isomorphic to $A_{\mathfrak{m}} \bowtie^{f_{\mathfrak{m}}} \mathfrak{b}_{S_{\mathfrak{m}}}$ and it is a valuation domain, by assumption. Keeping in mind that $f_{\mathfrak{m}}$ is surjective, Remark 1.66 implies $\mathfrak{b}_{S_{\mathfrak{m}}} = \{0\}$. It follows immediately, again by Proposition 1.9(3), that $(A \bowtie^f \mathfrak{b})_{\mathfrak{m}'^f}$ is isomorphic to $A_{\mathfrak{m}}$. Thus A has weak global dimension at most 1 (by Proposition 1.49(2)). The proof is now complete. $\square$

1.68 Lemma. *We preserve the notation of Proposition 1.9. If the set of all the ideals of* $A \bowtie^f \mathfrak{b}$ *is totally ordered by inclusion, then* $f(f^{-1}(\mathfrak{b})) = \{0\}$*.*

Proof. Let $a \in f^{-1}(\mathfrak{b})$. Then $(a, 0), (0, f(a)) \in A \bowtie^f \mathfrak{b}$ and, by assumption, the ideals $((a, 0)), ((0, f(a))$ of $A \bowtie^f \mathfrak{b}$ are comparable. If $((a, 0)) \subseteq ((0, f(a)))$, there are $\alpha \in A, \beta \in \mathfrak{b}$ such that

$$(a, 0) = (\alpha, f(\alpha) + \beta)(0, f(a)),$$

then $a = 0$ and, a fortiori, $f(a) = 0$.

If $((0, f(a))) \subseteq ((a, 0))$, then there are elements $\alpha' \in A, \beta' \in \mathfrak{b}$ such that

$$(0, f(a)) = (\alpha', f(\alpha') + \beta')(a, 0),$$

and then $f(a) = 0$. This proves completely the statement. $\square$

1.69 Proposition. *We preserve the notation of Proposition 1.9 and assume that, for each* $\mathfrak{m} \in \mathrm{Max}(A) \cap V(f^{-1}(\mathfrak{b}))$*, the map* $f_{\mathfrak{m}} : A_{\mathfrak{m}} \longrightarrow B_{S_{\mathfrak{m}}}$ *is surjective. Then, the following conditions are equivalent.*

(i) $A \bowtie^f \mathfrak{b}$ *is an arithmetical ring.*

(ii) *A is an arithmetical ring,* $\mathfrak{b}_{S_\mathfrak{m}} = \{0\}$*, for each* $\mathfrak{m} \in \mathrm{Max}(A) \cap V(f^{-1}(\mathfrak{b}))$*, and, for any* $\mathfrak{n} \in \mathrm{Max}(B) \backslash V(\mathfrak{b})$ *the set of all the ideals of* $B_\mathfrak{n}$ *is totally ordered by inclusion.*

Proof. (i)$\Longrightarrow$(ii). By [46, Theorem 1], for each localization of $A \bowtie^f \mathfrak{b}$ at the maximal ideals, the set of all the ideals of such a localization is totally ordered by inclusion. Thus, by Proposition 1.49(2), Lemma 1.68 and surjectivity of the maps $f_\mathfrak{m}$, it follows $\mathfrak{b}_{S_\mathfrak{m}} = \{0\}$, for each $\mathfrak{m} \in \mathrm{Max}(A) \cap V(f^{-1}(\mathfrak{b}))$. Keeping in mind also Proposition 1.9(3), it follows $(A \bowtie^f \mathfrak{b})_{\mathfrak{m}'^f} = A_\mathfrak{m}$, for each $\mathfrak{m} \in \mathrm{Max}(A)$, and thus A is an arithmetical rings. Since $A \bowtie^f \mathfrak{b}$ is arithmetical, [46, Theorem 1] and Proposition 1.49(1) imply that in each localization $B_\mathfrak{n}$ ($\mathfrak{n} \in \mathrm{Max}(B) \backslash V(\mathfrak{b})$) the set of the ideals is totally ordered by inclusion.

(ii)$\Longrightarrow$(i). As before, apply [46, Theorem 1], Proposition 1.9(3) and the local structure of $A \bowtie^f \mathfrak{b}$ (Proposition 1.49). □

Now, recall the following standard facts.

1.70 Definition. *Let A be a ring and M be an A−module*

(1) *We say that M is* finitely presented *if there exist positive integers r, s and an exact sequence of A−modules of the type*

$$A^r \longrightarrow A^s \longrightarrow M \longrightarrow 0$$

(2) *We say that M is* a coherent module *if it is finitely generated and every finitely generated submodule of M is finitely presented.*

(3) *We say that A is* a coherent ring *if it is a coherent module over itself.*

To give conditions to make $A \bowtie^f \mathfrak{b}$ a semihereditary rings, we want to use the following characterization.

1.71 Theorem. ([32, Corollary 4.2.19]) *Let A be a ring. Then, A is semi–hereditary if and only if A is coherent and the weak global dimension of A is at most 1.*

1.72 Example. (1) Every Noetherian ring is a coherent ring.

(2) By Theorems 1.71 and 1.58, every Prüfer domain is a coherent ring.

(3) If A is a Noetherian ring and $\mathbf{T}$ is a (possibly infinite) collection of indeterminates over A, then $A[\mathbf{T}]$ is a coherent ring (see [9, Chapter 1, Exercise 12(f)]).

Let $\phi : A \longrightarrow B$ be a ring homomorphism and let M be a $B-$module. We shall denote by $\cdot_\phi$ the scalar multiplication making M an $A-$module (that is $a \cdot_\phi m := \phi(a)m$, for each $a \in A, m \in M$).

The following facts about coherent modules will be useful.

1.73 THEOREM. *The following statements hold.*

(1) ([9, Pag. 43, Exercise 11(a)]) *Let A be a ring and*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of $A-$modules. If M' and M'' are coherent, then M is coherent.

(2) (M. Harris [35, Corollary 1.1]) *Let $\phi : A \longrightarrow B$ be a finite ring homomorphism and M be a $B-$module. If M is a coherent $A-$module, with the $A-$module structure induced by ϕ, then M is a coherent $B-$module.*

(3) ([32, Theorem 4.1.5]) *If $r : B \longrightarrow A$ is a ring retraction and B is a coherent ring, then A is a coherent ring.*

1.74 LEMMA. *We preserve the notation of Proposition 1.9. If $\mathfrak{b}$ is a finitely generated $A-$module (with the $A-$module structure induced by f), then the ring embedding $\iota : A \longrightarrow A \bowtie^f \mathfrak{b}$ is finite.*

Proof. Let $\{b_1, \ldots, b_n\} \subseteq \mathfrak{b}$ a finite set of generators of the $A-$module $\mathfrak{b}$, and fix an element $(a, f(a)+b) \in A \bowtie^f \mathfrak{b}$. Then, there exist elements $a_1, \ldots, a_n \in A$ such that $b = \sum_{i=1}^n a_i \cdot_f b_i = \sum_{i=1}^n f(a_i)b_i$. It follows immediately that

$$(a, f(a) + b) = a \cdot_\iota (1, 1) + \sum_{i=1}^{n} a_i \cdot_\iota (0, b_i).$$

This proves that $\{(1, 1), (0, b_1), \ldots, (0, b_n)\} \subseteq A \bowtie^f \mathfrak{b}$ is a finite set of generators of $A \bowtie^f \mathfrak{b}$ as an $A-$module (with the structure induced by ι), i.e. ι is finite. $\square$

1.75 Proposition. *We preserve the notation of Proposition 1.9. Then, the following statements hold.*

(1) *If $A \bowtie^f \mathfrak{b}$ is a coherent ring, then A is coherent.*

(2) *If A is a coherent ring and $\mathfrak{b}$ is a coherent $A-$module (with the structure induced by f), then $A \bowtie^f \mathfrak{b}$ is a coherent ring.*

Proof. Statement (1) follows by Example 1.21(2) and Theorem 1.73(3).

(2). We begin by noticed that, since $\mathfrak{b}$ is, in particular, a finitely generated $A-$module, the ring embedding ι is finite, by Lemma 1.74. Let $\pi_{_A} : A \bowtie^f \mathfrak{b} \longrightarrow A$, $\pi_{_B} : A \bowtie^f \mathfrak{b} \longrightarrow B$ be the projections. Then, $\pi_{_A}$ (resp. $\pi_{_B}$) induces on A (resp. $\mathfrak{b}$) a structure of $A \bowtie^f \mathfrak{b}-$module. With these structures, we have the following short exact sequence

$$0 \longrightarrow \mathfrak{b} \xrightarrow{i} A \bowtie^f \mathfrak{b} \xrightarrow{\pi_A} A \longrightarrow 0,$$

of $A \bowtie^f \mathfrak{b}-$modules, where $i : \mathfrak{b} \longrightarrow A \bowtie^f \mathfrak{b}$ is defined by $\beta \mapsto (0, \beta)$, for each $\beta \in \mathfrak{b}$. Let $\iota : A \hookrightarrow A \bowtie^f \mathfrak{b}$ be the ring enbedding such that $a \mapsto (a, f(a))$, for each $a \in A$. On the $A \bowtie^f \mathfrak{b}-$module $\mathfrak{b}$, the map ι induces the following scalar multiplication

$$a \cdot_\iota \beta := (a, f(a)) \cdot_{\pi_B} \beta = \pi_{_B}((a, f(a)))\beta = f(a)\beta \qquad (a \in A, \beta \in \mathfrak{b})$$

It follows that the structure of $A-$module given to $\mathfrak{b}$ by ι is the same structure induced on $A \bowtie^f \mathfrak{b}$ by f. By using the fact that $\mathfrak{b}$ is a coherent $A-$module and applying Theorem 1.73(2), it follows that $\mathfrak{b}$ is a coherent $A \bowtie^f \mathfrak{b}-$module. Moreover, ι induces to the $A \bowtie^f \mathfrak{b}-$module A the following scalar multiplication

$$a \cdot_\iota \alpha := (a, f(a)) \cdot_{\pi_A} \alpha = \pi_{_A}((a, f(a)))\alpha = a\alpha \qquad (a, \alpha \in A)$$

Thus ι induces on A its natural structure of module over itself. Since A, by assumption, is a coherent ring, it follows that it is a coherent $A \bowtie^f \mathfrak{b}-$module, again by Theorem 1.73(2). Then $A \bowtie^f \mathfrak{b}$ is a coherent $A \bowtie^f \mathfrak{b}-$module, by Theorem 1.73(1), that is, $A \bowtie^f \mathfrak{b}$ is a coherent ring. $\square$

1.76 Corollary. *We preserve the notation of Propositions 1.9 and 1.65, and assume that $\mathfrak{b}_{S_\mathfrak{m}} = \{0\}$, for each maximal ideal $\mathfrak{m}$ of A containing $f^{-1}(\mathfrak{b})$. If A is a semi–hereditary ring (resp. semi–hereditary and Noetherian ring),*

$B_{\mathfrak{n}}$ *is a valuation domain, for each* $\mathfrak{n} \in \mathrm{Max}(B)\backslash V(\mathfrak{b})$ *and* $\mathfrak{b}$ *is a coherent* $A-$*module (resp. finitely generated* $A-$*module), with the structure induced by* f*, then* $A \bowtie^f \mathfrak{b}$ *is a semi–hereditary ring.*

Proof. Apply Theorem 1.71 and Proposition 1.75, keeping in mind that, if A is a Noetherian ring, an $A-$module is coherent if and only if it is finitely generated. □

Recall that an integral domain A is *almost Dedekind* if $A_{\mathfrak{m}}$ is a DVR for each maximal ideal $\mathfrak{m}$ of A. Thus, in particular, an almost Dedekind domain is a Prüfer domain.

1.77 Example. Let A be a non Noetherian almost Dedekind domain having at least two distinct principal maximal ideals $\mathfrak{m} := (m), \mathfrak{n} := (n)$, let $B := A/(\mathfrak{m}\cap\mathfrak{n})$, let $f : A \longrightarrow B$ be the canonical projection and let $\mathfrak{b} := \mathfrak{m}/(\mathfrak{m}\cap\mathfrak{n})$. Trivially, $f^{-1}(\mathfrak{b}) = \mathfrak{m}$ and, since $f(n) \in S_{\mathfrak{m}}$, it follows that $\mathfrak{b}_{S_{\mathfrak{m}}} = \{0\}$. Let $\overline{\mathfrak{n}} := \mathfrak{n}/(\mathfrak{m}\cap\mathfrak{n})$ be the unique maximal ideal of B not containing $\mathfrak{b}$. Obviously, the localization $B_{\overline{\mathfrak{n}}}$ is isomorphic to the field $A/\mathfrak{n}$. Moreover, the natural map $p : A \longrightarrow \mathfrak{b}$, $a \mapsto f(am)$ is clearly $A-$linear, surjective and $\mathrm{Ker}(p) = \mathfrak{n}$. This shows that $\mathfrak{b}$ is finitely presented as an $A-$module. Then, keeping in mind that A is a coherent ring, being it a Prüfer domain, and applying [9, Exercise 12 (a)(β)], it follows that $\mathfrak{b}$ is a coherent $A-$module. Then $A \bowtie^f \mathfrak{b}$ is a semihereditary ring, by Corollary 1.76.

1.78 Example. Preserve the notation of Proposition 1.9. The fact that $A \bowtie^f \mathfrak{b}$ is semi–hereditary does not imply, in general, the conditions $\mathfrak{b}$ coherent as an $A-$module and $\mathfrak{b}_{S_{\mathfrak{m}}} = \{0\}$, for each $\mathfrak{m} \in \mathrm{Max}(A) \cap V(f^{-1}(\mathfrak{b}))$. For example, let T be an indeterminate over $\mathbb{Q}$, and let $A := \mathbb{Z}$, $B := \mathbb{Q}[T]$, $\mathfrak{b} := T\mathbb{Q}[T]$, $f : A \longrightarrow B$ the inclusion. Then $A \bowtie^f \mathfrak{b}$ is isomorphic to the ring $\mathbb{Z} + T\mathbb{Q}[T]$, by Example 1.11. Moreover, by [41, Theorem 1.3], it follows easily that $A \bowtie^f \mathfrak{b}$ is a Prüfer domain (i.e. a semi–hereditary domain). But, clearly, $\mathfrak{b}$ is not finitely generated as an $A-$module and $\mathfrak{b}_{S_{\mathfrak{m}}} \neq \{0\}$, for each $\mathfrak{m} \in \mathrm{Max}(A)$.

1.79 Corollary. *We preserve the notation of Proposition 1.9 and assume that* $\mathfrak{b}$ *is a coherent* $A-$*module and that* $f_{\mathfrak{m}}$ *is a surjective ring homomorphism, for each* $\mathfrak{m} \in \mathrm{Max}(A) \cap V(f^{-1}(\mathfrak{b}))$*. Then, the following conditions are equivalent.*

(i) $A \bowtie^f \mathfrak{b}$ *is a semi–hereditary ring.*

(ii) *A is a semi–hereditary ring, $B_{\mathfrak{n}}$ is a valuation domain, for each* $\mathfrak{n} \in \mathrm{Max}(B) \backslash V(\mathfrak{b})$ *and* $\mathfrak{b}_{S_{\mathfrak{m}}} = \{0\}$*, for each* $\mathfrak{m} \in \mathrm{Max}(A) \cap V(f^{-1}(\mathfrak{b}))$.

Proof. (ii)$\Longrightarrow$(i). It is the statement of Corollary 1.76.

(i)$\Longrightarrow$(ii). By Proposition 1.75(1), A is a coherent ring. Then, it sufficies to apply Theorem 1.71 and Proposition 1.67 to complete the proof. □

We recall the following key result.

1.80 Theorem. *Let A be a ring, T be an indeterminate over A and $f(T) \in A[T]$. Then the following statements hold.*

(1) (H. Tsang, [61]) *If $c(f)$ is locally principal, then f is a Gauss polynomial. In particular, f is a Gauss polynomial provided that its content is an invertible ideal of A.*

(2) (T. Lucas, [53]) *If $c(f)$ is a regular ideal of A and f is a Gauss polynomial, then $c(f)$ is invertible.*

1.81 Lemma. *Let $r : B \longrightarrow A$ be a ring retraction, and T be an indeterminate over B. If $\sum_{i=0}^{n} b_i T^i$ is a Gauss polynomial over B, then $\sum_{i=0}^{n} r(b_i) T^i$ is a Gauss polynomial over A.*

Proof. It follows by the proof of [5, Theorem 2.1(1)]. □

1.82 Proposition. *We preserve the notation of Proposition 1.9. If $A \bowtie^f \mathfrak{b}$ is a Prüfer ring and $f(\mathrm{Reg}(A)) \subseteq \mathrm{Reg}(B)$, then A is a Prüfer ring.*

Proof. Let $\mathfrak{a} := (a_0, \ldots, a_n)$ be a regular and finitely generated ideal of A, and consider the polynomial $p(X) := \sum_{i=0}^{n} a_i T^i \in A[T]$. Pick a regular element $a \in \mathfrak{a}$. Then, keeping in mind that $f(\mathrm{Reg}(A)) \subseteq \mathrm{Reg}(B)$, it is easily checked that $(a, f(a))$ is a regular element of the finitely generated ideal $\mathfrak{a}^{\bowtie} := ((a_0, f(a_0)), \ldots, (a_n, f(a_n))$ of $A \bowtie^f \mathfrak{b}$. Since $A \bowtie^f \mathfrak{b}$ is a Prüfer ring, it follows that $\mathfrak{a}^{\bowtie}$ is an invertible ideal of $A \bowtie^f \mathfrak{b}$, and thus the polynomial $p_{\bowtie}(T) := \sum_{i=0}^{n} (a_i, f(a_i)) T^i \in A \bowtie^f \mathfrak{b}[T]$, whose content is clearly $\mathfrak{a}^{\bowtie}$, is a Gauss polynomial over $A \bowtie^f \mathfrak{b}$, by Theorem 1.80(1). Let $\pi_{_A} : A \bowtie^f \mathfrak{b} \longrightarrow A$ be the projection such that $(a, f(a)+b) \mapsto a$. Then $p(T) = \sum_{i=0}^{n} \pi_{_A}((a_i, f(a_i))) T^i$. Since $\pi_{_A}$ is a ring retraction (Example 1.22(2)), it follows that $p(T)$ is a Gauss polynomial over A, by Lemma 1.81. Thus its content, that is exactly the regular ideal $\mathfrak{a}$, is invertible, by Theorem 1.80(2). This completes the proof. □

1.83 Remark. We preserve notation of Proposition 1.9. The fact that $A \bowtie^f \mathfrak{b}$ is a Prüfer ring does not imply, in general, that A is a Prüfer ring. For an example, see [5, Example 2.3], keeping in mind Remark 1.16.

The following result is obtained by modifing the proof of [8, Theorem 1].

1.84 Proposition. *Let $\phi : A \longrightarrow B$ be a surjective ring homomorphism. If A is a Prüfer ring and $\mathrm{Ker}(\phi)$ is a regular ideal of A, then $\mathfrak{a}(\mathfrak{b} \cap \mathfrak{c}) = \mathfrak{ab} \cap \mathfrak{ac}$, for all ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ of B. In particular, B is a Prüfer ring.*

Proof. Let $\mathfrak{d} := \mathrm{Ker}(\phi)$ and let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be ideals of B. To prove the equality $\mathfrak{a}(\mathfrak{b} \cap \mathfrak{c}) = \mathfrak{ab} \cap \mathfrak{ac}$, it suffices to show that $\mathfrak{ab} \cap \mathfrak{ac} \subseteq \mathfrak{a}(\mathfrak{b} \cap \mathfrak{c})$. If $\overline{x} \in \mathfrak{ab} \cap \mathfrak{ac}$, then there are elements $\overline{a}_i \in \mathfrak{a}, \overline{b}_i \in \mathfrak{b}, \overline{\alpha}_j \in \mathfrak{a}, \overline{c}_j \in \mathfrak{c}$, with $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$, such that $\overline{x} = \sum_{i=1}^{n} \overline{a}_i \overline{b}_i = \sum_{j=1}^{m} \overline{\alpha}_j \overline{c}_j$. For each $i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$, choose elements $a_i \in \phi^{-1}(\overline{a}_i), b_i \in \phi^{-1}(\overline{b}_i), \alpha_j \in \phi^{-1}(\overline{\alpha}_j), c_j \in \phi^{-1}(\overline{c}_j)$, and set $\mathfrak{a}' := \phi^{-1}(\mathfrak{a}), \mathfrak{b}' := \phi^{-1}(\mathfrak{b}), \mathfrak{c}' := \phi^{-1}(\mathfrak{c})$. If $x := \sum_{i=1}^{n} a_i b_i$, it is immediate that $x - \sum_{j=1}^{m} \alpha_j c_j \in \mathfrak{d}$. Therefore $x \in (\mathfrak{a}'\mathfrak{c}' + \mathfrak{d}) \cap \mathfrak{a}'\mathfrak{b}'$. Keeping in mind Theorem 1.60 and the fact that $\mathfrak{d}$ is a regular ideal of A, we have

$$(\mathfrak{a}'\mathfrak{c}'+\mathfrak{d})\cap\mathfrak{a}'\mathfrak{b}' = (\mathfrak{a}'\mathfrak{c}'\cap\mathfrak{a}'\mathfrak{b}')+(\mathfrak{d}\cap\mathfrak{a}'\mathfrak{b}') \subseteq (\mathfrak{a}'\mathfrak{c}' \cap \mathfrak{a}'(\mathfrak{b}' + \mathfrak{d}))+\mathfrak{d} = \mathfrak{a}'(\mathfrak{c}'\cap(\mathfrak{b}'+\mathfrak{d}))+\mathfrak{d}$$

Thus, there are elements $a'_h \in \mathfrak{a}', b'_h \in \mathfrak{b}', d_h \in \mathfrak{d}$, with $h \in \{1, \ldots, r\}$ such that $b'_h + d_h \in \mathfrak{c}'$, for each h, and $x = \sum_{h=1}^{r} a'_h(b'_h + d_h) + d$, for some $d \in \mathfrak{d}$. It follows immediately that $\overline{x} = \sum_{h=1}^{r} f(a'_h)f(b'_h) \in \mathfrak{a}(\mathfrak{b} \cap \mathfrak{c})$. Now the first statement is clear. The fact that B is a Prüfer ring follows by the previous statement and Theorem 1.60. □

1.85 Corollary. *Preserve the notation of Proposition 1.9 and assume that $A \bowtie^f \mathfrak{b}$ is a Prüfer ring. Then the following statements hold.*

(1) *If $\{0\} \times \mathfrak{b}$ is a regular ideal of $A \bowtie^f \mathfrak{b}$, then A is a Prüfer ring.*

(2) *If $f^{-1}(\mathfrak{b}) \times \{0\}$ is a regular ideal of $A \bowtie^f \mathfrak{b}$, then $f(A) + \mathfrak{b}$ is a Prüfer ring.*

Proof. It suffices to apply Propositions 1.9(3) and 1.84. □

1.86 Remark. Let $f : A \longrightarrow B$ be a ring homomorphism, S be a multiplicative subset of A and $\mathfrak{b}$ be an ideal of B. Consider the multiplicative subset $T := f(S) + \mathfrak{b}$ of B (see Remark 1.48) and let $f_S : A_S \longrightarrow B_T$ be the ring homomorphism induced by f. Then, a straightforward verification shows that $f_S^{-1}(\mathfrak{b}B_T) = f^{-1}(\mathfrak{b})A_S$.

The following result shows that when $\mathfrak{b}$ and $f^{-1}(\mathfrak{b})$ are regular ideals, then $A \bowtie^f \mathfrak{b}$ satisfies Prüfer–like conditions only in the trivial case.

1.87 Proposition. *We preserve the notation of Proposition 1.9, and assume that $f^{-1}(\mathfrak{b})$ and $\mathfrak{b}$ are regular ideals. Then, the following conditions are equivalent.*

(i) *$A \bowtie^f \mathfrak{b}$ is a locally Prüfer ring.*

(ii) *A, B are locally Prüfer rings and $\mathfrak{b} = B$.*

Proof. (ii)$\Longrightarrow$(i). If $\mathfrak{b} = B$, then $A \bowtie^f \mathfrak{b}$ is trivially equal to $A \times B$. Then condition (i) follows immediately keeping in mind that $(A \times B)_{\mathfrak{m}\times B} \cong A_{\mathfrak{m}}$ and $(A \times B)_{A\times\mathfrak{n}} \cong B_{\mathfrak{n}}$, for each $(\mathfrak{m}, \mathfrak{n}) \in \mathrm{Max}(A) \times \mathrm{Max}(B)$ (without any assumption on the regularity of $f^{-1}(\mathfrak{b})$ and $\mathfrak{b}$).

(i)$\Longrightarrow$(ii). Assume, by contradiction, that $\mathfrak{b}$ is a proper ideal of $\mathfrak{b}$, and take a maximal ideal $\mathfrak{n}$ of B containing $\mathfrak{b}$. Then, take a maximal ideal $\mathfrak{m}$ of A containing the prime ideal $f^{-1}(\mathfrak{n}) \supseteq f^{-1}(\mathfrak{b})$. If $S_{\mathfrak{m}} := f(A\backslash\mathfrak{m}) + \mathfrak{b}$, $\mathfrak{b}_{S_{\mathfrak{m}}} := \mathfrak{b}B_{S_{\mathfrak{m}}}$ and $f_{\mathfrak{m}} : A_{\mathfrak{m}} \longrightarrow B_{S_{\mathfrak{m}}}$ is the ring homomorphism induced by f, then the localization of $A \bowtie^f \mathfrak{b}$ at the maximal ideal $\mathfrak{m}'^f$ is isomorphic to $A_{\mathfrak{m}} \bowtie^{f_{\mathfrak{m}}} \mathfrak{b}_{S_{\mathfrak{m}}}$, in view of Proposition 1.49(2). Note that, by Lemma 1.50 and Remark 1.86, the conductor of $A_{\mathfrak{m}} \bowtie^{f_{\mathfrak{m}}} \mathfrak{b}_{S_{\mathfrak{m}}}$ into $A_{\mathfrak{m}} \times B_{S_{\mathfrak{m}}}$ is $f^{-1}(\mathfrak{b})A_{\mathfrak{m}} \times \mathfrak{b}_{S_{\mathfrak{m}}}$, and it is a regular ideal of $A_{\mathfrak{m}} \times B_{S_{\mathfrak{m}}}$, since $f^{-1}(\mathfrak{b})$ and $\mathfrak{b}$ are regular ideals, by assumption. By using Proposition 1.51, it follows that $A_{\mathfrak{m}} \times B_{S_{\mathfrak{m}}}$ is an overring of the local Prüfer ring $A_{\mathfrak{m}} \bowtie^{f_{\mathfrak{m}}} \mathfrak{b}_{S_{\mathfrak{m}}}$. Then $A_{\mathfrak{m}} \times B_{S_{\mathfrak{m}}}$ is a localization of $A_{\mathfrak{m}} \bowtie^{f_{\mathfrak{m}}} \mathfrak{b}_{S_{\mathfrak{m}}}$, by [10, Lemma 3.6]. It follows immediately that $B_{S_{\mathfrak{m}}} = \{0\}$ and, by Remark 1.48(2), $f^{-1}(\mathfrak{b}) \nsubseteq \mathfrak{m}$, a contradiction. This proves that $\mathfrak{b} = B$ and $A \bowtie^f \mathfrak{b} = A \times B$. The fact that A and B are locally Prüfer follows by what stated in (ii)$\Longrightarrow$(i). □

1.88 Corollary. *We preserve the notation of Proposition 1.9, assume that $f^{-1}(\mathfrak{b}), \mathfrak{b}$ are regular ideals, and let $n \in \{1, 2, 3, 4\}$. Then, the following conditions are equivalent.*

(i) *$A \bowtie^f \mathfrak{b}$ satisfies Prüfer–like condition* (P-n) *(see Definition 1.60).*

(ii) *A, B satisfy Prüfer–like condition* (P-n)*, and $\mathfrak{b} = B$.*

Proof. (ii)$\Longrightarrow$(i). As before, we have $A \bowtie^f \mathfrak{b} = A \times B$. Then, it suffices to apply [3, Theorem 3.4].

(i)$\Longrightarrow$(ii). If $A \bowtie^f \mathfrak{b}$ satisfies Prüfer–like condition (P-n), it is, in particular, a locally Prüfer ring, and then $\mathfrak{b} = B$, by Proposition 1.87. Moreover A and B satisfy Prüfer–like condition (P-n), again by [3, Theorem 3.4]. $\square$

1.89 Corollary. *Let A be a ring, $\mathfrak{a}$ be a regular ideal of A and $n \in \{1, 2, 3, 4\}$. Then, $A \bowtie \mathfrak{a}$ is a locally Prüfer ring (resp., $A \bowtie \mathfrak{a}$ satisfies Prüfer–like condition* P-n*) if and only if A is a locally Prüfer ring (resp., A satisfies Prüfer–like condition* P-n*) and $\mathfrak{a} = A$.*

Proof. Apply Corollary 1.88, keeping in mind Example 1.10. $\square$

1.90 Proposition. *We preserve the notation of Proposition 1.9 and set $S_{\mathfrak{p}} := S_{(\mathfrak{p},f,\mathfrak{b})} := f(A \setminus \mathfrak{p}) + \mathfrak{b}$, for each prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$. Assume that $\mathfrak{b}$ and $f^{-1}(\mathfrak{b})$ are regular ideals. If $A \bowtie^f \mathfrak{b}$ is a Prüfer ring, then A, B and $\dfrac{A}{f^{-1}(\mathfrak{b})}$ are Prüfer rings, and $\dfrac{A_{\mathfrak{p}}}{f^{-1}(\mathfrak{b})A_{\mathfrak{p}}} \times \dfrac{B_{S_{\mathfrak{p}}}}{\mathfrak{b}B_{S_{\mathfrak{p}}}}$ is isomorphic to an overring of $\dfrac{A_{\mathfrak{p}}}{f^{-1}(\mathfrak{b})A_{\mathfrak{p}}}$, for each prime (maximal) ideal $\mathfrak{p}$ of A containing $f^{-1}(\mathfrak{b})$.*

Proof. We begin by describing how the fiber product diagram

$$\begin{tikzcd}
A \bowtie^f \mathfrak{b} \arrow[r, two heads] \arrow[d, hook] & \frac{A}{f^{-1}(\mathfrak{b})} \arrow[d, hook, "\breve{u}"] \\
A \times B \arrow[r, two heads, "\breve{v}"] & \frac{A}{f^{-1}(\mathfrak{b})} \times \frac{B}{\mathfrak{b}}
\end{tikzcd}$$

localizes (see Remark 1.18 for the notation). Let $\mathfrak{p}$ be a prime ideal of A, and let $f_{\mathfrak{p}} : A_{\mathfrak{p}} \longrightarrow B_{S_{\mathfrak{p}}}$ be the ring homomorphism induced by f. Then, by Remarks 1.18 and 1.86, the ring $A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} \mathfrak{b}_{S_{\mathfrak{p}}} \cong (A \bowtie^f \mathfrak{b})_{\mathfrak{p}'^f}$ (Proposition 1.49(2)) arises from the following fiber product diagram

$$\begin{tikzcd}
A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} \mathfrak{b}_{S_{\mathfrak{p}}} \arrow[r, two heads] \arrow[d, hook] & \frac{A_{\mathfrak{p}}}{f^{-1}(\mathfrak{b})A_{\mathfrak{p}}} \arrow[d, hook, "\breve{u}_{\mathfrak{p}}"] \\
A_{\mathfrak{p}} \times B_{S_{\mathfrak{p}}} \arrow[r, two heads, "\breve{v}_{\mathfrak{p}}"] & \frac{A_{\mathfrak{p}}}{f^{-1}(\mathfrak{b})A_{\mathfrak{p}}} \times \frac{B_{S_{\mathfrak{p}}}}{\mathfrak{b}_{S_{\mathfrak{p}}}}
\end{tikzcd}$$

where $\breve{u}_{\mathfrak{p}}, \breve{v}_{\mathfrak{p}}$ are the ring homomorphisms induced by $\breve{u}$ and $\breve{v}$. By assumption, the conductor $\mathfrak{b}_1 := f^{-1}(\mathfrak{b}) \times \mathfrak{b}$ of $A \bowtie^f \mathfrak{b}$ into $A \times B$ (see Lemma 1.50) is regular. Then, the statement is a consequence of [10, Theorem 4.2(1)] and [31, Proposition 3]. $\square$

Now, we will give sufficient conditions to make $A \bowtie^f \mathfrak{b}$ a total ring of fraction (and, in particular, a Prüfer ring).

1.91 PROPOSITION. *Let A be a total ring of fractions (i.e. $A = \mathrm{Tot}(A)$), $f : A \longrightarrow B$ be a ring homomorphism and $\mathfrak{b}$ be an ideal of B contained in the Jacobson radical $\mathrm{Jac}(B)$ of B. Assume that at least one of the following conditions holds.*

(a) *$\mathfrak{b}$ is contained in $f(A)$.*

(b) *$\mathfrak{b}$ is a torsion $A-$module (with the $A-$module structure inherited by f).*

Then $A \bowtie^f \mathfrak{b}$ is a total ring of fraction (and it is, in particular, a Prüfer ring).

Proof. Let $(a, f(a) + b)$ be a non invertible element of $A \bowtie^f \mathfrak{b}$. The goal is to show that $(a, f(a) + b)$ is a zerodivisor of $A \bowtie^f \mathfrak{b}$. Since $\mathfrak{b} \subseteq \mathrm{Jac}(B)$, by Proposition 1.36 it follows that

$$\mathrm{Max}(A \bowtie^f \mathfrak{b}) = \{\mathfrak{m}'^f : \mathfrak{m} \in \mathrm{Max}(A)\}.$$

Thus, there exists a maximal ideal $\mathfrak{m}$ of A such that $(a, f(a) + b) \in \mathfrak{m}'^f$, that is $a \in \mathfrak{m}$. Since A is a total ring of fractions, it follows that a is a zerodivisor of A. Hence, we can pick a nonzero element $\alpha \in A$ such that $a\alpha = 0$. The following two cases may occour.

- *Condition (a) holds.* If $\alpha \in \mathrm{Ann}_A(\mathfrak{b})$, then it follows immediately that $(a, f(a) + b)(\alpha, f(\alpha)) = (0, 0)$. Otherwise, let $\beta \in \mathfrak{b}$ be an element such that $f(\alpha)\beta \neq 0$. Since $\mathfrak{b} \subseteq f(A)$, there is an element $x \in f^{-1}(\mathfrak{b})$ such that $f(x) = \beta$. Of course, $\alpha x \neq 0$ and $(\alpha x, 0) \in A \bowtie^f \mathfrak{b}$, since $\alpha x \in f^{-1}(\mathfrak{b})$. It follows $(a, f(a) + b)(\alpha x, 0) = (0, 0)$.

- *Condition (b) holds.* Since $\mathfrak{b}$ is a torsion $A-$module, there exists a regular element $x_0 \in A$ such that $f(x_0)b = 0$. Of course, $\alpha x_0 \neq 0$, since $\alpha \neq 0$. Then $(a, f(a) + b)(\alpha x_0, f(\alpha x_0)) = (0, 0)$.

The conclusion is now clear. $\square$

1.5 Krull dimension of $A \bowtie^f \mathfrak{b}$

Now we want to discuss about the Krull dimension of $A \bowtie^f \mathfrak{b}$. We start with an easy observation.

1.92 Proposition. *We preserve the notation of Proposition 1.9. Then* $\dim(A \bowtie^f \mathfrak{b}) = \max\{\dim(A), \dim(f(A)+\mathfrak{b})\}$*. In particular, if f is surjective, then* $\dim(A \bowtie^f \mathfrak{b}) = \max\{\dim(A), \dim(B)\} = \dim(A)$*.*

Proof. By Lemma 1.53 and [49, Theorem 48], it follows immediately that $\dim(A \bowtie^f \mathfrak{b}) = \dim(A \times (f(A) + \mathfrak{b}))$. Thus, the conclusion is an easy consequence of the fact that $\mathrm{Spec}(A \times (f(A) + \mathfrak{b}))$ is canonically homeomorphic to the disjoint union of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(f(A) + \mathfrak{b})$. The last statement is straightforward. $\square$

As Proposition 1.29, also last result has moderate interest in many cases, since the Krull dimension is related to the Krull dimension of $f(A) + \mathfrak{b}$, that is not easy to compute. Moreover, $A \bowtie^f \mathfrak{b} \cong f(A) + \mathfrak{b}$, if $f^{-1}(\mathfrak{b}) = \{0\}$ (Proposition 1.9(3)).

In the following case, it is easy to evaluate $\dim(A \bowtie^f \mathfrak{b})$.

1.93 Proposition. *We preserve the notation of Proposition 1.9. Let $f_\diamond : A \longrightarrow B_\diamond := f(A)+\mathfrak{b}$ the ring homomorphism induced from f. If we assume that $f_\diamond$ is integral (e.g., f is integral), then* $\dim(A \bowtie^f \mathfrak{b}) = \dim(A)$*.*

Proof. Apply Lemma 1.55 and [49, Theorem 48]. $\square$

We proceed our investigation looking for upper and lower bounds of the Krull dimension of $A \bowtie^f \mathfrak{b}$. By Proposition 1.36, we know that $\mathrm{Spec}(A \bowtie^f \mathfrak{b}) = X \cup U$, where $X := \mathrm{Spec}(A)$ and $U := \mathrm{Spec}(B) \setminus V(\mathfrak{b})$ (for the sake of simplicity, we will identify X and U with their homeomorphic images in $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$). Furthermore, again from Proposition 1.36, we deduce that ideals of the form $\overline{\mathfrak{q}}^f$ can be contained in ideals of the form $\mathfrak{p}'^f$, but not vice versa. Therefore, chains in $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$ are obtained by juxtaposition of two types of chains, one from U "on the bottom" and the other one from X "on the top" (where either one or the other may be empty or a single element). It follows immediately that both $\dim(X) = \dim(A)$ and $\dim(U)$ are lower bounds for $\dim(A \bowtie^f \mathfrak{b})$ and $\dim(A)+\dim(U)+1$ is an upper bound for $\dim(A \bowtie^f \mathfrak{b})$ (where, conventionally, if $U = \emptyset$, then $\dim(\emptyset) = -1$).

1.94 Remark. We preserve the notation of Proposition 1.9, and assume that $\mathfrak{b} \subseteq \mathrm{Jac}(B)$. By Proposition 1.36(5), we get that $U := \mathrm{Spec}(B) \backslash V(\mathfrak{b})$ does not contain maximal elements of $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$. Hence, in this case, $1 + \dim(U) \leq \dim(A \bowtie^f \mathfrak{b})$.

Let us define the following subset of U:

$$\mathcal{Y}_{(f,\mathfrak{b})} := \left\{ \mathfrak{q} \in U : f^{-1}(\mathfrak{q} + \mathfrak{b}) = \{0\} \right\};$$

it is obvious that $\mathcal{Y}_{(f,\mathfrak{b})}$ is *stable under generizations*, i.e., $\mathfrak{q} \in \mathcal{Y}_{(f,\mathfrak{b})}$, $\mathfrak{q}' \in \mathrm{Spec}(B)$ and $\mathfrak{q}' \subseteq \mathfrak{q}$ imply $\mathfrak{q}' \in \mathcal{Y}_{(f,\mathfrak{b})}$. Hence

$$\dim(\mathcal{Y}_{(f,\mathfrak{b})}) = \sup\left\{ \mathrm{ht}_B(\mathfrak{q}) : \mathfrak{q} \in \mathcal{Y}_{(f,\mathfrak{b})} \right\}$$

and we will denote this integer by $\delta_{(f,\mathfrak{b})}$.

1.95 Proposition. *We preserve the notation of Proposition 1.9; let* $U = \mathrm{Spec}(B) \setminus V(\mathfrak{b})$ *and* $\delta_{(f,\mathfrak{b})} = \dim(\mathcal{Y}_{(f,\mathfrak{b})})$.

(1) *Let* $\mathfrak{q} \in \mathrm{Spec}(B)$, *then* $f^{-1}(\mathfrak{q} + \mathfrak{b}) = \{0\}$ *if and only if* $\overline{\mathfrak{q}}^f$ $(= (A \times \mathfrak{q}) \cap A \bowtie^f \mathfrak{b})$ *is contained in* $\mathfrak{b}_0$ $(= \{0\} \times \mathfrak{b})$.

(2) *for every* $\mathfrak{q} \in \mathcal{Y}_{(f,\mathfrak{b})}$, *the corresponding prime* $\overline{\mathfrak{q}}^f$ *of* $A \bowtie^f \mathfrak{b}$ *is contained in every prime of the form* $\mathfrak{p}'^f$.

(3) $\max\{\dim(U), \dim(A) + 1 + \delta_{(f,\mathfrak{b})}\} \leq \dim(A \bowtie^f \mathfrak{b})$.

Proof. (1) Assume that $f^{-1}(\mathfrak{q}+\mathfrak{b}) = \{0\}$. If $(a, f(a)+b) \in \overline{\mathfrak{q}}^f$, with $a \in A$ and $b \in \mathfrak{b}$, then $f(a) + b \in \mathfrak{q}$, and so $a \in f^{-1}(\mathfrak{q} + \mathfrak{b}) = \{0\}$, i.e., $a = 0$. Therefore, $(a, f(a) + b) = (0, b) \in \mathfrak{b}_0$. Conversely, if $a \in f^{-1}(\mathfrak{q} + \mathfrak{b})$, i.e., $f(a) = q + b$ for some $q \in \mathfrak{q}$ and $b \in \mathfrak{b}$, then $f(a) - b \in Q$, and so $(a, f(a) - b) \in \overline{\mathfrak{q}}^f \subseteq \mathfrak{b}_0$, thus $a = 0$.

(2) By Proposition 1.36(1), we have that every ideal of the form $\mathfrak{p}'^f$ contains $\mathfrak{b}_0$. The conclusion follows immediately.

(3) By the observation preceding Remark 1.94, it is enough to show that $\dim(A) + 1 + \delta_{(f,\mathfrak{b})} \leq \dim(A \bowtie^f \mathfrak{b})$. If $\mathcal{Y}_{(f,\mathfrak{b})} = \emptyset$ the statement is obvious. Otherwise, let $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_r$ be a maximal chain in $\mathcal{Y}_{(f,\mathfrak{b})}$, thus $r = \delta_{(f,\mathfrak{b})}$. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_m$ be a chain realizing $\dim(A)$. By (2) we obtain that

$$\overline{\mathfrak{q}}_0^f \subset \ldots \subset \overline{\mathfrak{q}}_r^f \subset \mathfrak{p}_0'^f \subset \ldots \subset \mathfrak{p}_m'^f,$$

is a chain in $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$. $\square$

1.96 Remark. (a) In the situation of Proposition 1.95, note that, if $\mathfrak{b}$ is contained in the nilradical of B, i.e., if $V(\mathfrak{b}) = \mathrm{Spec}(B)$, then $\delta_{(f,\mathfrak{b})} = \dim(U) = -1$. Therefore, Proposition 1.95(3) gives $\dim(A) \leq \dim(A \bowtie^f \mathfrak{b})$. But, in this (trivial) case, we can say more, precisely that $\mathrm{Spec}(A)$ is homeomorphic to $\mathrm{Spec}(A \bowtie^f \mathfrak{b})$, via the homeomorphism $\pi_A^* : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A \bowtie^f \mathfrak{b})$ (see Propositions 1.9, 1.17 and 1.36).

(b) Note that, if $\mathfrak{b} \not\subseteq \mathrm{Jac}(B)$, the inequality $1 + \dim(U) \leq \dim(A \bowtie^f \mathfrak{b})$ from Remark 1.94 can be false, as the following Example 1.97 will show.

(c) If we assume that $\mathfrak{b} \neq \{0\}$ and that $A \bowtie^f \mathfrak{b}$ and B are integral domains, then, by Proposition 1.25, $f^{-1}(\mathfrak{b}) = \{0\}$ and the subset $\mathcal{Y}_{(f,\mathfrak{b})}$ of $\mathrm{Spec}(B)$, defined in the previous proposition, is nonempty, since $(0) \in \mathcal{Y}_{(f,\mathfrak{b})}$, and so $\delta_{(f,\mathfrak{b})} \geq 0$. The following Example 1.102 will show that $\delta_{(f,\mathfrak{b})}$ may be arbitrarily large. Note that $\delta_{(f,\mathfrak{b})}$ may be equal to -1 even if $\mathfrak{b} \neq \{0\}$, $f^{-1}(\mathfrak{b}) = \{0\}$, but B is not an integral domain. It is sufficient to take B equal to a local zero-dimensional ring not a field, $\mathfrak{b}$ equal to its maximal ideal, A any subring of B such that $\mathfrak{b} \cap A = (0)$, and f be the natural embedding of A in B (e.g., $B := K[X]/(X^2)$, where K is a field and X an indeterminate over K, and A any domain contained in K). In this case, $\mathrm{Spec}(B) = V(\mathfrak{b})$ and so $\delta_{(f,\mathfrak{b})} = -1$.

(d) Note that, in the situation of Proposition 1.95(1), we can have $\overline{\mathfrak{q}}^f \subseteq \mathfrak{b}_0$ $(= \{0\} \times \mathfrak{b})$ with $\mathfrak{q} \supsetneq \mathfrak{b}$. For instance, let $A := K$, $B := K[X,Y]$, $\mathfrak{q} := (X,Y)B$, $\mathfrak{b} := XB$, and let $f : A = K \hookrightarrow K[X,Y] = B$ be the natural embedding, where K is a field and X and Y two indeterminates over K. In this case, $A \bowtie^f \mathfrak{b} \cong A + J = K + XK[X,Y]$ (Proposition 1.9(3)). Clearly, $f^{-1}(\mathfrak{q}) = f^{-1}(\mathfrak{q} + \mathfrak{b}) = f^{-1}(\mathfrak{b}) = \{0\}$ and $\overline{\mathfrak{q}}^f = \mathfrak{b}_0 \cong XK[X,Y]$.

1.97 Example. Let K be a field and X and Y two indeterminates over K. Set $B := K(X)[Y]_{(Y)} \cap K(Y)[X]_{(X)}$. It is well known that B is a one-dimensional semilocal domain, having two maximal ideals

$$\mathfrak{M} := YK(X)[Y]_{(Y)} \cap B, \qquad \mathfrak{N} := XK(Y)[X]_{(X)} \cap B.$$

Let $\mathfrak{b} := \mathfrak{M}$, $A := K$ and let f be the natural embedding of A in B. Clearly, $f^{-1}(\mathfrak{b}) = \mathfrak{M} \cap K = \{0\}$. In this situation, $\mathfrak{N} \in \mathrm{Spec}(B) \setminus V(\mathfrak{b})$ and so $\dim(U) = 1$. It is easy to see that $A \bowtie^f \mathfrak{b} \cong K + \mathfrak{M}$ (Proposition 1.9(3)) is a one-dimensional local domain. Therefore, in this case, we have $2 = 1 + \dim(U) > 1 = \dim(A \bowtie^f \mathfrak{b})$.

1.98 Corollary. *We preserve the notation of Propositions 1.9 and 1.95, and assume that* $\mathfrak{b} \subseteq \mathrm{Jac}(B)$ *and that* $\delta_{(f,\mathfrak{b})} \geq 0$ *(e.g.,* $A \bowtie^f \mathfrak{b}$ *and* B *are integral domains). Then*

$$1 + \max\{\dim(A) + \delta_{(f,\mathfrak{b})},\ \dim(U)\} \leq \dim(A \bowtie^f \mathfrak{b})\,.$$

Proof. Apply Remark 1.94 and Proposition 1.95. □

The following observations will be useful for Remark 1.104.

1.99 Remark. We preserve the notation of Proposition 1.9, and let $\mathfrak{q}$ be a prime ideal of B.

(1) By Proposition 1.95(1), it follows immediately that $\overline{\mathfrak{q}}^f := (A \times \mathfrak{q}) \cap A \bowtie^f \mathfrak{b} \subsetneq \mathfrak{b}_0 := \{0\} \times \mathfrak{b}$ if and only if $\mathfrak{q} \in \mathcal{Y}_{(f,\mathfrak{b})}$ (as defined in Proposition 1.95), i.e., $f^{-1}(\mathfrak{q}+\mathfrak{b}) = \{0\}$ and $\mathfrak{q} \not\supseteq \mathfrak{b}$. Therefore, $\mathcal{Y}_{(f,\mathfrak{b})}$ is homeomorphic to $\{\mathfrak{h} \in \mathrm{Spec}(A \bowtie^f \mathfrak{b}) : \mathfrak{h} \subsetneq \mathfrak{b}_0\}$.

(2) If $A \bowtie^f \mathfrak{b}$ and B are integral domains and $\mathfrak{b} \neq \{0\}$ then, in this situation, $\mathfrak{b}_0 = \{0\}'^f \in \mathrm{Spec}(A \bowtie^f \mathfrak{b})$ and $f^{-1}(\mathfrak{b}) = \{0\}$ by Proposition 1.25. Therefore, $\mathfrak{q} = \{0\} \in \mathcal{Y}_{(f,\mathfrak{b})}(\neq \emptyset)$ and $\overline{\mathfrak{q}}^f = f^{-1}(\mathfrak{b}) \times \{0\} = \{0\} \subsetneq \mathfrak{b}_0$; thus, if $\mathrm{ht}_{A\bowtie^f\mathfrak{b}}(\mathfrak{b}_0) < \infty$, $\delta_{(f,\mathfrak{b})}(= \dim \mathcal{Y}_{(f,\mathfrak{b})}) = \mathrm{ht}_{A\bowtie^f\mathfrak{b}}(\mathfrak{b}_0) - 1$.

The next goal is to determine upper bounds to $\dim(A \bowtie^f \mathfrak{b})$, possibly sharper than $\dim(A) + \dim(U) + 1$.

To do this, the following Proposition is the crucial step.

1.100 Proposition. *With the notation of Definition 1.1, assume* β *surjective. Let* $\mathfrak{h}'$ *and* $\mathfrak{h}''$ *be prime ideals of* D *such that* $\mathfrak{h}' \subsetneq \mathfrak{h}''$*. Assume that* $\mathfrak{h}' \in \mathrm{Spec}(D) \setminus V(\mathrm{Ker}(p_A))$*,* $\mathfrak{h}'' \in V(\mathrm{Ker}(p_A))$*, and that* $\mathfrak{h}'$ *and* $\mathfrak{h}''$ *are adjacent prime ideals. Then, there exist two prime ideals* $\mathfrak{q}'$ *and* $\mathfrak{q}''$ *of* B*, with* $\mathfrak{q}' \subsetneq \mathfrak{q}''$*, and moreover such that* $\mathfrak{q}' \notin V(\mathrm{Ker}(\beta))$*,* $p_B^{-1}(\mathfrak{q}') = \mathfrak{h}'$*, and* $p_B^{-1}(\mathfrak{q}'') = \mathfrak{h}''$*.*

Proof. First of all, take the unique prime ideal $\mathfrak{q}'$ of B such that $p_B^{-1}(\mathfrak{q}') = \mathfrak{h}'$ (see Theorem 1.6(1)).

Now note that, for each ideal $\mathfrak{b}$ of B, the equality $p_B^{-1}(\mathfrak{b} + \mathrm{Ker}(\beta)) = p_B^{-1}(\mathfrak{b}) + \mathrm{Ker}(p_A)$ holds. It follows that the set

$$\mathcal{S}(\mathfrak{q}', \mathfrak{h}'') := \{\mathfrak{b} \subseteq B : \mathfrak{b} \text{ is an ideal of } B, \mathfrak{q}' + \mathrm{Ker}(\beta) \subseteq \mathfrak{b} \text{ and } p_B^{-1}(\mathfrak{b}) \subseteq \mathfrak{h}''\}$$

is nonempty (it contains $\mathfrak{q}' + \mathrm{Ker}(\beta)$). Moreover the inclusion makes $\mathcal{S}(\mathfrak{q}', \mathfrak{h}'')$ an inductive partially ordered set. Then, by Zorn's Lemma, $\mathcal{S}(\mathfrak{q}', \mathfrak{h}'')$ contains a maximal element $\mathfrak{q}''$, which is easy to see that is a prime ideal of B. Since $\mathfrak{h}'' \supseteq p_{_B}^{-1}(\mathfrak{q}'') \supseteq p_{_B}^{-1}(\mathfrak{q}) + \mathrm{Ker}(p_{_A}) \supsetneq \mathfrak{h}'$ and $\mathfrak{h}', \mathfrak{h}''$ are adjacent prime ideals, we have $p_{_B}^{-1}(\mathfrak{q}'') = \mathfrak{h}''$. □

1.101 Theorem. *Let $f : A \longrightarrow B$, $\mathfrak{b}$, and $A \bowtie^f \mathfrak{b}$ be as in Proposition 1.9. With the notation of Proposition 1.95, assume that $A \bowtie^f \mathfrak{b}$ has finite Krull dimension. Then*

$$\begin{aligned} \dim(A \bowtie^f \mathfrak{b}) &\leq \max\big\{\dim(A),\ \dim(A/f^{-1}(\mathfrak{b})) + \min\{\dim(B),\ 1+\dim(U)\}\big\} \\ &\leq \min\big\{\dim(A)+\dim(U)+1,\ \max\{\dim(A),\ \dim(A/f^{-1}(\mathfrak{b}))+\dim(B)\}\big\}. \end{aligned}$$

Proof. We can assume that $\mathrm{Spec}(B) \neq V(\mathfrak{b})$, because otherwise we already know that $\dim(A \bowtie^f \mathfrak{b}) = \dim(A)$ (Remark 1.96(a)) and so the inequalities hold.

Let $\mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \ldots \subset \mathfrak{h}_n$ be a chain of prime ideals of $A \bowtie^f \mathfrak{b}$ realizing $\dim(A \bowtie^f \mathfrak{b})$. Two extreme cases are possible.

(1) If $\mathfrak{h}_0 \supseteq \{0\} \times \mathfrak{b}$ then, by Proposition 1.36(1), the chain $\mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \ldots \subset \mathfrak{h}_n$ induces a chain of prime ideals of A of length n. From Proposition 1.95(2), we conclude that $\dim(A \bowtie^f \mathfrak{b}) = \dim(A)$.

(2) If $\mathfrak{h}_n \not\supseteq \{0\} \times \mathfrak{b}$. From Proposition 1.36(2), the chain $\mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \ldots \subset \mathfrak{h}_n$ induces a chain of prime ideals of U of length n. From Proposition 1.95(2), we conclude that $\dim(A \bowtie^f \mathfrak{b}) = \sup\{\mathrm{ht}(\mathfrak{q}) : \mathfrak{q} \in U\} = \dim(U)$.

We now consider the general case.

(3) Let t be the maximum index such that $\mathfrak{h}_t \not\supseteq \{0\} \times \mathfrak{b}$, with $0 \leq t \lneq n$. According to the notations of Proposition 1.36, rewrite the given chain as follows:

$$\overline{\mathfrak{q}}_0^{f} \subset \overline{\mathfrak{q}}_1^{f} \subset \ldots \subset \overline{\mathfrak{q}}_t^{f} \subset \mathfrak{p}_{t+1}'^{f} \subset \mathfrak{p}_{t+2}'^{f} \subset \ldots \subset \mathfrak{p}_n'^{f},$$

where $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_t$ is an increasing chain of prime ideals of B, with $\mathfrak{q}_t \not\supseteq \mathfrak{b}$ (Proposition 1.36(2)), and $\mathfrak{p}_{t+1} \subset \mathfrak{p}_{t+2} \subset \ldots \subset \mathfrak{p}_n$ is an increasing chain of prime ideals of A (Proposition 1.36(1)). Furthermore, by Proposition 1.100, we can find a prime ideal $\mathfrak{q}$ in $V(\mathfrak{b})$ ($\subseteq \mathrm{Spec}(B)$) such that the prime ideal $\mathfrak{h}_{t+1} = \mathfrak{p}_{t+1}'^{f}$ coincides also with the restriction to $A \bowtie^f \mathfrak{b}$ of the prime ideal $A \times \mathfrak{q}$ of $A \times B$, i.e., $\mathfrak{h}_{t+1} = \mathfrak{p}_{t+1}'^{f} = \overline{\mathfrak{q}}^{f}$. It follows immediately that $\mathfrak{p}_k \in V(f^{-1}(\mathfrak{b}))$, for $t+1 \leq k \leq n$. Therefore, $\dim(A \bowtie^f \mathfrak{b}) = (1+t)+(n-t-1)$ with $1+t \leq \min\{1+\dim(U),\ \dim(B)\}$ and $n-t-1 \leq \dim(A/f^{-1}(\mathfrak{b}))$.

Finally, it is obvious that $\min\{\dim(B),\ 1+\dim(U)\} \leq \dim(B)$ and that $\dim(A/f^{-1}(\mathfrak{b})) + \min\{\dim(B),\ 1+\dim(U)\} \leq \dim(A)+\dim(U)+1$. □

1.102 Example. Let V be a valuation domain with maximal ideal $\mathfrak{M}$ such that $V = K+\mathfrak{M}$, where K is a field isomorphic to the residue field $V/\mathfrak{M}$. Let D be an integral domain with quotient field K, and set $B := D+\mathfrak{M}$. Assume that $\dim(V) = n \geq 1$ and that $\mathfrak{Q}$ is a prime ideal of V with $\mathrm{ht}_V(\mathfrak{Q}) = t+1$, $n \geq t+1 \geq 0$. Set $\mathfrak{b} := \mathfrak{Q} \cap B$. By the well known properties of the "$D+\mathfrak{M}$ constructions", $B_{\mathfrak{M}} = V$ [29, Exercise 13(1), page 203], so $\mathfrak{b}$ is a prime ideal of B and $\mathrm{ht}_B(\mathfrak{b}) = t+1$. More precisely, if $(0) \subset \mathfrak{Q}_1 \subset \mathfrak{Q}_2 \subset \cdots \subset \mathfrak{Q}_t \subset \mathfrak{Q}_{t+1} = \mathfrak{Q}$ is the chain of prime ideals of V realizing the height of $\mathfrak{Q}$, then $\mathfrak{q}_0 := (0) \subset \mathfrak{q}_1 := \mathfrak{Q}_1 \cap B \subset \mathfrak{q}_2 := \mathfrak{Q}_2 \cap B \subset \cdots \subset \mathfrak{q}_t := \mathfrak{Q}_t \cap B \subset \mathfrak{q}_{t+1} := \mathfrak{Q}_{t+1} \cap B = \mathfrak{b}$ is the chain (in B) realizing $\mathrm{ht}_B(\mathfrak{b})$. Set $A := D$ and let $f : A = D \hookrightarrow D+\mathfrak{M} = B$ be the canonical embedding. Clearly, $f^{-1}(\mathfrak{b}) = \{0\}$ and so it is easy to verify that, in the present situation,

$$\begin{aligned}\mathcal{Y}_{(f,\mathfrak{b})} &:= \{\mathfrak{q} \in \mathrm{Spec}(B) : \mathfrak{q} \notin V(\mathfrak{b}), f^{-1}(\mathfrak{q}+\mathfrak{b}) = \{0\}\} \\ &= \{\mathfrak{q}_k : 0 \leq k \leq t\} = \mathrm{Spec}(B) \backslash V(\mathfrak{b}) = U\end{aligned}$$

(see also [29, Exercise 12(1), page 202]). Therefore, $\delta_{(f,\mathfrak{b})} = t = \dim(U)$. Moreover, if $m := \dim(D)$ $(= \dim(A))$ then, again by the well known properties of the "$D+\mathfrak{M}$ constructions", $\dim(B) = m+n$ [29, Exercise 12(4), page 203]. Henceforth, in the present example, we have $\max\{\dim(A)+1+\delta_{(f,J)},\ 1+\dim(U)\} = \dim(A)+1+\delta_{(f,\mathfrak{b})} = m+1+t$.

On the other hand, since $f^{-1}(\mathfrak{b}) = \{0\}$, clearly $A/f^{-1}(\mathfrak{b}) = A$ and so $\max\{\dim(A),\ \dim(A/f^{-1}(\mathfrak{b}))+\min\{\dim(B),\ 1+\dim(U)\}\} = \dim(A) + \min\{\dim(B),\ 1+\dim(U)\} = m+\min\{m+n,\ 1+t\}$. Since $n \geq t+1$, then $\min\{m+n,\ 1+t\} = 1+t$. Furthermore, by the fact that $f^{-1}(\mathfrak{b}) = \{0\}$, we have $A \bowtie^f \mathfrak{b} \cong A+\mathfrak{b} = D+\mathfrak{b}$ (Proposition 1.9(3)). Therefore, from Proposition 1.95(3) and Theorem 1.101, we deduce that $\dim(D+\mathfrak{b}) = m+1+t$.

Let $A \subset B$ be an arbitrary ring extension. We will apply the previous results to the polynomial rings of the form $A+XB[X]$ and we will show that the bounds given by Fontana, Izelgue and Kabbaj [21, Theorem 2.1] in the very special case where B and A are integral domains coincide to the bounds obtained specializing the general setting of amalgamated algebras.

1.103 Corollary. *Let $A \subseteq B$ be a ring extension and X an indeterminate over B. Set*

$$\delta'_{(A,B)} := \sup\{\mathrm{ht}_{B[X]}(\mathfrak{Q}) : \mathfrak{Q} \in \mathrm{Spec}(B[X]), X \notin \mathfrak{Q}, (\mathfrak{Q}+XB[X]) \cap A = \{0\}\}.$$

Then

$$\max\{\dim(A)+1+\delta'_{(A,B)},\ \dim(B[X,X^{-1}])\} \leq \dim(A+XB[X]) \leq \\ \leq \dim(A)+\dim(B[X])\,.$$

Proof. Let $B' := B[X]$ and $\mathfrak{b}' := XB[X]$. As observed in Example 1.11, we know that $A \bowtie^{\sigma'} \mathfrak{b}' \cong A + XB[X]$. From the definitions, it is easy to see that $\delta_{(\sigma',\mathfrak{b}')} = \delta'_{(A,B)}$. Moreover, since

$$\dim(B[X,X^{-1}]) = \sup\{\mathrm{ht}_{B[X]}(\mathfrak{Q}) : \mathfrak{Q} \in \mathrm{Spec}(B[X]), X \notin \mathfrak{Q}\} = \dim(U)$$

(where U, in this case, is homeomorphic to $\mathrm{Spec}(B[X])\setminus V(\mathfrak{b}')$) and $\sigma'^{-1}(\mathfrak{b}') = A \cap XB[X] = \{0\}$, the conclusion follows from Proposition 1.95(3) and Theorem 1.101. □

1.104 Remark. Let $A \subseteq B$ integral domains and and let $N := A\setminus\{0\}$. In [21, Theorem 2.1], Fontana, Izelgue and Kabbaj proved that

$$\max\{\dim(A)+\dim(N^{-1}B[X]),\, \dim(B[X])\} \leq \dim(A+XB[X]) \leq \\ \leq \dim(A)+\dim(B[X])\,.$$

By [21, Theorem 1.2(a) and Lemma 1.3], we know that

$$\dim(N^{-1}B[X]) = \mathrm{ht}_{A+XB[X]}(XB[X]) = 1+\lambda'_{(A,B)},$$

where

$$\lambda'_{(A,B)} := \sup\{\dim\left(B[X]_{\mathfrak{q}[X]}\right) : \mathfrak{q} \in \mathrm{Spec}(B),\ \mathfrak{q} \cap A = (0)\}\,.$$

From Remark 1.99(iii) and the proof of Corollary 1.103, we deduce the equality $\mathrm{ht}_{A+XB[X]}(XB[X]) = 1+\delta'_{(A,B)} = 1+\lambda'_{(A,B)}$, hence $\delta'_{(A,B)} = \lambda'_{(A,B)}$; moreover, we have $\dim B[X] = \dim B[X,X^{-1}]$, by [1, Proposition 1.14]. Therefore, in particular, we reobtain Fontana, Izelgue and Kabbaj's result on the dimension of the integral domain $A+XB[X]$. This fact provides further evidence on the sharpness of the bounds obtained in Proposition 1.95(3) and Theorem 1.101, in the general setting of amalgamated algebras.

We consider now the case of power series rings of the type $A + XB[\![X]\!]$ for arbitrary ring extensions $A \subset B$.

1.105 Corollary. *Let $A \subset B$ be a ring extension and X an indeterminate over B. Set*

$$\delta''_{(A,B)} := \sup\{\mathrm{ht}_{B[\![X]\!]}(\mathfrak{Q}) : \mathfrak{Q} \in \mathrm{Spec}(B[\![X]\!]) \setminus V(X), (\mathfrak{Q} + XB[\![X]\!]) \cap A = \{0\}\}.$$

Then

$$\max\{\dim(A) + 1 + \delta''_{(A,B)},\ 1 + \dim(B[\![X]\!][X^{-1}])\} \leq \dim(A + XB[\![X]\!]) \leq \\ \leq 1 + \dim(A) + \dim(B[\![X]\!][X^{-1}]).$$

Proof. Keeping in mind the statements and the notation of Example 1.11, it follows immediately that $\delta_{(\sigma'', XB[\![X]\!])} = \delta''_{(A,B)}$. Moreover, recalling that U, in this case, is homeomorphic to $\mathrm{Spec}(B[\![X]\!]) \setminus V(X)$, it is easy to see that $\dim(U) = \dim(B[\![X]\!][X^{-1}])$. Finally, note that $\min\{\dim(B[\![X]\!]),\ 1 + \dim(U)\} = 1 + \dim(U)$, since every maximal ideal of $B[\![X]\!]$ contains X [2, Chapter 1, Exercise 5(iv)]. The conclusion is now a straightforward consequence of Remark 1.94 and Theorem 1.101. □

1.106 Remark. By applying Corollary 1.105 and Remark 1.96, it follows that, if B is an integral domain, then

$$1 + \max\{\dim(A) + \delta''_{(A,B)},\ \dim(B[\![X]\!][X^{-1}])\} \leq \dim(A + XB[\![X]\!]) \leq \\ \leq 1 + \dim(A) + \dim(B[\![X]\!][X^{-1}]).$$

Now, we can compare our lower bound with that given by Dobbs and Khalis's Theorem [17, Theorem 11]. Set

$$\lambda''_{(A,B)} := \sup\{\dim\big(B[\![X]\!]_{\mathfrak{q}[\![X]\!]}\big) : \mathfrak{q} \in \mathrm{Spec}(B),\ \mathfrak{q} \cap A = (0)\}.$$

Then they prove that

$$1 + \max\{\dim(A) + \lambda''_{(A,B)},\ \dim(B[\![X]\!][X^{-1}])\} \leq \dim(A + XB[\![X]\!]) \leq \\ \leq 1 + \dim(A) + \dim(B[\![X]\!][X^{-1}]).$$

It is clear that $\dim\big(B[\![X]\!]_{\mathfrak{q}[\![X]\!]}\big) = \mathrm{ht}_{B[\![X]\!]}(\mathfrak{q}[\![X]\!])$. Moreover, it is immediately seen that, if $\mathfrak{q} \in \mathrm{Spec}(B)$ and $\mathfrak{q} \cap A = (0)$, then $(\mathfrak{q}[\![X]\!] + XB[\![X]\!]) \cap A = (0)$. Since the set $\{\mathfrak{q}[\![X]\!] \in \mathrm{Spec}(B[\![X]\!]) : \mathfrak{q} \in \mathrm{Spec}(B) \text{ and } \mathfrak{q} \cap A = \{0\}\}$ is a subset of $\{\mathfrak{Q} \in \mathrm{Spec}(B[\![X]\!]) : X \notin \mathfrak{Q} \text{ and } (\mathfrak{Q} + XB[\![X]\!]) \cap A = \{0\}\}$, we have $\lambda''_{(A,B)} \leq \delta''_{(A,B)}$. It is natural to ask, as in the polynomial case: does $\lambda''_{(A,B)} = \delta''_{(A,B)}$ hold? At the moment, the answer to this question is open.

However, by [17, Theorem 7], we observe that the answer could be negative if

$$\mathrm{ht}_{A+XB[\![X]\!]}(XB[\![X]\!]) = 1 + \delta''_{(A,B)}$$

and $\lambda''_{(A,B)} \lneq \sup\{\mathrm{ht}_{B[\![X]\!]}(\mathfrak{Q}) : \mathfrak{Q} \in \mathbf{\Lambda}_{(A,B)}\}$, where $\mathbf{\Lambda}_{(A,B)}$, as in [17, Theorem 7], is defined to be $\mathbf{\Lambda}_{(A,B)} = \{\mathfrak{Q} \in \mathrm{Spec}(B[\![X]\!]) : X \notin \mathfrak{Q},\ \mathfrak{Q} \subset (\mathfrak{q}, X),$ for some $\mathfrak{q} \in \mathrm{Spec}(B)$ with $\mathfrak{q} \cap A = (0)\}$.

1.107 Example. It is possible to construct an infinite dimensional ring of the type $A \bowtie^f \mathfrak{b}$, where A is a finite dimensional ring. In this situation, B must be a infinite dimensional ring by Theorem 1.101. For instance, let $A := \mathbb{C}$ be the field of complex numbers, let Y be an indeterminate over $\mathbb{C}$, and let $R := \mathbb{C}[\{Y^{1/n} : n \in \mathbb{N}\backslash\{0\}\}]$. Consider the maximal ideal $\mathfrak{M}$ of R generated by the set $\{Y^{1/n} : n \in \mathbb{N}\backslash\{0\}\}$. Set $B := R_{\mathfrak{M}}$, and consider the ring $A + XB[\![X]\!]$ ($\cong A \bowtie^{\sigma''} XB[\![X]\!]$, according to notation of Example 1.11). Then, by [17, Example 3], B is a one-dimensional non-discrete valuation domain and $\mathrm{ht}_{A+XB[\![X]\!]}(XB[\![X]\!]) = \infty$, and thus $\dim(A + XB[\![X]\!]) = \infty$.

The next two examples show that the upper bound and lower bound of Theorem 1.101 and Proposition 1.95(3) are "sharp", in the sense that $\dim(A \bowtie^f \mathfrak{b})$ may be equal to each of the two numerical terms appearing in the first inequality (respectively, in the inequality) of Theorem 1.101 (respectively, Proposition 1.95(3)).

1.108 Example. Let A be a valuation domain such that $\dim(A) = n \geq 3$, let $\{0\} \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals of A realizing $\dim(A)$, and let $x_h \in \mathfrak{p}_{h+1}\backslash\mathfrak{p}_h$, with $1 \leq h \leq n-2$ and $(x_h) \neq \mathfrak{p}_{h+1}$. Since A is a valuation domain, it is easily seen that $V(x_h) = V(\mathfrak{p}_{h+1})$, and thus $\dim(A/(x_h)) = \dim(A/\mathfrak{p}_{h+1}) = n - (h+1)$. Set $B := A/(x_h)$, $f : A \twoheadrightarrow B$ the canonical projection, $\mathfrak{q}_k := \mathfrak{p}_k/(x_h)$ for $h+1 \leq k \leq n$, and $\mathfrak{b} := \mathfrak{q}_{h+j}$ for some $1 \leq j \leq n-h$. In this case, by Proposition 1.92, $\dim(A \bowtie^f \mathfrak{b}) = \dim(A \times B) = \max\{\dim(A), \dim(B)\} = \dim(A) = n$. Note also that $\dim(A/f^{-1}(\mathfrak{b})) = n - (h+j) \leq n - (h+1) = \dim(B)$, and

$$\dim(U) = \begin{cases} -1\,, & \text{if } j = 1, \\ j-2\,, & \text{if } 1 < j \leq n-h. \end{cases}$$

It is also easy to see that $f^{-1}(\mathfrak{q} + \mathfrak{b}) \neq \{0\}$ for all $\mathfrak{q} \in \mathrm{Spec}(B)$ and so in this case $\delta_{(f,\mathfrak{b})} = -1$, for all $1 \leq j \leq n-h$. Moreover, in the present situation,

$A \bowtie^f \mathfrak{b}$ is a local ring, but it is not an integral domain since $f^{-1}(\mathfrak{b}) \neq \{0\}$ (see Proposition 1.25).

Consider now a chain $\mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \ldots \subset \mathfrak{h}_n$ of prime ideals of $A \bowtie^f \mathfrak{b}$ realizing $\dim(A \bowtie^f \mathfrak{b})$. Two cases are possible.

• If $1 \lneq j \leq n-h$, then the previous chain (realizing $\dim(A \bowtie^f \mathfrak{b})$) is of the type:

$$\begin{aligned} ((0) \neq) \mathfrak{p}_0'^f \subset \mathfrak{p}_1'^f \subset \ldots \subset \mathfrak{p}_h'^f \subset & \\ \subset \mathfrak{p}_{h+1}'^f = \overline{\mathfrak{q}}_{h+1}^f \subset \ldots \subset \mathfrak{p}_{h+j-1}'^f = \overline{\mathfrak{q}}_{h+j-1}^f \subset & \\ \subset \mathfrak{p}_{h+j}'^f \subset \ldots \subset \mathfrak{p}_n'^f & \end{aligned}$$

(where $\mathfrak{p}_k'^f = \overline{\mathfrak{q}}_k^f$ also for $h+j \leq k \leq n$, but in this case $\mathfrak{q}_k \supseteq \mathfrak{b}$);

• If $j = 1$, then the previous chain realizing $\dim(A \bowtie^f \mathfrak{b})$ is of the type:

$$((0) \neq) \mathfrak{p}_0'^f \subset \mathfrak{p}_1'^f \subset \ldots \subset \mathfrak{p}_h'^f \subset \ldots \subset \mathfrak{p}_n'^f$$

and none of the $\mathfrak{p}_k'^f$ is equal to a $\overline{\mathfrak{q}}_k^f$ for $\mathfrak{q}_k \not\supseteq \mathfrak{b}$.

In the present example, the inequality of Corollary 1.94 gives back the inequality $\max\big\{\dim(A)+1+\delta_{(f,\mathfrak{b})},\ 1+\dim(U)\big\} = \max\big\{n+1-1,\ 1+(j-2)\big\} \leq n = \dim(A \bowtie^f \mathfrak{b})$. The first inequality of Theorem 1.101 gives $\dim(A \bowtie^f \mathfrak{b}) = n \leq \max\big\{n,\ n-(h+j)+\min\{n-(h+1),\ 1+(j-2)\}\big\} = \max\big\{\dim(A), \dim(A/f^{-1}(J)) + \min\{\dim(B),\ 1+\dim(U)\}\big\}$.

1.109 Example. Let K be a field and let V and W be two incomparable finite dimensional valuation domains having same field of quotients F. Assume that V and W are K-algebras, that $V = K + \mathfrak{M}$ and $W = K + \mathfrak{N}$ where $\mathfrak{M}$ (respectively, $\mathfrak{N}$) is the maximal ideal of V (respectively, W), and that $\dim(V) = m \geq 1$ and $\dim(W) = n \geq 1$. Set $T := V \cap W$. It is well known that T is a finite dimensional Bézout domain with quotient field F and with two maximal ideals $\mathfrak{m} := \mathfrak{M} \cap T$ and $\mathfrak{n} := \mathfrak{N} \cap T$ such that $T_{\mathfrak{m}} = V$ and $T_{\mathfrak{n}} = W$, and so $\dim(T) = \max\{m, n\}$ [49, Theorem 101]. Let D be an integral domain of Krull dimension d with quotient field K. Since D is embedded naturally in V $(= K + \mathfrak{M})$ and W $(= K + \mathfrak{N})$, we have also a natural embedding $\iota : D \hookrightarrow T$.

In this situation, using the standard notation of the $A \bowtie^f \mathfrak{b}$ construction, when $A := D$, $B := T$, $\mathfrak{b} := \mathfrak{m}$, and $f := \iota$, we have that the ring $D + \mathfrak{m}$ (subring of T) is canonically isomorphic to $D \bowtie^\iota \mathfrak{m}$, by Example 1.14. Moreover, $f^{-1}(\mathfrak{b}) = \mathfrak{m} \cap D = \{0\}$ and so $\dim(A/f^{-1}(\mathfrak{b})) = \dim(D) = d$, and $\dim(U) = \max\{m-1, n\}$.

It is easy to verify that if $(0) = \mathfrak{Q}_0 \subset \mathfrak{Q}_1 \subset ... \subset \mathfrak{Q}_m = \mathfrak{M}$ are the prime ideals of V, and thus we have the following equality

$$\{\mathfrak{Q}_k \cap B : 0 \leq k \leq m-1\} = \{\mathfrak{Q} \in \operatorname{Spec}(B) \backslash V(\mathfrak{b}) : f^{-1}(\mathfrak{Q} + \mathfrak{b}) = \{0\}\}.$$

Therefore, $\delta_{(f,\mathfrak{b})} = m - 1$. On the other hand, it is easy to verify that $\dim(D + \mathfrak{m}) = \max\{m + d, n\}$.

In the present example, the inequality of Proposition 1.95(3) gives back the inequality $\max\big\{\dim(A) + 1 + \delta_{(f,\mathfrak{b})},\ \dim(U)\big\} = \max\big\{d + 1 + m - 1,\ \max\{m-1, n\}\big\} \leq \max\{m + d, n\} = \dim(A \bowtie^f \mathfrak{b}) = \dim(D + \mathfrak{m})$. Therefore, if $n > m + d$, then $\dim(A \bowtie^f \mathfrak{b}) = \dim(U)$. By the first inequality of Theorem 1.101, it follows that $\dim(A \bowtie^f \mathfrak{b}) = \max\{m + d, n\} \leq \max\big\{d,\ d + \min\{\max\{m, n\},\ 1 + \max\{m - 1, n\}\}\big\} = \max\big\{\dim(A),\ \dim\left(A/f^{-1}(\mathfrak{b})\right) + \min\{\dim(B),\ 1 + \dim(U)\}\big\}$. Therefore, if $m + d \leq n$, then $n = \dim(D + M) = \dim(D \bowtie^{\iota} \mathfrak{m}) = d + \min\big\{\max\{m, n\},\ 1 + \max\{m - 1, n\}\big\}$.

Chapter 2

The ultrafilter topology on an Abstract Riemann Surface

2.0 Notations and preliminaries

We preserve the notation and the conventions given in Section 1.0 of Chapter 1. If X is a set, we shall denote by $\mathscr{B}(X)$ (resp. $\mathscr{B}_{\mathtt{fin}}(X)$) the collection of all the subsets (resp. finite subsets) of X. As usual in Set Theory, we recall that a nonempty collection $\mathcal{F}$ of subsets of a set X has *the finite intersection property* if $\bigcap \mathcal{G}$ is nonempty, for each nonempty and finite subcollection $\mathcal{G}$ of $\mathcal{F}$. Recall that a (non necessarily Hausdorff) topological space is *compact* if each open cover of it admits a finite subcover. If X is a topological space and $Y \subseteq X$, we shall denote by $\mathring{Y}$ and $\mathrm{Ad}(Y)$ the interior of Y and the closure of Y, respectively.

Warning: since Chapter 3 is independent from Chapter 2, instead of giving "ad hoc" proofs in the special setting of Chatper 2, we will use some of the results of Chapter 3 in order to provide simple (and general) proofs of some statements of the present chapter.

2.0.1 Filters and ultrafilters on sets

2.1 Definition. *Let X be a set.*

(1) *A nonempty collection $\mathscr{F}$ of subsets of X is said to be a* filter on X *if the following conditions are satisfied:*

(a) $\emptyset \notin \mathscr{F}$;

(b) *if* $F, G \in \mathscr{F}$, *then* $F \cap G \in \mathscr{F}$;

(c) *if* $F, G \in \mathscr{B}(X)$, $F \subseteq G$, *and* $F \in \mathscr{F}$, *then* $G \in \mathscr{F}$.

(2) *A maximal element of the set of all the filters on* X, *partially ordered by the inclusion* $\subseteq$, *is said to be* an ultrafilter on X. *The set of all the ultrafilters on* X *will be denoted by* βX.

In the next lemma, we collect same basic properties of filters and ultrafilters needed in this chapter.

2.2 LEMMA. *Let* X *be a set.*

(1) *If* $\mathscr{F}$ *is a filter on* X, *then there is an ultrafilter* $\mathscr{U}$ *on* X *such that* $\mathscr{F} \subseteq \mathscr{U}$.

(2) *If* $\mathscr{G}$ *is a collection of subsets of* X *with the finite intersection property, then there is a filter* $\mathscr{F}$ *on* X *such that* $\mathscr{G} \subseteq \mathscr{F}$.

(3) *Let* $f : X \longrightarrow Y$ *be a map and* $\mathscr{U}$ *an ultrafilter [respectively,* $\mathscr{F}$ *a filter] on* Y. *If* f *is injective and* $f(X) \in \mathscr{U}$ *[respectively,* $f(X) \in \mathscr{F}$*], then*

$$\mathscr{U}^f := \{f^{-1}(Z) : Z \in \mathscr{U}\} \text{ [respectively, } \mathscr{F}^f := \{f^{-1}(Z) : Z \in \mathscr{F}\}\text{]}$$

is an ultrafilter [respectively, a filter] on X. *In particular, if* X *is a subset of* Y *and* f *is the inclusion map, then the set*

$$\mathscr{U}^X := \{Z \cap X : Z \in \mathscr{U}\} \text{ [respectively, } \mathscr{F}^X := \{Z \cap X : Z \in \mathscr{F}\}\text{]}$$

is an ultrafilter [respectively, a filter] on X. *Moreover, in this case,* $\mathscr{U}^X \subseteq \mathscr{U}$ *[respectively,* $\mathscr{F}^X \subseteq \mathscr{F}$*].*

(4) *Let* $f : X \longrightarrow Y$ *be a map and let* $\mathscr{U}$ *be an ultrafilter [respectively,* $\mathscr{F}$ *be a filter] on* X, *then*

$$\mathscr{U}_f := \{Z \in \mathscr{B}(Y) : f^{-1}(Z) \in \mathscr{U}\}$$
$$\text{[respectively, } \mathscr{F}_f := \{Z \in \mathscr{B}(Y) : f^{-1}(Z) \in \mathscr{F}\}\text{]}$$

is an ultrafilter [respectively, a filter] on Y. *In particular, if* X *is a subset of* Y, f *is the inclusion map and* $\mathscr{U}$ *is an ultrafilter [respectively,* $\mathscr{F}$ *is a filter] on* X, *then the set*

$$\mathscr{U}_Y := \{Z \in \mathscr{B}(Y) : Z \cap X \in \mathscr{U}\}$$
$$\text{[respectively, } \mathscr{F}_Y := \{Z \in \mathscr{B}(Y) : Z \cap X \in \mathscr{F}\}\text{]}$$

is an ultrafilter [respectively, a filter] on Y. Moreover, in this case, $\mathscr{U} \subseteq \mathscr{U}_Y$ [respectively, $\mathscr{F} \subseteq \mathscr{F}_Y$].

(5) *If $\mathscr{F}$ is a filter on X, then the following conditions are equivalent.*

(i) *$\mathscr{F}$ is an ultrafilter.*

(ii) *If $Y, Z \in \mathscr{B}(X)$ and $Y \cup Z \in \mathscr{F}$, then either $Y \in \mathscr{F}$ or $Z \in \mathscr{F}$.*

(iii) *If $Y \in \mathscr{B}(X)$, then either $Y \in \mathscr{F}$ or $X \backslash Y \in \mathscr{F}$.*

Proof. (1) is proved in [45, Theorem 7.5]. (2) Note that the collection

$$\mathscr{F}(\mathscr{G}) := \{Z \in \mathscr{B}(X) : Z \supseteq \bigcap \mathscr{G}', \text{ for some } \mathscr{G}' \subseteq \mathscr{G},\, \mathscr{G}' \neq \emptyset \text{ and finite}\}$$

is a filter on X and, precisely, it is the smallest filter on X containing $\mathscr{G}$ (see also [45, Lemma 7.2(iii)]). (3) is an easy consequence of definitions and [45, Exercise 7.1]. The first part of (4) is given in [45, Exercise 7.5]. The second part of (4) is a straightforward consequence of the first one. Finally, (5) is proved in [45, Lemma 7.4 and Exercise 7.3]. □

2.3 Remark. Let X be a set.

(1) For each $x \in X$, the collection of sets

$$\beta_X^x := \beta^x := \{Z \subseteq X : x \in Z\}$$

is an ultrafilter on X (by Lemma 2.2(5,iii)), and we call it *the trivial ultrafilter on X, centered on X.*

(2) An ultrafilter $\mathscr{U}$ on X is trivial if and only if it contains a finite set. As a matter of fact, if Y is a finite subset of X and $\mathscr{U}$ is an ultrafilter on X such that $Y \in \mathscr{U}$, then, keeping in mind that Lemma 2.2(5,ii), it follows that $\mathscr{U}$ contains a singleton, say $\{x\}$, and hence $\mathscr{U} = \beta^x$.

(3) X has only trivial ultrafilters if and only if it is finite. As a matter of fact, if X is finite and $\mathscr{U}$ is an ultrafilter on X, then, $\mathscr{U}$ is trivial by the previous statement. Conversely, if X is infinite, then the collection $\mathcal{F}$ of all the subsets of X with finite complement (in X) has the finite intersection property. Thus, by Lemma 1.9(1,2), there is an ultrafilter $\mathscr{U}$ on X containing $\mathcal{F}$ and it is nontrivial, since $X \setminus \{x\} \subseteq \mathcal{F} \subseteq \mathscr{U}$, for each $x \in X$.

2.0.2 The Zariski topology on collections of valuation domains

Let K be a field and A be a subring of K. We will denote by $\mathrm{Zar}(K|A)$ the set of all the valuation domains whose quotient field is K, containing A. If A_1 is the fundamental subring of K, then we shall denote simply by $\mathrm{Zar}(K)$ the set $\mathrm{Zar}(K|A_1)$.

As it is well known, Zariski [62] (or, [63, Volume II, Chapter VI, §1, page 110]) introduced and studied the set $Z := \mathrm{Zar}(K|A)$ together with a topological structure defined by taking, as a basis for the open sets, the subsets $B_F^Z := \{V \in Z : V \supseteq F\}$, for F varying in $\mathscr{B}_{\mathtt{fin}}(K)$, i.e., if $F := \{x_1, x_2, \dots, x_n\} \subseteq K$, then

$$B_F^Z = \mathrm{Zar}(K|A[x_1, x_2, \dots, x_n]).$$

This topology is called the *Zariski topology on* $Z = \mathrm{Zar}(K|A)$ and Z, equipped with this topology, denoted also later by Z^{zar}, is usually called the *(abstract) Zariski-Riemann surface of* K *over* A.

When no confusion can arise, we will simply denote by B_F the open set B_F^Z, and, with a small abuse of notation, by B_x the open set $B_{\{x\}}$, for $x \in K$. We will call the basis $\{B_F : F \in \mathscr{B}_{\mathtt{fin}}(K)\}$ *the natural basis of* $\mathrm{Zar}(K|A)^{\mathrm{zar}}$.

2.0.3 The preorder induced by a topology

If X is a topological space, it is well known that the topology induces on X the following preorder

$$x \preccurlyeq y :\Longleftrightarrow y \in \mathrm{Ad}(\{x\})$$

It is immediately seen that $\preccurlyeq$ is a partial order on X if and only if X satisfies the axiom T_0. Moreover, $\preccurlyeq$ is the trivial partial order if and only if X satisfies the axiom T_1. For each subset Y of X, set

$$Y^{\mathrm{sp}} := \{x \in X : [x \preccurlyeq y, \text{ for some } y \in Y]\}$$

$$Y^{\mathrm{gen}} := \{x \in X : [y \preccurlyeq x, \text{ for some } y \in Y]\}$$

We say that Y^{sp} is *the closure of* X *under specializations* and that Y^{gen} is *the generic closure of* Y.

For each subset Y of X, it is immediately seen that $Y \subseteq Y^{\mathrm{sp}} \cap Y^{\mathrm{gen}}$. We say that Y is *closed under specialization* (resp., *generically closed*) if

$Y = Y^{\mathrm{sp}}$ (resp. $Y = Y^{\mathrm{gen}}$). It is straightforward that closed subsets (resp., open subsets) of X are closed under specializations (resp., generically closed).

If $X = \mathrm{Zar}(K|A)$, for some field K and some subring A of K, and Y is a subset of X, we will call *the Zariski–generic closure of* Y the generic closure of Y, with respect to the Zariski topology.

2.0.4 Semistar operations on integral domains and their Kronecker function ring

Let A be an integral domain and let K be the quotient field of A. We denote by $\overline{\boldsymbol{F}}(A)$ the set of all the nonzero A-submodules of K, and by $\boldsymbol{f}(A)$ the set of all the nonzero finitely generated A-submodules of K.

2.4 Definition. *Let A be an integral domain and let K be the quotient field of A.*

(1) *A map* $\star : \overline{\boldsymbol{F}}(A) \longrightarrow \overline{\boldsymbol{F}}(A)$, $E \mapsto E^\star$, *is called* a semistar operation on A *if, for each* $0 \neq x \in K$ *and for all* $E, F \in \overline{\boldsymbol{F}}(A)$, *the following properties hold:*

($\star_1$) $(xE)^\star = xE^\star$;

($\star_2$) $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

($\star_3$) $E \subseteq E^\star$ *and* $(E^\star)^\star = E^\star$.

(2) *A* semistar operation of finite type $\star$ *on A is a semistar operation such that, for every* $E \in \overline{\boldsymbol{F}}(A)$,

$$E^\star = E^{\star_f} := \bigcup \{F^\star : F \in \boldsymbol{f}(A),\ F \subseteq E\}\,.$$

(3) *An* `e.a.b.` semistar operation $\star$ *on A is a semistar operation such that, for all* $F, G, H \in \boldsymbol{f}(A)$, $(FG)^\star \subseteq (FH)^\star$ *implies* $G^\star \subseteq H^\star$.

(4) *Let $\star$ be a semistar operation on A. A valuation domain* $V \in \mathrm{Zar}(K|A)$ *is called a* $\star$–valuation overring of A *if* $F^\star \subseteq FV$, *for each* $F \in \boldsymbol{f}(A)$. *We denote by* $\mathrm{Zar}^\star(K|A)$ *the collection of all the $\star$–valutation overring of A.*

Important classes of examples of semistar operations are obtained as follows.

2.5 Proposition. *Let A be an integral domain, K be its quotient field and let $\mathcal{S}$ be a nonempty collection of overrings of A. Then, the following statements hold.*

(1) (M. Fontana, J. Huckaba [22, Theorem 1.2(C)]). *The function*

$$\wedge_{\mathcal{S}} : \overline{\boldsymbol{F}}(A) \to \overline{\boldsymbol{F}}(A) \quad E \mapsto \bigcap \{ES : S \in \mathcal{S}\}$$

is a semistar operation on A.

(2) (M. Fontana, A. Loper [25, Proposition 7]) *If $\mathcal{S} \subseteq \mathrm{Zar}(K|A)$, then the semistar operation $\wedge_{\mathcal{S}}$ is* `e.a.b.`.

Preserve the notation of the previous Proposition. We say that a semistar operation $\star$ on A is *complete* if $\star = b(\star) := \wedge_{\mathrm{Zar}^\star(K|A)}$. For any semistar operation $\star$ on A, it is easily seen that $F^{b(\star)}V = F^\star V$, for each $F \in \boldsymbol{f}(A)$ and $V \in \mathrm{Zar}^\star(K|A)$. Thus, $b(b(\star)) = b(\star)$ and $b(\star)$ is a complete semistar operation. The *b–operation* on A is the `e.a.b.` semistar operation defined by $b := \wedge_{\mathrm{Zar}(K|A)}$ and, obviously, $b \leq b(\star)$ (i.e., $E^b \subseteq E^{b(\star)}$ for each $E \in \overline{\boldsymbol{F}}(A)$) for all semistar operations $\star$ on A.

2.6 Proposition (M. Fontana, A. Loper [23]). *Let A be an integral domain, K be its quotient field, and $\star$ be an* `e.a.b.` *semistar operation on A. Set*

$$\mathrm{Kr}(A, \star) := \{f/g \in K(T) : f, g \in A[T],\ g \neq 0,\ \textit{and}\ c(f)^\star \subseteq c(g)^\star\}$$

Then, the following statements hold.

(1) $\mathrm{Kr}(A, \star)$ *is a Bézout domain whose quotient field $K(T)$, and we will call it* the $\star$–Kronecker function ring of A.

(2) *If $Y \subseteq \mathrm{Zar}(K|A)$, then* $\mathrm{Kr}(A, \wedge_Y) = \bigcap \{V(T) : V \in Y\}$.

2.0.5 The constructible topology on $\mathrm{Spec}(A)$

Let A be a ring. We shall denote by $\mathrm{Spec}(A)^{\mathrm{zar}}$ the set of all the prime ideals of A, endowed with the Zariski topology (see Section 1.0 of Chapter 1). For each $a \in A$, set $D_a := \mathrm{Spec}(A) \setminus V(aA)$. We will call the sets D_a *the principal open sets of* $\mathrm{Spec}(A)$. It is well known that the collection of all the principal open sets of $\mathrm{Spec}(A)$ is a basis of open sets for $\mathrm{Spec}(A)^{\mathrm{zar}}$, that $\mathrm{Spec}(A)^{\mathrm{zar}}$ is always compact and satisfies the axiom T_0. Moreover, the axiom T_1 and the

Hausdorff axiom are equivalent on $\mathrm{Spec}(A)^{\mathrm{zar}}$. More precisely, $\mathrm{Spec}(A)^{\mathrm{zar}}$ is a Hausdorff topological space if and only if A is zero dimensional (see [30, Theorem 3.6] or [56, Théorème 1.3]). This means that the Zariski topology on $\mathrm{Spec}(A)$ is very coarse. Thus many authors have considered on $\mathrm{Spec}(A)$ a topology that is finer (or equal) to the Zariski topology, known as *the constructible topology* ([28, pages 337-339] or [2, Chapter 3, Exercises 27, 28 and 30]) or as the *patch topology* [40]. More precisely, let $\mathcal{U}(A)$ be the set of all the open and compact subspaces of $\mathrm{Spec}(A)^{\mathrm{zar}}$. It is obvious that the boolean subalgebra $\mathcal{B}_{\mathrm{cons}}$ of $\mathscr{B}(\mathrm{Spec}(A))$ generated by $\mathcal{U}(A)$ can be taken as a basis for a (unique) topology on $\mathrm{Spec}(A)$. Then, we call this topology *the constructible topology on* $\mathrm{Spec}(A)$, and call all the elements of $\mathcal{B}_{\mathrm{cons}}$ *the constructible subsets of* $\mathrm{Spec}(A)$. We shall denote the set $\mathrm{Spec}(A)$ endowed with the constructible topology by $\mathrm{Spec}(A)^{\mathrm{cons}}$. Since all the principal open sets of $\mathrm{Spec}(A)$ are compact, they are constructible. It follows that the constructible topology is finer or equal than the Zariski topology. In the following result, we collect some properties of the constructibile topology.

2.7 THEOREM. ([2, Chapter 3, Exercises 27, 28] and [28, (**0**.2.3.11), (**0**.2.4.1) and (**I**.7.2.12)]) *Let A be a ring. Then, the following statements hold.*

(1) $\mathrm{Spec}(A)^{\mathrm{cons}}$ *is a compact, Hausdorff and totally disconnected topological space.*

(2) $\mathcal{B}_{\mathrm{cons}}$ *is the collection of all the clopen subsets of* $\mathrm{Spec}(A)^{\mathrm{cons}}$*. Moreover, the constructible topology is the coarsest topology on* $\mathrm{Spec}(A)$ *for which* $\mathcal{B}_{\mathrm{cons}}$ *is a collection of clopen sets.*

(3) *A subset Y of* $\mathrm{Spec}(A)^{\mathrm{cons}}$ *is closed if and only if there exists a ring homomorphism $f : A \longrightarrow B$ such that $Y = f^*(\mathrm{Spec}(B))$.*

(4) *If* $\mathrm{Spec}(A)^{\mathrm{zar}}$ *is Noetherian, then the constructible subsets of* $\mathrm{Spec}(A)$ *are exactly the finite unions of locally finite subsets of* $\mathrm{Spec}(A)^{\mathrm{zar}}$*.*

2.0.6 The ultrafilter topology on $\mathrm{Spec}(A)$

Let A be a ring. Recently, Fontana and Loper in [26] have defined a topology on $\mathrm{Spec}(A)$ by using the notion of an ultrafilter. Let Y be a subset of $\mathrm{Spec}(A)$ and let $\mathscr{U}$ be an ultrafilter on Y. If we set

$$\mathfrak{p}_{\mathscr{U}} := \{a \in A : V(a) \cap Y \in \mathscr{U}\},$$

then it is easily seen, by an argument similar to that given in [11, Lemma 2.4], that $\mathfrak{p}_{\mathscr{U}}$ is a prime ideal of A. We call $\mathfrak{p}_{\mathscr{U}}$ an *ultrafilter limit point of* Y *in* $\mathrm{Spec}(A)$. This notion of ultrafilter limit points of collections of prime ideals has been used to great effect in several recent papers [11], [51], and [52].

If $\mathscr{U}$ is a trivial ultrafilter on Y, that is $\mathscr{U} = \{Z \in \mathscr{B}(Y) : \mathfrak{p} \in Z\}$ $(=: \beta_Y^{\mathfrak{p}})$ for some $\mathfrak{p} \in Y$, then it is straightforward that $\mathfrak{p}_{\mathscr{U}} = \mathfrak{p} \in Y$ [26, page 2918]. On the other hand, if $\mathscr{U}$ is nontrivial, it may happen that $\mathfrak{p}_{\mathscr{U}} \notin Y$. For example, let $A := \mathbb{Z}$ and let $\mathscr{U}$ be *any* nontrivial ultrafilter on $Y := \mathrm{Max}(\mathbb{Z})$. Then, it is immediate that $\mathfrak{p}_{\mathscr{U}} = \{0\}$ (in fact, if $x \in \mathfrak{p}_{\mathscr{U}}$, then the set $V(x) \cap Y$ belongs to $\mathscr{U}$ and it is infinite, by Remark 2.3(2), thus $x = 0$). That motivates the following definition. We say that a subset Y of $\mathrm{Spec}(A)$ is *ultrafilter closed* if Y contains all of its ultrafilter limit points. It is not hard to see that the ultrafilter closed subsets of $\mathrm{Spec}(A)$ define a topology on the $\mathrm{Spec}(A)$, called *the ultrafilter topology on* X [26, Definition 1]. We denote by $\mathrm{Spec}(A)^{\mathrm{ultra}}$ the set of prime ideals of A endowed with the ultrafilter topology. One of the main results of a recent paper be Fontana and Loper is the following.

2.8 Theorem. (M. Fontana, A. Loper [26, Theorem 8]) *Let A be a ring. Then, the ultrafilter topology and the constructible topology on* $\mathrm{Spec}(A)$ *are identical.*

The following result will be useful in the following.

2.9 Proposition. *Let A be a ring and let Y be a subset of* $\mathrm{Spec}(A)$. *Then,*

$$\mathrm{Ad}^{\mathrm{ultra}}(Y) = \{\mathfrak{p}_{\mathscr{U}} : \mathscr{U} \in \beta Y\}$$

Proof. If $\mathcal{P}$ is the basis of the principal open sets of $\mathrm{Spec}(A)^{\mathrm{zar}}$, then the ultrafilter topology is identical to the $\mathcal{P}-$ultrafilter topology (Remark 3.6(3)). The conclusion is now clear, by Example 3.1(2) and Proposition 3.11. □

2.1 The ultrafilter topology on $\mathrm{Zar}(K|A)$

Let K be a field and A a subring of K. Taking as starting point the situation on the prime spectrum of a ring, the next goal is a study of some distinguished topologies on the space $Z := \mathrm{Zar}(K|A)$ that are finer than the Zariski topology.

We start by recalling a very useful fact.

2.10 PROPOSITION. *Let K be a field and A a subring of K. If Y is a nonempty subset of Z :=Zar$(K|A)$ and $\mathscr{U}$ is an ultrafilter on Y, then $A_{\mathscr{U},Y} := A_{\mathscr{U}} := \{x \in K : B_x \cap Y \in \mathscr{U}\}$ is a valuation domain belonging to Z. We will call $A_{\mathscr{U},Y}$* the ultrafilter limit point of Y, with respect to $\mathscr{U}$*.*

Proof. By [11, Lemma (2.9)], $A_{\mathscr{U}}$ is a valuation domain having K as quotient field. It remains to show that $A \subseteq A_{\mathscr{U}}$. This follow immediately noting that, for every $x \in A$, we have $B_x = Z$, and hence $B_x \cap Y = Y \in \mathscr{U}$. □

2.11 REMARK. The previous statement shows that, if $Y \subseteq Z :=$ Zar$(K|A)$, we have a canonical map:

$$\pi_Y : \beta Y \longrightarrow Z\,, \quad \mathscr{U} \mapsto A_{\mathscr{U},Y} := \{x \in K : B_x \cap Y \in \mathscr{U}\},$$

and, in this case, $Y \subseteq \pi_Y(\beta Y)$, since for each $V \in Y$, taking the trivial ultrafilter $\beta_Y^V \in \beta(Y)$, we have $A_{\beta_Y^V,Y} = V$.

The previous remark leads naturally to the following crucial definition of this section.

2.12 DEFINITION. *Let K be a field and A a subring of K. A subset Y of* Zar$(K|A)$ *is called* stable for ultrafilters *if $A_{\mathscr{U},Y} \in Y$, for each $\mathscr{U} \in \beta Y$ (or, equivalently, with the notation of Remark 2.11, $\pi_Y(\beta Y) = Y$).*

2.13 PROPOSITION. *Let K be a field, A a subring of K and $Z :=$* Zar$(K|A)$*. Then, the collection of all subsets of Z stable for ultrafilters is the family of closed sets for a topology on Z called* the ultrafilter topology *of the Zariski-Riemann surface Z.*

Proof. The empty set and Z are clearly stable for ultrafilters. Now, consider two subsets C', C'' of Z stable for ultrafilters, set $Y := C' \cup C''$, and let $\mathscr{U}$ be an ultrafilter on Y. By Lemma 2.2(5), we can assume, without loss of generality, that $C' \in \mathscr{U}$. Then, $\mathscr{U}' := \mathscr{U}^{C'} := \{Z \cap C' : Z \in \mathscr{U}\}$ is an ultrafilter on C', by Lemma 2.2(3). We want to show that $A_{\mathscr{U}} = A_{\mathscr{U}'}$. Let $x \in A_{\mathscr{U}'}$. Then, $B_x \cap C' \in \mathscr{U}' \subseteq \mathscr{U}$ (by Lemma 2.2(3)). Since $B_x \cap C' \subseteq B_x \cap Y$, it follows immediately that $B_x \cap Y \in \mathscr{U}$ and hence $x \in A_{\mathscr{U}}$ $(= \{x \in K : B_x \cap Y \in \mathscr{U}\})$. Conversely, let $x \in A_{\mathscr{U}}$. Since $B_x \cap Y \in \mathscr{U}$, we have $B_x \cap C' = (B_x \cap Y) \cap C' \in \mathscr{U}'$. Hence, $x \in A_{\mathscr{U}'}$ $(= \{x \in K : B_x \cap C' \in \mathscr{U}'\})$ and so $A_{\mathscr{U}} = A_{\mathscr{U}'}$. As C' is stable for ultrafilters, we have $A_{\mathscr{U}} = A_{\mathscr{U}'} \in C' \subseteq Y$ and so Y is also stable for ultrafilters. By induction, we easily deduce that the union of a finite family of subsets stable for ultrafilters is still stable for ultrafilters. Now, let $\mathscr{C}$ be any collection of subsets stable for ultrafilters in Z and set $Y := \bigcap \mathscr{C}$. Let $\mathscr{U}$ be an ultrafilter on Y. For every $C \in \mathscr{C}$, clearly $Y \subseteq C$ and so, by Lemma 2.2(4), $\mathscr{U}_C := \{W \in \mathscr{B}(C) : W \cap Y \in \mathscr{U}\}$ is an ultrafilter on C. Moreover, as before, it is easily seen that $A_{\mathscr{U}} = A_{\mathscr{U}_C} \in C$. This proves that $A_{\mathscr{U}} \in \bigcap \mathscr{C}$, and thus every intersection of subsets of Z stable for ultrafilters is still stable for ultrafilters. $\square$

As above, let $Z := \mathrm{Zar}(K|A)$, we denote by Z^{ultra} [respectively, Z^{zar}] the space of valuation domains of K containing A equipped with the ultrafilter topology [respectively, with the with the Zariski topology].

We note that the next results on the ultrafilter topology on $\mathrm{Zar}(K|A)$ will be proved in the next chapter, in a more general setting. Moreover, Proposition 2.13 may be also deduced from the next chapter. We gave the proof just to illustrate the use of ultrafilters.

2.14 THEOREM. *Let K be a field and A be a subring of K. The following properties hold.*

(1) *The ultrafilter topology is finer than the Zariski topology on Z.*

(2) *For each subset S of K, the set $B_S^Z (:= B_S := \{V \in Z : V \supseteq S\})$ is closed in the ultrafilter topology. In particular, the basic open sets of the Zariski topology are clopen in the ultrafilter topology.*

(3) *We denote by $Z^\sharp$ the set Z endowed with the* $\sharp$-topology*, defined as the coarsest topology for which the set B_F is clopen, for each finite subset F of K. Then, $Z^\sharp$ is a Hausdorff topological space.*

(4) *The $\sharp$-topology is the coarsest topology on Z for which the closed subsets and the open and compact subsets of Z^{zar} are closed, i.e. the collection of sets*

$$\mathcal{C}^\sharp := \left\{ B_F \ ; \bigcap \{Z \backslash B_G : G \in \mathcal{G}\} : F, G \in \mathscr{B}_{\mathtt{fin}}(K),\, \mathcal{G} \subseteq \mathscr{B}_{\mathtt{fin}}(K) \right\}$$

is a subbasis of closed subsets of $Z^\sharp$.

(5) *Z^{ultra} is a compact, Hausdorff and totally disconnected topological space.*

(6) *The $\sharp$-topology and the ultrafilter topology on Z are identical.*

Proof. If $\mathcal{Q} := \{B_F : F \in \mathscr{B}_{\texttt{fin}}(K)\}$ is the natural basis of Z^{zar}, then the ultrafilter topology on Z is identical to the $\mathcal{Q}-$ultrafilter topology (Remark 3.6(4)).

(1) follows by Proposition 3.16(1).

(2) follows by Proposition 2.10, keeping in mind that $B_S = \mathrm{Zar}(K|A[S])$.

(3) Let V, W be distinct valuation domains in Z. Without loss of generality, we can assume that there exists an element $x \in V \setminus W$. Then, B_x and $Z \setminus B_x$ are (disjoint) open neighborhood, in the $\sharp$-topology, of V and W, respectively.

(4) It is clear that each set in $\mathcal{C}^\sharp$ is closed in the $\sharp$-topology and every topology in which the sets of type B_F (for $F \in \mathscr{B}_{\texttt{fin}}(K)$) are clopen must be finer than the topology having $\mathcal{C}^\sharp$ as subbasis for the closed sets. Conversely, it is obvious that, in this last topology, each set of type B_F (for $F \in \mathscr{B}_{\texttt{fin}}(K)$) is clopen.

(5) Let $\mathscr{U}$ be an ultrafilter on Z. By Example 3(3), $Z_{\mathcal{Q}}(\mathscr{U}) = \{A_{\mathscr{U}}\}$. Then, compactness of Z^{ultra} follows by Theorem 3.12. Moreover, Z^{ultra} is Hausdorff and totally disconnected by Proposition 3.16(2).

(6) follows immediately by (5) and Proposition 3.16(3). □

2.15 Remark. Note that an immediate consequence of Theorem 2.14(1,5) is the following classical result: *if K is a field and A is a subring of K, then* $\mathrm{Zar}(K|A)^{\text{zar}}$ *is a compact topological space* ([63, Chapter VI, Theorem (40)]).

2.16 Proposition. *Let K be a field, A be a subring of K and $Z :=$ $\mathrm{Zar}(K|A)$. Denote by $\mathrm{Ad}^{\text{ultra}}(Y)$ the closure of a subset of Y of Z^{ultra}. Then,*

$$\mathrm{Ad}^{\text{ultra}}(Y) = \{A_{\mathscr{U}} : \mathscr{U} \in \beta Y\}$$

Proof. As in the beginning of the proof of Theorem 2.14, let $\mathcal{Q}$ be the natural basis of Z^{zar}. Since, for each ultrafilter $\mathscr{U}$ on Y, $Y_{\mathcal{Q}}(\mathscr{U}) = \{A_{\mathscr{U}}\}$ (Example 3.1(3)), then the conclusion follows immediately by applying Remark 3.6(4) and Proposition 3.11. □

2.2 The map $\mathrm{Zar}(K/A)^{\text{ultra}} \longrightarrow \mathrm{Spec}(A)^{\text{ultra}}$

As is well known, if K is a field and A is a subring of K, we can construct a map $\gamma : \mathrm{Zar}(K|A) \to \mathrm{Spec}(A)$ sending a valutation ring $V \in \mathrm{Zar}(K|A)$, with maximal ideal M_V, to the prime ideal $M_V \cap A$ of A, called *the center of* V *over* A. It is well known (by an application of Zorn's Lemma) that γ is a surjective map.

Moreover, if we consider $Z := \mathrm{Zar}(K|A)$ and $X := \mathrm{Spec}(A)$ as topological spaces both endowed with the Zariski topology, then, by [15, Lemma (2.1)], the map $\gamma : Z^{\mathrm{zar}} \longrightarrow X^{\mathrm{zar}}$ is continuous, since $\gamma^{-1}(D_a) = B_{a^{-1}}$, for each nonzero $a \in A$. Moreover, $\gamma : Z^{\mathrm{zar}} \longrightarrow X^{\mathrm{zar}}$ is also a closed map, essentially by [15, Theorem (2.5)]. In particular, $\gamma : Z^{\mathrm{zar}} \longrightarrow X^{\mathrm{zar}}$ is a homeomorphism if and only if γ is injective (i.e., if and only if for each $\mathfrak{p} \in \mathrm{Spec}(A)$ there exists a unique valuation domain of K dominating $A_{\mathfrak{p}}$). In particular, if A is a Prüfer domain with quotient field K, then $\gamma : Z^{\mathrm{zar}} \longrightarrow X^{\mathrm{zar}}$ is a homeomorphism.

The next goal is to study the map γ when $Z := \mathrm{Zar}(K|A)$ and $X := \mathrm{Spec}(A)$ are both equipped with the ultrafilter topology.

2.17 Theorem. *Let K be a field and A a subring of K. Then, the surjective map $\gamma : \mathrm{Zar}(K|A)^{\mathrm{ultra}} \longrightarrow \mathrm{Spec}(A)^{\mathrm{ultra}}$ is continuous and closed.*

Proof. Set as usual $Z := \mathrm{Zar}(K|A)$ and $X := \mathrm{Spec}(A)$. Since Z^{ultra} is compact, by Theorem 2.14(5), and $X^{\mathrm{ultra}} = X^{\mathrm{cons}}$ is Hausdorff (and compact), by [18, Chapter XI, Theorem 2.1]), it is enough to show that γ is continuous. Set $\mathcal{Q} := \{B_F : F \in \mathscr{V}_{\mathtt{fin}}(K)\}$, $\mathcal{P} := \{D_a : a \in A\}$. Since, for each $a \in A \setminus \{0\}$, $\gamma^{-1}(D_a) = B_{a^{-1}}$, it follows that $\{\gamma^{-1}(Y) : Y \in \mathcal{P}\} \subseteq \mathcal{Q}$. Thus the conclusion is an immediate consequence of Remark 3.6(3,4) and Proposition 3.15. $\square$

2.18 Remark. Note that, by the previous Theorem 2.17, if A is a Prüfer domain, the map $\gamma : \mathrm{Zar}(K|A)^{\mathrm{ultra}} \longrightarrow \mathrm{Spec}(A)^{\mathrm{ultra}}$ is a homeomorphism, since in this case γ is injective, as observed before.

2.3 Kronecker function rings and Zariski-Riemann surfaces

Recall that a *spectral space* (or *coherent space*) is a topological space homeomorphic to the prime spectrum of a ring equipped with the Zariski topology (see [40] and [47]). In 1969, M. Hochster gave the following characterization of spectral spaces.

2.19 Theorem. (M. Hochster [40, Proposition 4]) *Let X be a topological space. The following conditions are equivalent.*

(i) *X is a spectral space.*

(ii) *X is compact, admits a basis of open and compact subsets closed under finite intersection, and each irreducible subspace of X has a unique generic point.*

Let K be a field and A be *any* subring of K. In this section, we will show, in a constructive way, that both $\mathrm{Zar}(K|A)^{\mathrm{zar}}$ and $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$ are spectral spaces. Let K be a field and T an indeterminate over K. For every $W \in \mathrm{Zar}(K(T))$, it is well known that $V := W \cap K \in \mathrm{Zar}(K)$ [29, Theorem 19.16(a)] and conversely, for each $V \in \mathrm{Zar}(K)$, there are infinitely many valuation domains W of $K(T)$ such that $W \cap K = V$, called *extensions of V to $K(T)$* [29, Proposition 20.11]. Among the extensions of a valuation V of K to $K(T)$, there is a distinguished one, called *the trivial extension of V to $K(T)$*, which is $V(T) := V[T]_{\mathfrak{m}[T]}$, where $\mathfrak{m}$ is the maximal ideal of V [29, Proposition 18.7].

2.20 Proposition. *Let K be a field and T an indeterminate over K.*

(1) *The canonical map $\varphi : \mathrm{Zar}(K(T))^{\mathrm{zar}} \longrightarrow \mathrm{Zar}(K)^{\mathrm{zar}}$, $W \mapsto W \cap K$, is a continuous surjection.*

(2) *Let $\mathrm{Zar}_0(K(T)) := \{V(T) \in \mathrm{Zar}(K(T)) : V \in \mathrm{Zar}(K)\}$. Then,*

$$\varphi|_{\mathrm{Zar}_0(K(T))} : \mathrm{Zar}_0(K(T))^{\mathrm{zar}} \longrightarrow \mathrm{Zar}(K)^{\mathrm{zar}}$$

is a homeomorphism.

Proof. (1) The map φ is clearly surjective by the previous remarks. It is also a continuous map since, for each finite subset F of K and for each basic open set $B_F^{\mathrm{Zar}(K)}$ of $\mathrm{Zar}(K)^{\mathrm{zar}}$, it is straightforward to see that $\varphi^{-1}(B_F^{\mathrm{Zar}(K)}) = B_F^{\mathrm{Zar}(K(T))}$.

(2) It is obvious that $\varphi|_{\mathrm{Zar}_0(K(T))} : \mathrm{Zar}_0(K(T))^{\mathrm{zar}} \longrightarrow \mathrm{Zar}(K)^{\mathrm{zar}}$ is a bijection and, by (1), is a continuous map. The conclusion will follows if we show that the map $\varphi|_{\mathrm{Zar}_0(K(T))}$ is also open. Let $h \in K(T)\setminus\{0\}$, say

$$h := \frac{a_0 + a_1T + \ldots + a_rT^r}{b_0 + b_1T + \ldots + b_sT^s},$$

with a_i and b_j in K, for $i = 0, 1, \ldots, r$ and $j = 0, 1, \ldots, s$. Let $V(T)$ be a valutation domain in $\mathrm{Zar}_0(K(T))$, let v be the valuation on K defining V

and let v^* be the valuation associated to $V(T)$, i.e., $v^*(T) = v(1) = 0$ and, for each nonzero polynomial $f := a_0 + a_1T + \ldots + a_rT^r \in K[T]$, $v^*(f) := \min\{v(a_i) : i = 0, 1, \ldots, r\}$.

It is easy to see that $V(T) \in B_h^{\mathrm{Zar}(K(T))}$ if and only if $v^*(h) \geq 0$, that is, if and only if

$$\min\{v(a_i) : i = 0, 1, \ldots, r\} \geq \min\{v(b_j) : j = 0, 1, \ldots, s\}.$$

Now, for all $i \in \{0, 1, \ldots, r\}$ and $j \in \{0, 1, \ldots, s\}$ such that both a_i and b_j are nonzero, set:

$$F_{ij} := \left\{ \frac{a_i}{b_j},\ \frac{a_\lambda}{a_i},\ \frac{b_\mu}{b_j} : \lambda = 0, 1, \ldots, r,\ \mu = 0, 1, \ldots, s \right\}.$$

We claim that that $\varphi(B_h^{\mathrm{Zar}_0(K(T))}) = \bigcup_{i,j} B_{F_{ij}}^{\mathrm{Zar}(K)}$ (the argument will be similar to that given in the proof of [16, Lemma 1]). As a matter of fact, let $V \in \varphi(B_h^{\mathrm{Zar}_0(K(T))})$. This means that $h \in V(T)$ and, by the discussion above, this is equivalent to the inequality $v(a_{i^*}) \geq v(b_{j^*})$, where $v(a_{i^*}) = \min\{v(a_i) : i = 0, \ldots, r\}$ and $v(b_{j^*}) = \min\{v(b_j) : j = 0, \ldots, s\}$. It follows immediately that $V \in B_{F_{i^*j^*}}^{\mathrm{Zar}(K)}$. Conversely, let i, j such that $a_i, b_j \neq 0$ and such that $V \supseteq F_{ij}$. This means that $v(a_i) \geq v(b_j)$, $v(a_\lambda) \geq v(a_i)$ and $v(b_\mu) \geq v(b_j)$, for each $(\lambda, \mu) \in \{0, \ldots, r\} \times \{0, \ldots, s\}$. It follows, in particular, that $v(a_i) = \min\{v(a_\lambda) : \lambda = 0, \ldots, r\}$ and $v(b_j) = \min\{v(b_\mu) : \mu = 0, \ldots, s\}$. Then, the inequality $v(a_i) \geq v(b_j)$ and the discussion at the beginning of the proof of (2) imply $h \in V(T)$, that is, $V \in \varphi(B_h^{\mathrm{Zar}_0(K(T))})$. By the equality proved now, it follows that the continuous bijective map $\varphi|_{\mathrm{Zar}_0(K(T))}$ is open, and so it is a homeomorphism. $\square$

2.21 PROPOSITION. *We preserve the notation of Proposition 2.20 and, now, let* $\mathrm{Zar}(K(T))$ *and* $\mathrm{Zar}(K)$ *be endowed with the ultrafilter topology. Then, the canonical (surjective) mapping* $\varphi : \mathrm{Zar}(K(T))^{\mathrm{ultra}} \longrightarrow \mathrm{Zar}(K)^{\mathrm{ultra}}$ *is continuous and (hence) closed. In particular,* $\varphi|_{\mathrm{Zar}_0(K(T))}$ *is a homeomorphism of* $\mathrm{Zar}_0(K(T))^{\mathrm{ultra}}$ *onto* $\mathrm{Zar}(K)^{\mathrm{ultra}}$.

Proof. . Set

$$\mathcal{Q} := \{B_F^{\mathrm{Zar}(K)} : F \in \mathscr{B}_{\mathtt{fin}}(K)\}, \quad \mathcal{Q}' := \{B_F^{\mathrm{Zar}(K(T))} : F \in \mathscr{B}_{\mathtt{fin}}(K(T))\}$$

Since, as observed before, $\varphi^{-1}(B_F^{\mathrm{Zar}(K)}) = B_F^{\mathrm{Zar}(K(T))}$, it follows immediately that $\{\varphi^{-1}(Y) : Y \in \mathcal{Q}\} \subseteq \mathcal{Q}'$. Then, the first statement is a consequence of Remark 3.6(4) and Proposition 3.15. Note that φ is closed, again by [18, Chapter XI, Theorem 2.1]. Finally, last statement follows keeping in mind that the restriction of φ to $\mathrm{Zar}_0(K(T))$ is bijective. $\square$

Now, we recall the following key notion introduced by Halter-Koch [34, Definition 2.1], providing an axiomatic approach to the theory of Kronecker function rings.

2.22 Definition. *Let K be field, T an indeterminate over K, and S a subring of $K(T)$. We call S* a K–function ring *(after Halter-Koch) if T and T^{-1} belong to S and, for each nonzero polynomial $f \in K[T]$, $f(0) \in f(T)S$.*

We collect in the next proposition several properties of K–function ring that will be useful in the following.

2.23 Proposition. *Let K be a field, T an indeterminate over K and let S be a subring of $K(T)$. Assume that S is a K–function ring.*

(1) *If S' is subring of $K(T)$ containing S, then S' is also a K–function ring.*

(2) *If $\mathscr{S}$ is a nonempty collection of K–function rings (in $K(T)$), then $\bigcap \mathscr{S}$ is a K–function ring.*

(3) *S is a Bézout domain with quotient field $K(T)$.*

(4) *If $f := f_0 + f_1T + \ldots + f_rT^r \in K[T]$, then $(f_0, f_1, \ldots, f_r)S = fS$.*

(5) *For every valuation domain V of K, $V(T)$ is a K–function ring.*

Proof. (1), (2), (3) and (4) were proved in [34, Remarks at page 47 and Theorem (2.2)]. To prove (5), observe that, if v is the valuation associated to V and v^* is the trivial extension of v to $K(T)$ [29, page 218], then $v^*(T) = v(1) = 0$ or, equivalently, T is invertible in $V(T)$. Moreover, if $f := f_0 + f_1T + \ldots + f_rT^r \in K[T]$, then $v^*(f) \leq v(f_0) = v^*(f_0)$, and so $f(0) = f_0 \in fV(T)$. $\square$

The following fact provides a slight generalization of [36, Theorem 2.3] and its proof is similar to that given by O. Kwegna Heubo, which is based on the work by Halter-Koch [34].

2.24 Proposition. *Let K be a field, T an indeterminate over K and R a K–function ring. Then,* $\mathrm{Zar}(K(T)|R) = \mathrm{Zar}_0(K(T)|R)$ *(i.e., for every valuation domain* $W \in \mathrm{Zar}(K(T)|R)$, $W = (W \cap K)(T)$*).*

Proof. Let W be a valutation overring of R. First, observe that $V := W \cap K$ is a valuation ring of K [29, Theorem 19.16(a)]. Now, let v be a valuation of K defining V and let $f := f_0 + f_1 T + \ldots + f_r T^r \in K[T]$, $f \neq 0$. By Proposition 2.23(1), since $R \subseteq W$, W is a K–function ring. Let w be a valuation of $K(T)$ defining W, since T and T^{-1} belong to W, we have $w(T) = 0$. Moreover, $w|_K = v$ and so $w(f) \geq \inf\{w(f_i) : i = 0, 1, \ldots, r\} = \inf\{v(f_i) : i = 0, 1, \ldots, r\}$.

On the other hand, by Proposition 2.23(4), $fR = f_0 R + f_1 + \ldots + f_r R$, and thus $f_i \in fR$, for every $i = 0, 1, \ldots, r$. Since $R \subseteq W$, we have $f_i \in fW$ and thus $w(f_i) = v(f_i) \geq w(f)$, for every $i = 0, 1, \ldots, r$. Therefore, $w(f) = \inf\{v(f_i) : i = 0, 1, \ldots, r\}$. This proves that $w = v^*$, and hence W is the trivial extension of V in $K(T)$. $\square$

2.25 Proposition. *Let K be a field, T an indeterminate over K, and R a subring of $K(T)$. Then, the following conditions are equivalent.*

(i) *R is a $K-$function ring.*

(ii) *R is an integrally closed in $K(T)$ and* $\mathrm{Zar}(K(T)|R) = \mathrm{Zar}_0(K(T)|R)$*.*

Proof. (i)⇒(ii) is already known (Propositions 2.23(3) and 2.24).
(ii)⇒(i). Since R is integrally closed in $K(T)$, the equality $\mathrm{Zar}(K(T)|R) = \mathrm{Zar}_0(K(T)|R)$ imply $R = \bigcap \mathrm{Zar}_0(K(T)|R)$. Now, the conclusion is clear, by Proposition 2.23(2, 5). $\square$

2.26 Remark. Let K be a field, T an indeterminate over K, and set $R_0 := \bigcap\{V(T) : V \in \mathrm{Zar}(K)\}$. Then, by Propositions 2.23(2) and 2.24, it follows that

$$\mathrm{Zar}_0(K(T)) = \mathrm{Zar}(K(T)|R_0).$$

In particular, $\mathrm{Zar}_0(K(T))$ is a closed subspace of $\mathrm{Zar}(K(T))^{\mathrm{ultra}}$, in view of Theorem 2.14(2).

As a consequence of Propositions 2.20(2), 2.21 and 2.24, we deduce immediately the following.

2.27 Corollary. *Let K be a field, T an indeterminate over K and $R(\subseteq K(T))$ a K–function ring. Set $A_R := R \cap K$. Then, the canonical map $\varphi : \mathrm{Zar}(K(T)|R) \longrightarrow \mathrm{Zar}(K|A_R)$, $W \mapsto W \cap K$, is a homeomorphism with respect to both Zariski topologies and ultrafilter topologies.*

As an application of the previous corollary we reobtain in particular [36, Corollary 2.2, Proposition 2.7 and Corollary 2.9]. More precisely,

2.28 Corollary. *Let K be a field, A* any *subring of K and T an indeterminate over K. Then,*

(1) $\mathrm{Kr}(K|A) := \bigcap\{V(T) : V \in \mathrm{Zar}(K|A)\}$ *is a K–function ring.*

(2) *The canonical map $\varphi : \mathrm{Zar}(K(T)|\mathrm{Kr}(K|A))^{\mathrm{zar}} \longrightarrow \mathrm{Zar}(K|A)^{\mathrm{zar}}$, $W \mapsto W \cap K$, is a homeomorphism.*

(3) *The canonical map $\psi : \mathrm{Spec}(\mathrm{Kr}(K|A))^{\mathrm{zar}} \longrightarrow \mathrm{Zar}(K|A)^{\mathrm{zar}}$, $\mathfrak{p} \mapsto \mathrm{Kr}(K|A)_{\mathfrak{p}} \cap K$, is a homeomorphism. In particular, $\mathrm{Zar}(K|A)^{\mathrm{zar}}$ is a spectral space.*

Proof. (1) By Proposition 2.23(2 and 5), $\mathrm{Kr}(K|A)$ is a K–function ring (in $K(T)$).

(2) Let $R := \mathrm{Kr}(K|A)$, then clearly $A_R := R \cap K$ coincides with the integral closure $\bar{A}$ of A in K. Therefore, (2) follows from Corollary 2.27, since $\mathrm{Zar}(K|A) = \mathrm{Zar}(K|\bar{A})$.

(3) Recall that, if $\mathcal{A}$ is a Prüfer domain and $\mathcal{K}$ is the quotient field of $\mathcal{A}$, by [16, Proposition 2.2], $\mathrm{Zar}(\mathcal{K}|\mathcal{A})^{\mathrm{zar}}$ is canonically homeomorphic to $\mathrm{Spec}(\mathcal{A})^{\mathrm{zar}}$ (under the map $\mathcal{V} \mapsto \mathfrak{m}_{\mathcal{V}} \cap \mathcal{A}$, where $\mathfrak{m}_{\mathcal{V}}$ is the maximal ideal of the valuation domain $\mathcal{V}$ of $\mathcal{K}$ containing $\mathcal{A}$). Now, the conclusion follows immediately, since $\mathrm{Kr}(K|A)$ is a Prüfer domain with quotient field $K(T)$ (Proposition 2.23(3)). $\square$

2.29 Remark. Observe that the noteworthy progress provided by Corollary 2.28 concerns the case where A is a proper subfield of K. As a matter of fact, if A is an integrally closed domain and K is its quotient field, the statements (2) and (3) of Corollary 2.28 were already proved in [16, Theorem 2].

An "ultrafilter-type" version of the previous corollary can also be deduced from Corollary 2.27.

2.30 Corollary. *Let K be a field, A* any *subring of K, T an indeterminate over K, and let $\mathrm{Kr}(K|A)$ be as in Corollary 2.28.*

(1) *The canonical map* $\varphi : \mathrm{Zar}(K(T)|\mathrm{Kr}(K|A))^{\mathrm{ultra}} \longrightarrow \mathrm{Zar}(K|A)^{\mathrm{ultra}}$*, defined by* $W \mapsto W \cap K$*, is a homeomorphism.*

(2) *The canonical map* $\psi : \mathrm{Spec}(\mathrm{Kr}(K|A)))^{\mathrm{ultra}} \longrightarrow \mathrm{Zar}(K|A)^{\mathrm{ultra}}$*, defined by* $\mathfrak{p} \mapsto \mathrm{Kr}(K|A)_{\mathfrak{p}} \cap K$*, is a homeomorphism.*

Proof. (1) As observed in the proof of Corollary 2.28(1), $\mathrm{Kr}(K|A)$ is a K–function ring, hence this statement follows from Corollary 2.27.

(2) is a consequence of (1), Theorem 2.17 and Remark 2.18. □

2.4 Applications

In this section, we give some applications of the previous results to the representations of integrally closed domains as intersections of valuation overrings.

2.31 PROPOSITION. *Let K be a field, A be a subring of K and U be a subset of $Z := \mathrm{Zar}(K|A)$. Let Y' and Y'' be two subsets of a given subset U of Z and assume that their closures in U, with the subspace topology induced by the ultrafilter topology of Z, coincide, i.e.,* $\mathrm{Ad}^{\mathrm{ultra}}(Y') \cap U = \mathrm{Ad}^{\mathrm{ultra}}(Y'') \cap U$*. Then,* $\bigcap\{V' : V' \in Y'\} = \bigcap\{V'' : V'' \in Y''\}$*. In particular, for each subset Y of Z,*

$$\bigcap\{V : V \in Y\} = \bigcap\{W : W \in \mathrm{Ad}^{\mathrm{ultra}}(Y)\}.$$

Proof. Assume, by contradiction, that there is an element $x_0 \in \bigcap\{V' : V' \in Y'\} \setminus \bigcap\{V'' : V'' \in Y''\}$, and pick a valuation domain $V_0 \in Y''$ such that $x_0 \notin V_0$. By Theorem 2.14(2), the set $\Omega := U \setminus B_{x_0}$ is an open subset of U, with respect to the subspace topology induced by Z^{ultra}, and it contains V_0. Since $V_0 \in Y'' \subseteq \mathrm{Ad}^{\mathrm{ultra}}(Y'') \cap U = \mathrm{Ad}^{\mathrm{ultra}}(Y') \cap U$ and $V_0 \notin Y'$, then $\Omega \cap Y'$ is nonempty. This implies that there exists a valuation domain $V' \in Y'$ such that $x_0 \notin V'$, a contradiction. □

2.32 DEFINITION. *Let Σ be a collection of subrings of a field K, having K as quotient field. We say that Σ is locally finite if each nonzero element of K is noninvertible in at most finitely many of the rings belonging to Σ. Moreover, we say that the ring $\bigcap \Sigma$ is* a locally finite intersection of the collection Σ*.*

The following easy result will provide a class of integral domains for which the equality $\bigcap\{V' : V' \in Y'\} = \bigcap\{V'' : V'' \in Y''\}$ does not imply, in general, that $\mathrm{Ad}^{\mathrm{ultra}}(Y') = \mathrm{Ad}^{\mathrm{ultra}}(Y'')$.

2.33 Lemma. *Let K be a field and A be a subring of K. If Σ is an infinite and locally finite subset of $Z := \mathrm{Zar}(K|A)$, then $\mathrm{Ad}^{\mathrm{ultra}}(\Sigma) = \Sigma \cup \{K\}$.*

Proof. By Proposition 2.16 and Remark 2.11, it is enough to show that $K = A_{\mathscr{U}}$ ($= \{x \in K : B_x \cap \Sigma \in \mathscr{U}\}$), for every nontrivial ultrafilter $\mathscr{U}$ on Σ. By contradiction, assume that there exists an element $x_0 \in K \setminus A_{\mathscr{U}}$. Then, $\Sigma \setminus B_{x_0} \in \mathscr{U}$, and so it is infinite, since $\mathscr{U}$ is nontrivial (an ultrafilter containing a finite set is trivial, by Lemma 2.2(5)(ii)). This implies that x_0 is noninvertibile in infinitely many valuation domains belonging to Σ, a contradiction. $\square$

As a consequence of the previous lemma, we have that, if an integral domain admits two distinct infinite and locally finite representations as intersection of valutation domains, then the converse of Proposition 2.31 does not hold. An explicit example is given next.

2.34 Example. Let k be a field and let T_1, T_2, T_3 be three indeterminates. Let B be the two-dimensional, local domain $k(T_3)[T_1, T_2]_{(T_1,T_2)}$ and let $\mathfrak{m}_B$ be the maximal ideal of B, i.e., $B = k(T_3) + \mathfrak{m}_B$. Now, let V be (the rank 1 discrete) valuation domain defined by $V := k[T_3]_{(T_3)}$ and let A be the pullback domain given by

$$A := V + \mathfrak{m}_B = k[T_3]_{(T_3)} + (T_1, T_2)k(T_3)[T_1, T_2]_{(T_1,T_2)}.$$

Our goal is to represent A as a locally finite intersection of valuation domains in two different ways. In fact, we can use one description to generate an infinite number of different such representations.

First, note that B can be represented as an intersection of DVR's which are obtained by localizing at its height-one primes, i.e., $B = \bigcap\{B_{\mathfrak{p}} : \mathfrak{p} \in \mathrm{Spec}(B), \mathrm{ht}(\mathfrak{p}) = 1\}$. It is well known that this collection is locally finite. Now, note that A is a local domain with maximal ideal $\mathfrak{n}_A := T_3 k[T_3]_{(T_3)} + (T_1, T_2)k(T_3)[T_1, T_2]_{(T_1,T_2)}$. Choose any valuation overring W of A such that $\mathfrak{n}_W$ (the maximal ideal of W) is generated by T_3 and lies over the maximal ideal of A. It is easy to see that many such valuation domains of the field $k(T_1, T_2, T_3)$ exist (e.g., let W' be a valuation overring of B with maximal ideal $\mathfrak{m}'$ lying over $\mathfrak{m}_B$ and such that the residue field $W'/\mathfrak{m}'$ is canonically isomorphic to $k(T_3)$, which is the residue field $B/\mathfrak{m}_B$, then the domain $V+\mathfrak{m}'$, with V as in the previous paragraph, can serve as the desired domain W). Now, the intersection $R := \bigcap\{B_{\mathfrak{p}} : \mathfrak{p} \in \mathrm{Spec}(B), \mathrm{ht}(\mathfrak{p}) = 1\} \bigcap W$ is clearly a locally finite intersection. We claim that any choice, as above, of the domain W will yield $R = A$.

To prove our claim, we note first that it is obvious that R is an overring of A. So, we need to prove that $R \subseteq A$. Observe that the ideal $\mathfrak{m}_B$ is an ideal of A as well as of B. It follows easily that $\mathfrak{m}_B$ is a prime ideal of R, since $R \subset B$. Then, given an element $r \in R$, we can write $r = \psi + f$, where $\psi \in k(T_3)$ and $f \in \mathfrak{m}_B$. However, $f \in \mathfrak{m}_B \subseteq W$ and so $\psi \in W$. It is clear though that $W \cap k(T_3) = V$. It follows that $\psi \in V$ and so $r \in A$. Hence, we have proven that $R \subseteq A$.

The following Proposition is the key step to prove the main results of the section.

2.35 Proposition. *Let A be a Prüfer domain and K be the quotient field of A. Let Y be a subset of $Z := \mathrm{Zar}(K|A)$ such that $A = \bigcap\{V : V \in Y\}$, and let $\gamma : \mathrm{Zar}(K|A) \longrightarrow \mathrm{Spec}(A)$, be the canonical map (as in Theorem 2.17). Then, $\gamma^{-1}(\mathrm{Max}(A)) \subseteq \mathrm{Ad}^{\mathrm{ultra}}(Y)$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of A. Since A is a Prüfer domain, the t-operation on A coincides with the identity [29, Theorem 22.1(3)], thus obviously $\mathfrak{m}$ is a $t-$maximal $t-$ideal of A. Now, we are able to apply [11, Proposition 2.8(ii)] and, so, there exists an ultrafilter $\mathscr{U} \in \beta Y$ such that

$$\mathfrak{m} = \{x \in A : \gamma^{-1}(V(x)) \cap Y \in \mathscr{U}\}.$$

On the other hand, the collection of subsets

$$\mathscr{V} := \{X' \subseteq \gamma(Y) : \gamma^{-1}(X') \cap Y \in \mathscr{U}\}$$

of $\gamma(Y)$ is an ultrafilter on $\gamma(Y)$ (precisely, with the notation of Lemma 2.2(4), $\mathscr{V} = \mathscr{U}_\gamma$, where for simplicity we have still denoted by γ the map $\gamma|_Y : Y \to \gamma(Y)$) and, moreover,

$$\begin{aligned}
\mathfrak{p}_{\mathscr{V}} := &\ \{x \in A : V(x) \cap \gamma(Y) \in \mathscr{V}\} \\
= &\ \{x \in A : \gamma^{-1}(V(x) \cap \gamma(Y)) \cap Y \in \mathscr{U}\} \\
= &\ \{x \in A : \gamma^{-1}(V(x)) \cap Y \in \mathscr{U}\} = \mathfrak{m}\,.
\end{aligned}$$

Moreover, if $A_{\mathscr{U}}$ is the ultrafilter limit valuation domain of K associated to $\mathscr{U} \in \beta Y$ (i.e., $A_{\mathscr{U}} = \{x \in K : B_x \cap Y \in \mathscr{U}\}$), then by [11, Proposition 2.10(i)], we have $\gamma(A_{\mathscr{U}}) = \mathfrak{p}_{\mathscr{V}} = \mathfrak{m}$. Therefore, by applying Proposition

2.9, $\mathrm{Max}(A) \subseteq \mathrm{Ad}^{\mathrm{ultra}}(\gamma(Y))$ Since γ is continuous and closed with respect to the ultrafilter topology (Theorem 2.17), it follows that $\mathrm{Ad}^{\mathrm{ultra}}(\gamma(Y)) = \gamma(\mathrm{Ad}^{\mathrm{ultra}}(Y))$. Moreover, since A is a Prüfer domain, by [16, Proposition 2.2] γ is also injective and, hence, $\gamma^{-1}(\mathrm{Max}(A)) \subseteq \mathrm{Ad}^{\mathrm{ultra}}(Y)$. $\square$

Let A be a domain, K be the quotient field of A, and T be an indeterminate over K. For each subset Y of $Z := \mathrm{Zar}(K|A)$, we denote by $Y_0 := \{V(T) : V \in Y\}$. Recall that, in Corollary 2.28(1), we introduced a general form of the Kronecker function ring, by setting $\mathrm{Kr}(K|A) = \bigcap\{V(T) : V \in Z\} = \bigcap Z_0 =: \mathrm{Kr}(Z)$. Now, we can extend this notion for $Y \subseteq Z$, by setting

$$\mathrm{Kr}(Y) := \bigcap Y_0 = \bigcap\{V(T) : V \in Y\}$$

which is called the *K–function ring associated to* Y. In particular, $\mathrm{Kr}(Z) = \mathrm{Kr}(K|A)$. We recall that an integrally closed domain A is a *vacant domain* if, for each $Y \subseteq Z$ such that $A = \bigcap Y$, then $\mathrm{Kr}(Y) = \mathrm{Kr}(Z)$ [19, Definition 2.1.11].

2.36 Remark. Let K be a field and A be a subring of K. If Y is a subset of $\mathrm{Zar}(K|A)$, then $Y^{\uparrow} := \{V \in \mathrm{Zar}(K|A) : V \supseteq W, \text{ for some } W \in Y\}$ is the Zariski–generic closure of Y in $\mathrm{Zar}(K|A)^{\mathrm{zar}}$.

2.37 Theorem. *Let K be a field and C a closed subset of* $\mathrm{Zar}(K)^{\mathrm{ultra}}$. *Let* $(C^{\uparrow})_0 = \{W(T) : W \in C^{\uparrow}\}$. *Then,* $\mathrm{Zar}(K(T)|\mathrm{Kr}(C)) = (C^{\uparrow})_0$.

Proof. The inclusion $\supseteq$ is obvious. Conversely, let $\widetilde{W} \in \mathrm{Zar}(K(T)|\mathrm{Kr}(C))$. By Proposition 2.24, we can suppose that $\widetilde{W} = W(T)$, for some $W \in \mathrm{Zar}(K)$. We want to show that $W \supseteq V$, for some $V \in C$. Let $\varphi : \mathrm{Zar}(K(T)) \longrightarrow \mathrm{Zar}(K)$ be the canonical map (Corollary 2.20). Since the function $\varphi|_{\mathrm{Zar}_0(K(T))} : \mathrm{Zar}_0(K(T))^{\mathrm{ultra}} \longrightarrow \mathrm{Zar}(K)^{\mathrm{ultra}}$ is an homeomorphism (Proposition 2.21), then the set

$$\varphi|_{\mathrm{Zar}_0(K(T))}{}^{-1}(C) = \{V(T) : V \in C\} = C_0$$

is closed both in $\mathrm{Zar}_0(K(T))^{\mathrm{ultra}}$ and $\mathrm{Zar}(K(T))^{\mathrm{ultra}}$ (Remark 2.26). Let the map $\gamma : \mathrm{Zar}(K(T)|\mathrm{Kr}(C))^{\mathrm{ultra}} \longrightarrow \mathrm{Spec}(\mathrm{Kr}(C))^{\mathrm{ultra}}$ be as in Theorem 2.17. Since the Kronecker function ring $\mathrm{Kr}(C)$ is, in particular, a Prüfer domain with quotient field $K(T)$ (Proposition 2.23(3)) then, from Proposition 2.35, it follows immediately that $\gamma^{-1}(\mathrm{Max}(\mathrm{Kr}(C))) \subseteq C_0$. Set $A(C) := \bigcap\{V :$

$V \in C\}$. Now, by Zorn's Lemma, we can find a minimal valuation overring of $\mathrm{Kr}(C)$ which, by Proposition 2.24, is of the form $V'(T)$, for some $V' \in \mathrm{Zar}(K|A(C))$, such that $W(T) \supseteq V'(T)$. Then, by applying [29, Corollary 19.7] (and, again, Proposition 2.24), we have $\mathrm{Zar}_{\min}(\mathrm{Kr}(C)) \subseteq \gamma^{-1}(\mathrm{Max}(\mathrm{Kr}(C)))$. Since, by what we observed above, $\gamma^{-1}(\mathrm{Max}(\mathrm{Kr}(C))) \subseteq C_0$, then $V'(T) \in C_0$. $\square$

2.38 Remark. Preserving the notation and assumptions of Theorem 2.37, then

$$(C^{\uparrow})_0 = (C_0)^{\uparrow} := \{\widetilde{W} \in \mathrm{Zar}(K(T)) : \widetilde{W} \supseteq V(T), \text{ for some } V \in C\}.$$

As a matter of fact, $\{\widetilde{W} \in \mathrm{Zar}(K(T)) : \widetilde{W} \supseteq V(T)$, for some $V \in C\} = \{\widetilde{W} \in \mathrm{Zar}(K(T)|\mathrm{Kr}(C)) : \widetilde{W} \supseteq V(T)$, for some $V \in C\}$. By Proposition 2.24, we have $\mathrm{Zar}(K(T)|\mathrm{Kr}(C)) = \mathrm{Zar}_0(K(T)|\mathrm{Kr}(C))$, thus $(C_0)^{\uparrow} = \{W(T) \in \mathrm{Zar}(K(T)) : W \in \mathrm{Zar}(K), W(T) \supseteq V(T)$, for some $V \in C\} = (C^{\uparrow})_0$.

2.39 Remark. Let K be a field and A be a subring of K. If Y is a subset of $\mathrm{Zar}(K|A)$, then it is immediately seen that $\wedge_Y = \wedge_{Y^{\uparrow}}$. Moreover, if Y is a compact subset of $\mathrm{Zar}(K|A)^{\mathtt{Zar}}$, the subset $Y_{\mathtt{Min}}$, consisting of the minimal elements of Y, is nonempty and it is easy to see that $\wedge_{Y^{\uparrow}} = \wedge_Y = \wedge_{Y_{\mathtt{Min}}}$.

Now, we give an application of the ultrafilter topology for characterizing when two `e.a.b.` semistar operations of finite type are equal.

2.40 Theorem. *Let A be an integral domain with quotient field K and $Y', Y'' \subseteq \mathrm{Zar}(K|A)$. Then, the following conditions are equivalent.*

(i) $(\wedge_{Y'})_f = (\wedge_{Y''})_f$.

(ii) *The sets $\mathrm{Ad}^{\mathrm{ultra}}(Y'), \mathrm{Ad}^{\mathrm{ultra}}(Y'')$ have the same Zariski–generic closure in $\mathrm{Zar}(K|A)$, i.e., $\mathrm{Ad}^{\mathrm{ultra}}(Y')^{\uparrow} = \mathrm{Ad}^{\mathrm{ultra}}(Y'')^{\uparrow}$.*

Proof. Let T be an indeterminate over K. By [23, Remark 3.5(b)], it is enough to show that condition (ii) is equivalent to the following

(i′) $\mathrm{Kr}(A, \wedge_{Y'}) = \mathrm{Kr}(A, \wedge_{Y''})$.

(ii)$\Rightarrow$(i′). Assume that the equality $\mathrm{Ad}^{\mathrm{ultra}}(Y')^{\uparrow} = \mathrm{Ad}^{\mathrm{ultra}}(Y'')^{\uparrow}$ holds. Keeping in mind the notation introduced before Theorem 2.37 and applying

Corollary 2.30(1), it follows easily that, inside $\mathrm{Zar}(K(T))$, $\mathrm{Ad}^{\mathrm{ultra}}(Y_0')^{\uparrow} = \mathrm{Ad}^{\mathrm{ultra}}(Y_0'')^{\uparrow}$. By using Proposition 2.31 and Remark 2.39, we have

$$\begin{aligned}\bigcap Y_0' &= \bigcap \mathrm{Ad}^{\mathrm{ultra}}(Y_0') = \bigcap \mathrm{Ad}^{\mathrm{ultra}}(Y_0')^{\uparrow} = \bigcap \mathrm{Ad}^{\mathrm{ultra}}(Y_0'')^{\uparrow} \\ &= \bigcap \mathrm{Ad}^{\mathrm{ultra}}(Y_0'') = \bigcap Y_0'',\end{aligned}$$

and thus $\mathrm{Kr}(A, \wedge_{Y'}) = \mathrm{Kr}(A, \wedge_{Y''})$, in view of [23, Corollary 3.8].

(i′)⇒(ii). Set $B := \mathrm{Kr}(A, \wedge_{Y'}) = \mathrm{Kr}(A, \wedge_{Y''})$. By using [23, Corollary 3.8], Proposition 2.31, Theorem 2.37 and Remark 2.38, it follows that

$$\mathrm{Ad}^{\mathrm{ultra}}(Y_0')^{\uparrow} = \mathrm{Zar}(K(T)|B) = \mathrm{Ad}^{\mathrm{ultra}}(Y_0'')^{\uparrow},$$

and thus the conclusion is clear, again by Corollary 2.30(1). □

2.41 Corollary. *Let A be an integrally closed domain. Then, the following conditions are equivalent.*

(i) *A is a vacant domain.*

(ii) *For each representation $Y \subseteq \mathrm{Zar}(K|A)$ of A (i.e., $\bigcap Y = A$), then $\mathrm{Ad}^{\mathrm{ultra}}(Y)^{\uparrow} = \mathrm{Zar}(K|A)$.*

Proof. Set $Z := \mathrm{Zar}(K|A)$.

(i)⇒(ii). Assume A vacant and take a subset $Y \subseteq Z$ such that $\bigcap Y = A$. By [23, Proposition 3.3], we have $\mathrm{Kr}(A, \wedge_Y) = \mathrm{Kr}(Y) = \mathrm{Kr}(Z) = \mathrm{Kr}(A, b)$, and thus

$$(\wedge_Y)_f = b = \wedge_Z = (\wedge_Z)_f.$$

The conclusion follows immediately from Theorem 2.40.

(ii)⇒(i). Take a subset Y of $\mathrm{Zar}(K|A)$ such that $\bigcap Y = A$. By assumption and Theorem 2.40, it follows that $(\wedge_Y)_f = (\wedge_Z)_f = \wedge_Z = b$, and thus $\mathrm{Kr}(Y) = \mathrm{Kr}(A, \wedge_Y) = \mathrm{Kr}(A, \wedge_Z) = \mathrm{Kr}(Z)$. This proves that A is vacant. □

From the previous theorem, we deduce immediately the following

2.42 Corollary. *Let A be an integrally closed domain. If each representation of A is dense in $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$, then A is a vacant domain.*

2.43 Example. Let K be a field and T_1, T_2 two indeterminates over K. Consider the pseudo-valuation domain $A := K + T_2K(T_1)[T_2]_{(T_2)}$ with associated valuation domain $V := K(T_1)[T_2]_{(T_2)}$ of $K(T_1, T_2)$. Let $p : V \to K(T_1)$ be the canonical projection of V onto its residue field $K(T_1)$ and so $A = p^{-1}(K)$.

Then, by [29, Exercise 12, page 409], the domain A is a vacant domain. It is easily seen that the set $C := \{p^{-1}(W') : W' \in \mathrm{Zar}(K(T_1)|K)\} \subset \mathrm{Zar}(K(T_1, T_2)|A)$ is a representation of A, and that it is closed, with respect to the ultrafilter topology of $\mathrm{Zar}(K(T_1, T_2)|A)$, since $C = \{W \in \mathrm{Zar}(K(T_1, T_2)|A) : W \subseteq V\} = \bigcap_{z \in K(T_1,T_2) \setminus V} (\mathrm{Zar}(K(T_1, T_2)|A) \setminus B_z) = \mathrm{Ad}^{\mathrm{zar}}(\{V\})$. Thus, the converse of the previous Corollary 2.42 does not hold in general .

Let T be an indeterminate over $K(T_1, T_2)$. Note that this example shows also that, in the statement of Theorem 2.37, we need to consider $C^{\uparrow}$ and not just C, since in this case

$$\mathrm{Zar}(K(T; T_1, T_2)|\mathrm{Kr}(A, \wedge_C)) = \mathrm{Zar}(K(T; T_1, T_2)|\mathrm{Kr}(C)) = (C^{\uparrow})_0 \supsetneq C_0.$$

The next result will be crucial in the last part of the section.

2.44 Lemma. *Let K be a field, A be a subring of K, and Y be a compact subspace of $\mathrm{Zar}(K|A)^{\mathrm{zar}}$. Then, $Y^{\uparrow}$ is a closed (or, equivalently, compact) subspace of $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$.*

Proof. Let $\mathcal{Q}$ be the natural basis of $\mathrm{Zar}(K|A)^{\mathrm{zar}}$. Recall that the ultrafilter topology on $\mathrm{Zar}(K|A)$ is identical to the $\mathcal{Q}-$ultrafilter topology (Remark 3.6(4)). It is now enough to apply Proposition 3.18. □

Now, we prove that the property of being "complete" for a semistar operation can be caracterized by a"compactness" property for a suitable subspace of the Zariski-Riemann surface.

2.45 Theorem. *Let A be an integral domain with quotient field K and $\star$ be a semistar operation on A. Then, the following conditions are equivalent.*

(i) *$\star$ is* `e.a.b.` *of finite type.*

(ii) *$\star$ is complete.*

(iii) *There exists a closed subset Y of $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$ such that $Y = Y^{\uparrow}$ and $\star = \wedge_Y$.*

(iv) *There exists a compact subspace Y' in $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$ such that $\star = \wedge_{Y'}$.*

(v) *There exists a compact subspace of Y'' of $\mathrm{Zar}(K|A)^{\mathrm{zar}}$ such that $\star = \wedge_{Y''}$.*

Proof. (i)$\Leftrightarrow$(ii) depends on the fact that if $\star$ is `e.a.b.`, then $\star_f = b(\star)$ [25, Proposition 9].

Let T be an indeterminate on K and let $\varphi : \mathrm{Zar}(K(T)) \longrightarrow \mathrm{Zar}(K)$ be the canonical surjective map, defined in Proposition 2.20.

(ii)$\Rightarrow$(iv). By [24, Theorem 3.5], $\mathrm{Zar}^\star(K|A) = \varphi(\mathrm{Zar}(K(T)|\mathrm{Kr}(A,\star)))$, and thus (by Proposition 2.21) it is closed in $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$ or, equivalently, compact, in the compact Hausdorff space $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$. Then, the conclusion follows by taking $Y' := \mathrm{Zar}^\star(K|A)$ (since, by definition, $b(\star) = \wedge_{\mathrm{Zar}^\star(K|A)}$).

(iv)$\Rightarrow$(ii). As observed above the compact subspaces of $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$ are exactly the closed subsets. Take a closed set Y' of $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$ such that $\star = \wedge_{Y'}$. Set $Y'^{\uparrow} := \{W \in \mathrm{Zar}(K|A) : W \supseteq V, \text{ for some } V \in Y'\}$. By [23, Corollary 3.8], we have $\mathrm{Kr}(A, \wedge_{Y'}) = \bigcap\{V(T) : V \in Y'\} =: \mathrm{Kr}(Y')$. On the other hand, since Y' is closed, by Theorem 2.37, it follows that $\mathrm{Zar}(K(T)|\mathrm{Kr}(A,\wedge_{Y'})) = (Y'^{\uparrow})_0 = \{W(T) : W \in Y'^{\uparrow}\}$. Therefore, as above (by [24, Theorem 3.5]), $\mathrm{Zar}^{\wedge_Y}(K|A) = \varphi(\mathrm{Zar}(K(T)|\mathrm{Kr}(A,\wedge_Y))) = Y^{\uparrow}$ and so $Y^{\uparrow}$ is also a closed subspace of $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$. Since by definition $b(\wedge_Y) = \wedge_{\mathrm{Zar}^{\wedge_Y}(K|A)} = \wedge_{Y^{\uparrow}}$, then the conclusion is immediate, by Remark 2.39.

(iii)$\Rightarrow$(iv) is trivial since, as observed above, closed coincides with compact in $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$.

(iv)$\Rightarrow$(v) is obvious, by Theorem 2.14(1).

(v)$\Rightarrow$ (iii). Take a set Y'' as stated in (v). Then, (iii) follows immediately from Lemma 2.44 and Remark 2.39, by taking $Y := Y''^{\uparrow}$. $\square$

2.46 Corollary. *Let A be an integral domain and K its quotient field. Let Y be a subset of $\mathrm{Zar}(K|A)$ and set $\widehat{Y} := \mathrm{Ad}^{\mathrm{ultra}}(Y)^{\uparrow}$. Then,*

$$(\wedge_Y)_f = \wedge_{\widehat{Y}} = \wedge_{\mathrm{Ad}^{\mathrm{ultra}}(Y)}.$$

Proof. In view of Lemma 2.44, $\widehat{Y}$ is closed, with respect to the ultrafilter topology. Thus $\wedge_{\widehat{Y}}$ is of finite type, by Theorem 2.45, and hence the equality $(\wedge_Y)_f = \wedge_{\widehat{Y}}$ follows immediately by Theorem 2.40 since $\mathrm{Ad}^{\mathrm{ultra}}(Y)^{\uparrow} = \widehat{Y}^{\uparrow}(= \mathrm{Ad}^{\mathrm{ultra}}(\widehat{Y}^{\uparrow}))$. Moreover, the semistar operation $\wedge_{\mathrm{Ad}^{\mathrm{ultra}}(Y)}$ is of finite type, by Theorem 2.45, and thus the last equality follows by applying Theorem 2.40. $\square$

The next example illustrates the possibility that the sets Y, Y' and Y'' in Theorem 2.45 can form a proper chain of sets.

2.47 EXAMPLE. Let k be a field and T_1, T_2 two indeterminates over k. Let A be the two-dimensional, integrally closed, local domain $k[T_1, T_2]_{(T_1,T_2)}$ with quotient field $K := k(T_1, T_2)$. Let $\star$ be the b–operation on A. It is well known that the b–operation is an `e.a.b.` operation of finite type. Hence, it satisfies the equivalent conditions of Theorem 2.45. Our goal is to show that there is a great deal of flexibility in the choice of the sets Y, Y' and Y'' in the theorem. First, note that if the valuation domains in $\mathrm{Zar}(K|A)$ are ordered by inclusion, then any chain is finite [29, Corollary 30.10] and, hence, obviously there are minimal elements. Any such minimal valuation overring V will necessarily have maximal ideal M_V lying over the maximal ideal (T_1, T_2) of A. The standard definition of the b–operation involves extending an ideal (or, more generally a sub-A-module of K) to all valuation overrings. It is clearly sufficient to extend to just those valuation overrings that are minimal. So, any subcollection of $\mathrm{Zar}(K|A)$ which contains all the minimal elements will generated the b–operation. It is not clear that the collection of minimal valuation overrings is closed under the Zariski or the ultrafilter topology.

- Consider the members of $\mathrm{Zar}(K|A)$ which do not contain the elements $\frac{1}{T_1}, \frac{1}{T_2}$. This is a closed, quasi-compact subset of $\mathrm{Zar}(K|A)^{\mathrm{zar}}$. It can also be thought of as being those valuation domains in $\mathrm{Zar}(K|A)$ whose maximal ideal dominates (T_1, T_2) in A. Hence, it contains the minimal valuation overrings and is sufficient to generate the b operation. We can let this collection be denoted by Y'' in Theorem 4.14.

- The set Y'', described above, is a (proper) closed subset of $\mathrm{Zar}(K|A)^{\mathrm{zar}}$. Hence, it is also closed in $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$. Moreover, any closed subset of $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$ is compact. Hence, to obtain our set Y', we can choose any closed subset of $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$ which contains Y''. Since any single point is closed in $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$, we can let Y' be the union of Y'' and any other single valuation overring, for example, the localization of A at a height-one prime.

- The set Y should contain all overrings of its members. An obvious choice then is to let Y be all of $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$. Since this is the entire space it is trivially closed (in $\mathrm{Zar}(K|A)^{\mathrm{ultra}}$) and generates the b–operation.

This then gives three different sets $Y'' \subset Y' \subset Y$ with the notation of Theorem 2.45, all associated with the same (semi)star operation.

Let K be a field, A be a subring of K and set, as usual, $Z := \mathrm{Zar}(K|A)$. Following [40], define *the inverse (or dual) topology of* $\mathrm{Zar}(K|A)$*, with respect to the Zariski topology,* the topology on $\mathrm{Zar}(K|A)$ having as basis of closed sets the collection of all the quasi-compact open subspaces of Z^{zar}. We shall denote by Z^{inv} the set Z endowed with the inverse topology. The name of this topology is due to the following fact, clear by [40, Proposition 8].

2.48 Proposition. *Let K be a field and let A be a subring of K. If $V, W \in \mathrm{Zar}(K|A)$, then $V \in \mathrm{Ad}^{\mathrm{zar}}(\{W\})$ (i.e., $W \preccurlyeq V$ in the ordering induced on Z by the Zariski topology) if and only if $W \in \mathrm{Ad}^{\mathrm{inv}}(\{V\})$ (i.e., $V \preccurlyeq' W$ in the ordering induced on Z by the inverse topology).*

2.49 Remark. Let K be a field and A be a subring of K. If C is a closed subset of $\mathrm{Zar}(K|A)^{\mathrm{inv}}$, then $C = C^{\uparrow}$. In fact, it is easy to see that the order relation " $\preccurlyeq'$ " on Z (with the inverse topology) coincides with the set-theoretic inclusion " $\subseteq$ " and, by Proposition 2.48, it follows that

$$\begin{aligned} C^{\uparrow} = &\ \{W \in \mathrm{Zar}(K|A) : W \in \mathrm{Ad}^{\mathrm{inv}}(\{V\}),\ \text{for some } V \in C\} \\ &\ \{W \in \mathrm{Zar}(K|A) : V \preccurlyeq' W,\ \text{for some } V \in C\}, \end{aligned}$$

that is, $C^{\uparrow}$ is the closure under specializations of C, with respect to the inverse topology. Then, it is enough to use the fact that each closed set in any topological space is stable under specializations.

2.50 Proposition. *Let K be a field and A be a subring of K. Then, for each subset Y of $Z := \mathrm{Zar}(K|A)$, we have $\mathrm{Ad}^{\mathrm{inv}}(Y) = (\mathrm{Ad}^{\mathrm{ultra}}(Y))^{\uparrow}$.*

Proof. Let Y be a subset of Z. Since each quasi-compact open subset of Z^{zar} is a finite union of sets of the form B_F (with $F \in \mathscr{B}_{\mathtt{fin}}(K)$), then it follows immediately from Theorem 2.14(2) that the inverse topology is coarser than the ultrafilter topology, and thus $\mathrm{Ad}^{\mathrm{ultra}}(Y) \subseteq \mathrm{Ad}^{\mathrm{inv}}(Y)$. Moreover, keeping in mind the previous Remark 2.49, we have $(\mathrm{Ad}^{\mathrm{ultra}}(Y))^{\uparrow} \subseteq \mathrm{Ad}^{\mathrm{inv}}(Y)$. Conversely, take a valuation domain $V \in \mathrm{Ad}^{\mathrm{inv}}(Y)$, and set

$$\mathcal{F}_V := \{(Z \setminus B_x) \cap \mathrm{Ad}^{\mathrm{ultra}}(Y) \mid x \in K \setminus V\}.$$

Given $x_1, x_2, \ldots, x_n \in K \setminus V$, then, by definition, $\Omega := \bigcap_{i=1}^{n}(Z \setminus B_{x_i})$ is an open neighborhood of V in Z^{inv}, and, since $V \in \mathrm{Ad}^{\mathrm{inv}}(Y)$, we have $\Omega \cap Y \neq \emptyset$. It follows immediately that $\mathcal{F}_V$ is a collection of closed subsets of the compact space $\mathrm{Ad}^{\mathrm{ultra}}(Y)$ (see Theorem 2.14(5)), with the finite intersection property.

Then, we can find a valuation domain $W \in \bigcap \mathcal{F}_V$ ($\subseteq \mathrm{Ad}^{\mathrm{ultra}}(Y)$). In order to conclude, we show that $W \subseteq V$. If not, let $y \in W \setminus V$. Then, the set $(Z \setminus B_y) \cap \mathrm{Ad}^{\mathrm{ultra}}(Y) \in \mathcal{F}_V$, and hence, in particular, $W \in (Z \setminus B_y) \cap \mathrm{Ad}^{\mathrm{ultra}}(Y) \subseteq Z \setminus B_y$. But, this contradicts the fact that $y \in W$ and, thus, the statement is completely proved. $\square$

By using the previous proposition, we can restate Corollaries 2.41 and 2.46 as follows:

2.51 COROLLARY. *Let A be an integrally closed domains and K be its quotient field. Then, the following conditions are equivalent.*

(i) *A is a vacant domain.*

(ii) *Each representation of A is dense in* $\mathrm{Zar}(K|A)$*, with respect to the inverse topology.*

2.52 COROLLARY. *Let A be an integral domain and K its quotient field. Let Y be a subset of* $\mathrm{Zar}(K|A)$*. Then,* $(\wedge_Y)_f = \wedge_{\mathrm{Ad}^{\mathrm{inv}}(Y)}$.

2.53 EXAMPLE. Let A be an integral domain, K be its quotient field and Y be a subset of $\mathrm{Zar}(K|A)$. It is not true, in general, that $(\wedge_Y)_f = \wedge_{\mathrm{Ad}^{\mathrm{zar}}(Y)}$. As a matter of fact, take, for example, a valuation overring W of A that is not minimal, pick a valuation overring V of A such that $V \subsetneq W$, and set $Y := \{W\}$. Then, by Theorem 2.45, $\wedge_Y$ is of finite type, but $\wedge_Y \neq \wedge_{\mathrm{Ad}^{\mathrm{zar}}(Y)}$. In fact, otherwise, by Theorem 2.40 it follows

$$\{W\}^{\uparrow} = \mathrm{Ad}^{\mathrm{ultra}}(Y)^{\uparrow} = \mathrm{Ad}^{\mathrm{ultra}}(\mathrm{Ad}^{\mathrm{zar}}(Y))^{\uparrow} = \{V' \in \mathrm{Zar}(K|A) : V' \subseteq W\}^{\uparrow},$$

but $V \in \{V' \in \mathrm{Zar}(K|A) : V' \subseteq W\}^{\uparrow} \setminus \{W\}^{\uparrow}$, a contradiction.

Chapter 3

Appendix

We preserve notation and conventions given in the previous chapters.

Let X be a set and $\mathcal{F}$ be a given nonempty collection of subsets of X. For each $Y \subseteq X$ and each ultrafilter $\mathscr{U}$ on Y, we define

$$Y_{\mathcal{F}}(\mathscr{U}) := \{x \in X : [\forall F \in \mathcal{F}, x \in F \iff F \cap Y \in \mathscr{U}]\}.$$

Since $\mathcal{F}$ will be always a fixed collection of sets, we will denote the set $Y_{\mathcal{F}}(\mathscr{U})$ simply by $Y(\mathscr{U})$, when no confusion can arise.

3.1 Example. Let X be a set, $\mathcal{F}$ be a nonempty collection of subsets of X and Y be a subset of X.

(1) If $y \in Y$ and β_Y^y is, as usual, the trivial ultrafilter on Y generated by y, then $y \in Y_{\mathcal{F}}(\beta_Y^y)$.

(2) Let A be a ring, Y be a subset of $\mathrm{Spec}(A)$ and $\mathscr{U}$ be an ultrafilter on Y. Set, as in 2.0.6,

$$\mathfrak{p}_{\mathscr{U}} := \{x \in A : V(x) \cap Y \in \mathscr{U}\}$$

Then, if $\mathcal{P} := \{D_a : a \in A\}$ is the collection of all the principal open subsets of $\mathrm{Spec}(A)$, then, by definitions, $Y_{\mathcal{P}}(\mathscr{U}) = \{\mathfrak{p}_{\mathscr{U}}\}$.

(3) Let K be a field and A be a subring of K. Let $\mathcal{Q} := \{B_F : F \in \mathscr{B}_{\mathtt{fin}}(K)\}$ be the natural basis of open sets of the Zariski topology of $\mathrm{Zar}(K|A)$. If Y is a subset of $\mathrm{Zar}(K|A)$ and $\mathscr{U}$ is an ultrafilter on Y, set as in Proposition 2.10,

$$A_{\mathscr{U}} := \{x \in K : B_x \cap Y \in \mathscr{U}\}$$

Then, as in (2), we have $Y_{\mathcal{Q}}(\mathscr{U}) = \{A_{\mathscr{U}}\}$.

3.2 Definition. *Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. Then, we say that a subset Y of X is* $\mathcal{F}-$stable under ultrafilters *if $Y(\mathscr{U}) \subseteq Y$, for each ultrafilter $\mathscr{U}$ on Y.*

3.3 Example. Let A be a ring (resp. K be a field and S be a subring of K). Keeping in mind the notation and the statements given in Example 3.1. it follows immediately that a subset Y of $\mathrm{Spec}(A)$ (resp. $\mathrm{Zar}(K|A)$) is $\mathcal{P}-$stable under ultrafilters (resp. $\mathcal{Q}-$stable under ultrafilters) if and only if it is closed in the ultrafilter topology of $\mathrm{Spec}(A)$ (resp. $\mathrm{Zar}(K|A)$).

The following easy and technical lemma will allow us to show that the ultrafilter topology is a very particular case of a more general construction.

3.4 Lemma. *Let X be a set, $\mathcal{F}$ be a given nonempty collection of subsets of X and $Y \subseteq Z \subseteq X$. Let $\mathscr{U}$ be an ultrafilter on Y, $T \in \mathscr{U}$ and, as in Lemma 2.2(3,4) of Chapter 2, set*

$$\mathscr{U}_T := \{U \cap T : U \in \mathscr{U}\} \qquad \mathscr{U}^Z := \{Z' \subseteq Z : Z' \cap Y \in \mathscr{U}\}$$

Then we have

$$Y(\mathscr{U}) = T(\mathscr{U}_T) = Z(\mathscr{U}^Z).$$

Proof. We shall prove only the inclusion $Y(\mathscr{U}) \subseteq T(\mathscr{U}_T)$. The others are shown with the same straightforward arguments. Let $x \in Y(\mathscr{U})$ and $F \in \mathcal{F}$. We need to show that $x \in F$ if and only if $F \cap T \in \mathscr{U}_T$. Assume $x \in \mathcal{F}$. Then, $F \cap Y \in \mathscr{U}$ and $F \cap T = (F \cap Y) \cap T \in \mathscr{U}_T$, by definition. Assume $F \cap T \in \mathscr{U}_T$. Since $\mathscr{U}_T \subseteq \mathscr{U}$ and $F \cap T \subseteq F \cap Y$, then $F \cap Y \in \mathscr{U}$ and thus $x \in F$. □

3.5 Proposition. *Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. Then, the family of all the subsets of X that are $\mathcal{F}-$stable under ultrafilters is the collection of the closed sets for a topology on X. We will call it* the $\mathcal{F}-$ultrafilter topology on X, *and denote by $X^{\mathcal{F}-\mathrm{ultra}}$ the set X endowed with the $\mathcal{F}-$ultrafilter topology.*

Proof. Let C, C_0 be $\mathcal{F}-$stable under ultrafilters subsets of X, and $\mathscr{U}$ be an ultrafilter on $Y := C \cup C_0$. By Lemma 2.2(3,5), we can assume that $C \in \mathscr{U}$. Then, by hypothesis and Lemma 3.4, we have $Y(\mathscr{U}) = C(\mathscr{U}_C) \subseteq C \subseteq Y$, and thus Y is $\mathcal{F}-$stable under ultrafilters.

Now, let $\mathcal{G}$ be a collection of $\mathcal{F}-$stable under ultrafilters subsets of X and let $\mathscr{U}$ be an ultrafilter on $Z := \bigcap \mathcal{G}$. For each $C \in \mathcal{G}$, we have $C(\mathscr{U}^C) = Z(\mathscr{U})$ (by Lemma 3.4), and thus $Z(\mathscr{U}) \subseteq Z$. This complete the proof. $\square$

3.6 Remark. Let X be a set.

(1) The $\mathcal{B}(X)-$ultrafilter topology on X is the discrete topology on X.

(2) The $\{X\}-$ultrafilter topology on X is the chaotic topology.

(3) Let A be a ring and $\mathcal{P}$ the collection of all the principal open subsets of $X := \mathrm{Spec}(A)$. Then, the $\mathcal{P}-$ultrafilter topology of X is equal to the ultrafilter topology studied in [26].

(4) Let K be a field, A be a subring of K and $\mathcal{Q}$ be the natural basis of open sets for the Zariski topology on $\mathrm{Zar}(K|A)$. Then, the $\mathcal{Q}-$ultrafilter topology is equal to the ultrafilter topology on $\mathrm{Zar}(K|A)$.

(5) If $\mathcal{F} \subseteq \mathcal{G}$ are collections of subsets of X, then the $\mathcal{G}-$ultrafilter topology is finer or equal than the $\mathcal{F}-$ultrafilter topology. Infact, for each subset Y of X and each ultrafilter $\mathscr{U}$ on Y, we have $Y_{\mathcal{G}}(\mathscr{U}) \subseteq Y_{\mathcal{F}}(\mathscr{U})$.

3.7 Proposition. *Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. Set*

$$\mathcal{F}_\sharp := \left\{\bigcap \mathcal{G} : \mathcal{G} \in \mathscr{B}_{\mathtt{fin}}(\mathcal{F})\right\} \quad \mathcal{F}^\sharp := \left\{\bigcup \mathcal{G} : \mathcal{G} \in \mathscr{B}_{\mathtt{fin}}(\mathcal{F})\right\}$$
$$\mathcal{F}^- := \{X \backslash F : F \in \mathcal{F}\}.$$

Then, the $\mathcal{F}-$ultrafilter topology, the $\mathcal{F}_\sharp-$ultrafilter topology and the $\mathcal{F}^\sharp-$ultrafilter topology are the same.

Proof. By Remark 3.6(5) and the obvious inclusion $\mathcal{F} \subseteq \mathcal{F}^\sharp$, it is enough to show that the $\mathcal{F}^\sharp-$ultrafilter topology is finer or equal than the $\mathcal{F}-$ultrafilter topology. Let Y be a $\mathcal{F}^\sharp-$stable under ultrafilter subset of X, $\mathscr{U}$ be an ultrafilter on Y, $x \in Y_{\mathcal{F}}(\mathscr{U})$, $\mathcal{G} := \{F_1, \ldots, F_n\} \in \mathscr{B}_{\mathtt{fin}}(\mathcal{F})$ and $G := \bigcap \mathcal{G}$. We want to show that $x \in G$ if and only if $G \cap Y \in \mathscr{U}$. If $x \in G$, then $F_i \cap Y \in \mathscr{U}$, for $i = 1, \ldots, n$, and thus $G \cap Y \in \mathscr{U}$. Since $G \cap Y \subseteq F_i \cap Y$, for each $i = 1, \ldots, n$, if $G \cap Y \in \mathscr{U}$, then it follows immediately that $F_i \cap Y \in \mathscr{U}$, for $i = 1, \ldots, n$, and thus $x \in G$, by definition. This prove that $Y_{\mathcal{F}}(\mathscr{U}) \subseteq Y_{\mathcal{F}^\sharp}(\mathscr{U})$. Thus it is clear that the $\mathcal{F}-$ultrafilter topology and the $\mathcal{F}_\sharp-$ultrafilter topology are the same. By a similar argument it can be shown that $Y_{\mathcal{F}}(\mathscr{U}) = Y_{\mathcal{F}^\sharp}(\mathscr{U}) = Y_{\mathcal{F}^-}(\mathscr{U})$. Thus the proof is complete. $\square$

3.8 Corollary. *Let X be a set, $\mathcal{F}$ be a nonempty collection of subsets of X and* $\mathrm{Bool}(\mathcal{F})$ *be the boolean subalgebra of $\mathcal{B}(X)$ generated by $\mathcal{F}$. Then the $\mathcal{F}-$ultrafilter topology and the* $\mathrm{Bool}(\mathcal{F})-$*ultrafilter topology are the same.*

Proof. Let Y be a nonempty subset of X and $\mathscr{U}$ be an ultrafilter on Y. Keeping in mind the proof of Proposition 3.7, it follows that $Y_{\mathcal{F}}(\mathscr{U}) = Y_{\mathcal{F}\cup\mathcal{F}^-}(\mathscr{U})$. Thus the $\mathcal{F}-$ultrafilter topology and the $(\mathcal{F}\cup\mathcal{F}^-)-$ultrafilter topology are the same. Since, obviously, $\mathrm{Bool}(\mathcal{F}) = \left((\mathcal{F}\cup\mathcal{F}^-)_\sharp\right)^\sharp$, the statement follows by using Proposition 3.7. $\square$

3.9 Proposition. *Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. Then, the following statements hold:*

(1) $\mathrm{Bool}(\mathcal{F}) \subseteq \mathrm{Clop}(X^{\mathcal{F}-\mathrm{ultra}})$.

(2) *If, for each couple of distinct points $x, y \in X$ there exists a set $F \in \mathcal{F}$ such that $x \in F$ and $y \notin F$, then $X^{\mathcal{F}-\mathrm{ultra}}$ is an Hausdorff and totally disconnected space, and $Y(\mathscr{U})$ has at most an element, for each $Y \subseteq X$ and $\mathscr{U} \in \beta Y$.*

Proof. Since $\mathrm{Clop}(X^{\mathcal{F}-\mathrm{ultra}})$ is a boolean algebra, it is enough to show that $\mathcal{F} \subseteq \mathrm{Clop}(X^{\mathcal{F}-\mathrm{ultra}})$. Pick a set $E \in \mathcal{F}$. If $\mathscr{U}$ is an ultrafilter on E and $x \in E(\mathscr{U})$, then the statement $x \in F \iff F \cap E \in \mathscr{U}$ holds for each $F \in \mathscr{F}$, and in particular for $F := E$. Then, $x \in E$. Thus E is closed in the $\mathcal{F}-$ultrafilter topology.

Now let $\mathscr{V}$ be an ultrafilter on $Z := X\setminus E$ and $x \in Z(\mathscr{V})$. The statement $x \in E \iff E \cap Z \in \mathscr{V}$ holds and thus $x \in Z$. Then, E is clopen. Thus (1) is proved.

(2) The fact that $X^{\mathcal{F}-\mathrm{ultra}}$ is an Hausdorff and totally disconnected space follows immediately by (1) and the extra assumption on $\mathcal{F}$. For the second part of (2), assume, by contradiction, that there exist distinct elements $x, y \in Y(\mathscr{U})$, and pick, by hypotesis, a set $F \in \mathcal{F}$, such that $x \in F$ and $y \notin F$. Thus, $\emptyset = (F \cap Y) \cap (Y\setminus F) \in \mathscr{U}$. $\square$

3.10 Proposition. *Let X be a set, $\mathcal{F}$ be a nonempty collection of subsets of X, $\emptyset \neq Y \subseteq X$ and $\mathscr{U}$ an ultrafilter on Y. Then, for each topology on X for which $\mathcal{F}$ is a collection of clopen sets, $Y(\mathscr{U})$ is closed. In particular, $Y(\mathscr{U})$ is closed in the $\mathcal{F}-$ultrafilter topology.*

Proof. Let $x_0 \in \mathrm{Ad}(Y(\mathscr{U}))$ and $E \in \mathcal{F}$. If $x_0 \in E$, then E is an open neighborhood of x_0, by assumption, and thus there exists an element $y_1 \in Y(\mathscr{U}) \cap E$. By definition, it follows that, for each $F \in \mathcal{F}$, $y_1 \in F \iff F \cap Y \in \mathscr{U}$, and thus $E \cap Y \in \mathscr{U}$, in particular. Conversely, assume $x_0 \notin E$. Then, $X \backslash E$ is an open neighborhood of x_0, and thus there exists $y_2 \in Y(\mathscr{U}) \backslash E$. Hence, we have $E \cap Y \notin \mathscr{U}$. This proves that $Y(\mathscr{U})$ is closed.

Last part of the statement follows immediately by Proposition 3.9(1). $\square$

3.11 PROPOSITION. *Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. Then, for each subspace Y of $X^{\mathcal{F}-\mathrm{ultra}}$, we have*

$$\mathrm{Ad}(Y) = \bigcup \{Y(\mathscr{U}) : \mathscr{U} \in \beta Y\}$$

Proof. Let $Y \subseteq X$, $\mathscr{U} \in \beta Y$, $x \in Y(\mathscr{U})$, and Ω be an open neighborhood of x. If $Y \cap \Omega = \emptyset$, then $Y \subseteq Z := X \backslash \Omega$ and, using Lemma 3.4, we have $Y(\mathscr{U}) = Z(\mathscr{U}^Z) \subseteq Z$, since Z is $\mathcal{F}-$stable under ultrafilters. Thus we get a contradiction, since $x \in \Omega$. The inclusion $\supseteq$ follows. Conversely, pick an element $x \in \mathrm{Ad}(Y)$, and set

$$\mathcal{G} := \{\Omega \cap Y : \Omega \text{ open neighborhood of } x\}.$$

It is clear that $\mathcal{G}$ is a collection of subsets of Y with the finite intersection property (since $x \in \mathrm{Ad}(Y)$), and thus (by Lemma 2.2(1)) there exists an ultrafilter $\mathscr{U}^*$ on Y containing $\mathcal{G}$. The conclusion will follow if we show that $x \in Y(\mathscr{U}^*)$. Fix $F \in \mathcal{F}$. If $x \in F$, then F is an open neighborhood of x, by Proposition 3.9(1), and thus $F \cap Y \in \mathcal{G} \subseteq \mathscr{U}^*$. Conversely, assume $F \cap Y \in \mathscr{U}^*$. If $x \notin F$, then $X \backslash F$ is an open neighborhood of x, again by Proposition 3.9, and thus $(X \backslash F) \cap Y \in \mathcal{G} \subseteq \mathscr{U}^*$. It follows that $\emptyset \in \mathscr{U}^*$, a contradiction. $\square$

3.12 THEOREM. *Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. Then, the following conditions are equivalent.*

(i) *$X^{\mathcal{F}-\mathrm{ultra}}$ is a compact topological space.*

(ii) *$X(\mathscr{U}) \neq \emptyset$, for each ultrafilter $\mathscr{U}$ on X.*

(iii) *If $\mathcal{H}$ is a subcollection of $\mathcal{G} := \mathcal{F} \cup \mathcal{F}^-$ with the finite intersection property, then $\bigcap \mathcal{H} \neq \emptyset$.*

Proof. (i)$\Longrightarrow$(iii). It is enough to use Proposition 3.9(1) and compactness of $X^{\mathcal{F}-\text{ultra}}$.

(iii)$\Longrightarrow$(ii). Let $\mathscr{U}$ be an ultrafilter on X. Assume, by contradiction, that $X(\mathscr{U}) = \emptyset$. This means that, for each $x \in X$ there exists a set $F_x \in \mathcal{F}$ such that exactly one of the following statements is true

(a) $x \in F_x$ and $F_x \notin \mathscr{U}$.

(b) $x \notin F_x$ and $F_x \in \mathscr{U}$.

Now, for each $x \in X$, set $C_x := X \setminus F_x$, if $x \in F_x$, and $C_x := F_x$, if $x \notin F_x$. Then, it is clear that $\mathcal{H} := \{C_x : x \in X\}$ is a subcollection of $\mathcal{G}$ and that it has the finite intersection property, since $\mathcal{H} \subseteq \mathscr{U}$. Thus, by assumption, there exists $x_0 \in \bigcap \mathcal{H}$. This is a contradiction, since $x \in X \setminus C_x$, for each $x \in X$.

(ii)$\Longrightarrow$(i). Let $\mathcal{C}$ be a collection of closed subsets of $X^{\mathcal{F}-\text{ultra}}$ with the finite intersection property. By Lemma 2.2(1), there exists an ultrafilter $\mathscr{U}^*$ on X such that $\mathcal{C} \subseteq \mathscr{U}^*$. By assumption, we can pick a point $x^* \in X(\mathscr{U}^*)$. Now, let $C \in \mathcal{C}$. Since $C \in \mathscr{U}^*$, we have $x^* \in X(\mathscr{U}^*) = C(\mathscr{U}^*{}_C) \subseteq C$, keeping in mind Lemma 3.4. Thus $x^* \in \bigcap \mathcal{C}$. This completes the proof. $\square$

3.13 Example. Let A be a ring. By Example 3.1(2), Remark 3.6(3) and Theorem 3.12 we get immediately the well known fact that the ultrafilter topology on $\mathrm{Spec}(A)$ is compact.

3.14 Proposition. *Let X be a set and $\mathcal{F}$ a nonempty collection of subsets of X such that, for each couple of distinct points $x, y \in X$, there exists a set $F \in \mathcal{F}$ such that $x \in F$ and $y \notin F$. If $X^{\mathcal{F}-\text{ultra}}$ is a compact topological space, then the $\mathcal{F}-$ultrafilter topology is the coarsest topology for which $\mathcal{F}$ is a collection of clopen sets.*

Proof. Denote by $X_\star$ the set X with the coarsest topology for which $\mathcal{F}$ is a collection of clopen sets. Then, the identity map $\mathbf{1} : X^{\mathcal{F}-\text{ultra}} \longrightarrow X_\star$ is continuous, by Proposition 3.9(1). Moreover, it is clear that $X_\star$ is an Hausdorff space, by assumption. Then, $\mathbf{1}$ is an homeomorphism. $\square$

3.15 Proposition. *Let X, Y be sets, $\mathcal{F}$ (resp. $\mathcal{G}$) be a nonempty collection of subsets of X (resp. Y). If $f : X \longrightarrow Y$ is a function such that $\{f^{-1}(G) : G \in \mathcal{G}\} \subseteq \mathcal{F}$, then $f : X^{\mathcal{F}-\text{ultra}} \longrightarrow Y^{\mathcal{G}-\text{ultra}}$ is a continuous function.*

Proof. Let C be a closed subset of Y, set $\Gamma := f^{-1}(C)$, and let $\mathscr{U}$ be an ultrafilter on Γ. Then, it sufficies to show that $\Gamma(\mathscr{U}) \subseteq \Gamma$. Let $g : f^{-1}(C) \longrightarrow C$ be the restriction of f to $f^{-1}(C)$. Now, note that the collection of sets $\mathscr{V} := \{D \subseteq C : f^{-1}(D) \in \mathscr{U}\}$ is an ultrafilter on C, since $\mathscr{V} = \mathscr{U}^g$ (see Lemma 2.2(3)). Now, take an element $x \in \Gamma(\mathscr{U})$, and fix a set $G \in \mathcal{G}$. If $f(x) \in G$, then $x \in f^{-1}(G) \in \mathcal{F}$ (by assumption), and thus $f^{-1}(G \cap C) = f^{-1}(G) \cap \Gamma \in \mathscr{U}$, since $x \in \Gamma(\mathscr{U})$. This proves that, if $f(x) \in G$, then $G \cap C \in \mathscr{V}$. Conversely, if $G \cap C \in \mathscr{V}$, then $f^{-1}(G) \cap \Gamma \in \mathscr{U}$ and, since $f^{-1}(G) \in \mathscr{U}$ and $x \in \Gamma(\mathscr{U})$, it follows $f(x) \in G$. This argument shows that $f(x) \in C(\mathscr{V})$ and, since C is closed, we have $f(x) \in C$. Then, the inclusion $\Gamma(\mathscr{U}) \subseteq \Gamma$ follows, and the statement is now clear. $\square$

3.1 Applications

An interesting case is when $\mathcal{F}$ is a basis of a topology on X.

3.16 Proposition. *Let X be a topological space, $\mathcal{T}$ the topology and $\mathcal{B}$ be a basis of open sets of X. Then, the following statements hold.*

(1) *The $\mathcal{B}-$ultrafilter topology on X is finer or equal to the given topology $\mathcal{T}$.*

(2) *If X satisfies the T_0 axiom, then the $\mathcal{B}-$ultrafilter topology is Hausdorff and totally disconnected. In particular, if X satisfies the T_0 axiom but it is not Hausdorff, then the $\mathcal{B}-$ultrafilter topology is strictly finer than the given topology $\mathcal{T}$.*

(3) *If X satisfies the T_0 axiom and $X^{\mathcal{B}-\mathrm{ultra}}$ is compact, then the $\mathcal{B}-$ultrafilter topology is the coarsest topology on X for which $\mathcal{B}$ is a collection of clopen sets.*

Proof. (1) and (2) are immediate consequences of Proposition 3.9. Statement (3) follows by applying Proposition 3.14. $\square$

The following example will show that, fixed a topological space X, the $\mathcal{B}-$ultrafilter topology depends on the choice of the basis $\mathcal{B}$.

3.17 Example. Let K be a field, $\{T_n : n \in \mathbb{N}\}$ be an infinite and countable collection of indeterminates over K and consider the ring $A := K[\{T_n : n \in \mathbb{N}\}]$. Set $X := \mathrm{Spec}(A)$ and endow this set with the Zariski topology. As usual, let $\mathcal{P} := \{D_f : f \in A\}$ be the basis of the principal open subsets of Spec(A), and let $\mathcal{T} := \{D(\mathfrak{a}) := X \backslash V(\mathfrak{a}) : \mathfrak{a} \text{ ideal of } A\}$ (clearly, $\mathcal{T}$ is a basis of X, being it the topology). We claim that the $\mathcal{P}-$ultrafilter topology (i.e. the usual ultrafilter topology on X) and the $\mathcal{T}-$ultrafilter topology are different. Let $\mathfrak{m}$ be the maximal ideal of A generated by the set $\{T_n : n \in \mathbb{N}\}$ and set $\mathcal{F} := \{V(T_n) : n \in \mathbb{N}\} \cup \{X \backslash \{\mathfrak{m}\}\}$. It is straightforward that $\mathcal{F}$ is a collection of subsets of X with the finite intersection property, and thus there exists an ultrafilter $\mathscr{U}$ on X containing $\mathcal{F}$, by virtue of Lemma 2.2(1,2) of Chapter 2. We claim that the set

$$X_{\mathcal{T}}(\mathscr{U}) := \{\mathfrak{p} \in X : [\text{for each ideal } \mathfrak{a} \text{ of } A, (\mathfrak{p} \in D(\mathfrak{a}) \Longleftrightarrow D(\mathfrak{a}) \in \mathscr{U})]\}$$

is empty. If not, let $\mathfrak{p} \in X_{\mathcal{T}}(\mathscr{U})$. Since $\mathcal{F} \subseteq \mathscr{U}$, it follows that $V(T_n) \in \mathscr{U}$, for each $n \in \mathbb{N}$, and thus $T_n \in \mathfrak{p}$, for each $n \in \mathbb{N}$ (by the definition of $X_{\mathcal{T}}(\mathscr{U})$). This proves that $\mathfrak{p} = \mathfrak{m}$. On the other hand, if we set $\mathfrak{a} := \mathfrak{m}$, we have obviously $\mathfrak{m} \notin D(\mathfrak{a})$, hence $D(\mathfrak{a}) \notin \mathscr{U}$, that is $V(\mathfrak{a}) = \{\mathfrak{m}\} \in \mathscr{U}$. It follows $\emptyset \in \mathscr{U}$, since $X - \{\mathfrak{m}\} \in \mathcal{F} \subseteq \mathscr{U}$, a contradiction. This argument proves that $X_{\mathcal{T}}(\mathscr{U})$ is empty, and thus $X^{\mathcal{T}-\text{ultra}}$ is not compact, by Theorem 3.12. It follows that the $\mathcal{T}-$ultrafilter topology and the $\mathcal{P}-$ultrafilter topology on X are not the same, since the $\mathcal{P}-$ultrafilter topology is compact (see Theorems 2.8 and 2.7(1)).

3.18 Proposition. *Let X be a topological space and $\mathcal{F}$ be a collection of subsets of X containing at least a basis of open sets of X. If Y is a compact subspace of X, then the generic closure of Y (with respect to the given topology) is closed, with respect to the $\mathcal{F}-$ultrafilter topology.*

Proof. Let us denote by Y_0 the generic closure of Y and assume, by contradiction, that Y_0 is not a closed subset of $X^{\mathcal{F}-\text{ultra}}$. By definition, there exist an ultrafilter $\mathscr{U}$ on Y_0 and a point $x_0 \in Y_0(\mathscr{U}) \backslash Y_0$. Let $\mathcal{B}$ be a basis of X contained in $\mathcal{F}$. By the definition of the set Y_0, for each $y \in Y$ there exists an open neighborhood Ω_y of y such that $x_0 \notin \Omega_y$. Without loss of generality, we can assume that $\Omega_y \in \mathcal{B}$. By compactness, the open cover $\{\Omega_y : y \in Y\}$ of Y admits a finite subcover, say $\{\Omega_{y_i} : i = 1, \ldots, n\}$. Now, by definition, it is immediately verified that $Y_0 \subseteq \bigcup\{\Omega_{y_i} : i = 1, \ldots, n\}$, that is $Y_0 = \bigcup\{\Omega_{y_i} \cap Y_0 : i = 1, \ldots, n\}$, and thus, by Lemma 2.2(5), it follows that $\Omega_{y_i} \cap Y_0 \in \mathscr{U}$, for some $i \in \{1, \ldots, n\}$. Keeping in mind that, by assumption, $\Omega_{y_i} \in \mathcal{F}$ and that $x_0 \in Y_0(\mathscr{U})$, it follows $x_0 \in \Omega_{y_i}$, a contradiction. $\square$

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