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# Properties of chains of prime ideals in an amalgamated algebra along an ideal

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ABSTRACT

Let $f : A \to B$ be a ring homomorphism and let $J$ be an ideal of $B$. In this paper, we study the amalgamation of $A$ with $B$ along $J$ with respect to $f$ (denoted by $A \bowtie^f J$), a construction that provides a general frame for studying the amalgamated duplication of a ring along an ideal, introduced and studied by D'Anna and Fontana in 2007, and other classical constructions (such as the $A + XB[X]$, the $A + XB[\![X]\!]$ and the $D + M$ constructions). In particular, we completely describe the prime spectrum of the amalgamated duplication and we give bounds for its Krull dimension.



## 1. Introduction

Let $A$ and $B$ be commutative rings with unity, let $J$ be an ideal of $B$ and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A,\ j \in J\}$$

called *the amalgamation of* $A$ *with* $B$ *along* $J$ *with respect to* $f$. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied in [9,6,10,20]). Moreover, several classical constructions (such as the $A + XB[X]$, the $A + XB[\![X]\!]$ and the $D + M$ constructions) can be studied as particular cases of the amalgamation [7, Examples 2.5 and 2.6] and other classical constructions, such as the Nagata's idealization (cf. [21, page 2], [17, Chapter VI, Section 25]), also called Fossum's trivial extension (cf. [15,19,4]), and the CPI extensions (in the sense of Boisen and Sheldon [5]) are strictly related to it [7, Example 2.7 and Remark 2.8].

On the other hand, the amalgamation $A \bowtie^f J$ is related to a construction proposed by D.D. Anderson in [1] and motivated by a classical construction due to Dorroh [12], concerning the embedding of a ring without identity in a ring with identity. An ample introduction on the genesis of the notion of amalgamation is given in [7, Section 2].

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [7, Section 4] (for a systematic study of these types of constructions, cf. [13]). This point of view allows us to deepen the study initiated in [7] and to provide an ample description of various properties of $A \bowtie^f J$, in connection with the properties of $A$, $J$ and $f$.

More precisely, in [7], we studied the basic properties of this construction (e.g., we provided characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and we characterized those distinguished pullbacks that can be expressed as an amalgamation. In this paper, we pursue the investigation on the structure of the rings of the form $A \bowtie^f J$, with

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particular attention to the prime spectrum, to the chain properties and to the Krull dimension. In particular, after recalling (in Section 2) some basic properties proved in [7,8], needed in the present paper, we start our investigation by deepening the study of chains of prime ideals in pullback constructions (Proposition 2.7).

In Section 3, we study the integral closure of $A \bowtie^f J$ in its total ring of fractions and, finally, in Section 4, we concentrate our attention to evaluate its Krull dimension. In particular, we provide upper and lower bounds for $\dim(A \bowtie^f J)$ (Proposition 4.4 and Theorem 4.9) and we show that these bounds, obtained in a such general setting, are so sharp that generalize, and possibly improve, analogous bounds established for the very particular cases of integral domains of the form $A + XB[X]$ [14] or $A + XB[\![X]\!]$ [11].

## 2. Preliminaries

Before beginning a systematic study of the ring $A \bowtie^f J$, we recall from our introductory paper [7] to the subject some basic properties of this construction.

**Proposition 2.1** ([7, Proposition 5.1]). *Let $f : A \to B$ be a ring homomorphism, $J$ an ideal of $B$ and set $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$.*

(1) *Let $\iota := \iota_{A,f,J} : A \to A \bowtie^f J$ be the natural ring homomorphism defined by $\iota(a) := (a, f(a))$, for all $a \in A$. Then $\iota$ is an embedding, making $A \bowtie^f J$ a ring extension of $A$ (with $\iota(A) = \Gamma(f)$ $(:= \{(a, f(a)) \mid a \in A\}$ subring of $A \bowtie^f J)$).*
(2) *Let $I$ be an ideal of $A$ and set $I \bowtie^f J := \{(i, f(i) + j) \mid i \in I, j \in J\}$. Then $I \bowtie^f J$ is an ideal of $A \bowtie^f J$, the composition of canonical homomorphisms $A \overset{\iota}{\hookrightarrow} A \bowtie^f J \twoheadrightarrow A \bowtie^f J/I \bowtie^f J$ is a surjective ring homomorphism and its kernel coincides with $I$.*

   *Hence, we have the following canonical isomorphism:*

$$\frac{A}{I} \cong \frac{A \bowtie^f J}{I \bowtie^f J}.$$

(3) *Let $p_A : A \bowtie^f J \to A$ and $p_B : A \bowtie^f J \to B$ be the natural projections of $A \bowtie^f J \subseteq A \times B$ into $A$ and $B$, respectively. Then $p_A$ is surjective and $\mathrm{Ker}(p_A) = \{0\} \times J$.*

   *Moreover, $p_B(A \bowtie^f J) = f(A) + J$ and $\mathrm{Ker}(p_B) = f^{-1}(J) \times \{0\}$. Hence, the following canonical isomorphisms hold:*

$$\frac{A \bowtie^f J}{(\{0\} \times J)} \cong A \quad \text{and} \quad \frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(A) + J.$$

(4) *Let $\gamma : A \bowtie^f J \to (f(A) + J)/J$ be the natural ring homomorphism, defined by $(a, f(a) + j) \mapsto f(a) + J$. Then $\gamma$ is surjective and $\mathrm{Ker}(\gamma) = f^{-1}(J) \times J$. Thus, there exists a natural isomorphism*

$$\frac{A \bowtie^f J}{f^{-1}(J) \times J} \cong \frac{f(A) + J}{J}.$$

   *In particular, when $f$ is surjective, we have the following natural isomorphism*

$$\frac{A \bowtie^f J}{f^{-1}(J) \times J} \cong \frac{B}{J}.$$

Recall that, in [7,8], we have shown that the ring $A \bowtie^f J$ can be represented as a pullback of natural ring homomorphisms and, using the notion of ring retraction, we have characterized the pullbacks that produce exactly rings of the form $A \bowtie^f J$ (see also Propositions 2.3 and 2.5). Now we will make some pertinent remarks and prove a new result on chains of prime ideals of pullbacks, that will be useful for our subsequent investigation of the ring $A \bowtie^f J$.

**Definition 2.2.** We recall that, if $\alpha : A \to C$, $\beta : B \to C$ are ring homomorphisms, the subring $D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the *pullback* (or *fiber product*) of $\alpha$ and $\beta$. In the following, we will denote by $p_A$ (respectively, $p_B$) the restriction to $\alpha \times_C \beta$ of the projection of $A \times B$ onto $A$ (respectively, $B$).

The following proposition is a straightforward consequence of the definitions.

**Proposition 2.3** ([7, Proposition 4.2]). *Let $f : A \to B$ be a ring homomorphism and $J$ be an ideal of $B$. If $\pi : B \to B/J$ is the canonical projection and $\check{f} := \pi \circ f$, then $A \bowtie^f J = \check{f} \times_{B/J} \pi$.*

Now, recall that a ring homomorphism $r : B \to A$ is called *a ring retraction* if there exists an (injective) ring homomorphism $i : A \to B$ such that $r \circ i = id_A$. In this case, we say also that $A$ is *a retract of* $B$.

**Example 2.4** ([7, Remark 4.6]). Let $f : A \to B$ be a ring homomorphism and $J$ an ideal of $B$. Then $A$ is a retract of $A \bowtie^f J$ and the map $p_A : A \bowtie^f J \to A$, defined in Proposition 2.1(3), is a ring retraction. In fact, we have $p_A \circ \iota = id_A$, where $\iota$ is the ring embedding of $A$ into $A \bowtie^f J$ (Proposition 2.1(1)).

The pullbacks of the form $A \bowtie^f J$ form a distinguished subclass of the class of pullbacks of ring homomorphisms, as described in the following proposition.

**Proposition 2.5** ([7, Proposition 4.7]). *Let $A, B, C, \alpha, \beta, p_A, p_B$ be as in Definition 2.2. Then, the following conditions are equivalent.*

(i) *$p_A : \alpha \times_C \beta \to A$ is a ring retraction.*
(ii) *There exist an ideal $J$ of $B$ and a ring homomorphism $f : A \to B$ such that $\alpha \times_C \beta = A \bowtie^f J$.*

Let $f : A \to B$ be a ring homomorphism, and set $X := \mathrm{Spec}(A)$, $Y := \mathrm{Spec}(B)$. Recall that $f^* : Y \to X$ denotes the continuous map (with respect to the Zariski topologies) naturally associated to $f$ (i.e., $f^*(Q) := f^{-1}(Q)$ for all $Q \in Y$). Let $S$ be a subset of $A$. Then, as usual, $V_X(S)$, or simply $V(S)$, if no confusion can arise, denotes the closed subspace of $X$, consisting of all prime ideals of $A$ containing $S$. We will denote by $\mathrm{Jac}(A)$ the Jacobson radical of a ring $A$ and we will call *local ring* a (not necessarily Noetherian) ring with a unique maximal ideal.

Now, we collect some results about the structure of the prime ideals of the ring $A \bowtie^f J$. The proof of the following proposition is based on well known properties of rings arising from pullbacks [13, Theorem 1.4] (for details, see [8]).

**Proposition 2.6.** *With the notation of Proposition 2.1, set $X := \mathrm{Spec}(A)$, $Y := \mathrm{Spec}(B)$, and $W := \mathrm{Spec}(A \bowtie^f J)$, and $J_0 := \{0\} \times J\ (\subseteq A \bowtie^f J)$. For all $P \in X$ and $Q \in Y$, set:*

$$P'^f := P \bowtie^f J := \{(p, f(p) + j) \mid p \in P,\ j \in J\},$$
$$\overline{Q}^f := \{(a, f(a) + j) \mid a \in A,\ j \in J,\ f(a) + j \in Q\}.$$

*Then, the following statements hold.*

(1) *The map $P \mapsto P'^f$ establishes a closed embedding of $X$ into $W$, so its image, which coincides with $V(J_0)$, is homeomorphic to $X$.*
(2) *The map $Q \mapsto \overline{Q}^f$ is a homeomorphism of $Y \setminus V(J)$ onto $W \setminus V(J_0)$.*
(3) *The prime ideals of $A \bowtie^f J$ are of the type $P'^f$ or $\overline{Q}^f$, for $P$ varying in $X$ and $Q$ in $Y \setminus V(J)$.*
(4) *Let $P \in \mathrm{Spec}(A)$. Then, $P'^f$ is a maximal ideal of $A \bowtie^f J$ if and only if $P$ is a maximal ideal of $A$.*
(5) *Let $Q$ be a prime ideal of $B$ not containing $J$. Then, $\overline{Q}^f$ is a maximal ideal of $A \bowtie^f J$ if and only if $Q$ is a maximal ideal of $B$.*

*In particular:*

$$\mathrm{Max}(A \bowtie^f J) = \{P'^f \mid P \in \mathrm{Max}(A)\} \cup \{\overline{Q}^f \mid Q \in \mathrm{Max}(B) \setminus V(J)\}.$$

The last result of this section concerns the chains of prime ideals in rings arising from pullbacks of rather general type.

**Proposition 2.7.** *With the notation of Definition 2.2, assume $\beta$ surjective. Let $H'$ and $H''$ be prime ideals of $D$ such that $H' \subsetneq H''$. Assume that $H' \in \mathrm{Spec}(D) \setminus V(\mathrm{Ker}(p_A))$, $H'' \in V(\mathrm{Ker}(p_A))$, and that $H'$ and $H''$ are adjacent prime ideals. Then, there exist two prime ideals $Q'$ and $Q''$ of $B$, with $Q' \subsetneq Q''$, and moreover such that $Q' \notin V(\mathrm{Ker}(\beta))$, $p_B^{-1}(Q') = H'$, and $p_B^{-1}(Q'') = H''$.*

**Proof.** Note that the existence (and uniqueness) of a prime ideal $Q'$ of $B$ such that $Q' \notin V(\mathrm{Ker}(\beta))$ and $p_B^{-1}(Q') = H'$ is well known [13, Theorem 1.4, Statement (c) of the proof].

On the other hand, note that $p_B^{-1}(L + \mathrm{Ker}(\beta)) = p_B^{-1}(L) + \mathrm{Ker}(p_A)$, for each ideal $L$ of $B$. Now, it is clear that the set

$$\mathcal{S}(Q') := \{L \text{ ideal of } B \mid Q' + \mathrm{Ker}(\beta) \subseteq L \text{ and } p_B^{-1}(L) \subseteq H''\}$$

is nonempty (it contains $Q' + \mathrm{Ker}(\beta)$) and inductive. Thus, by Zorn's lemma, $\mathcal{S}(Q')$ contains a maximal element $Q''$, which is easy to see that is a prime ideal of $B$. Since $H'' \supseteq p_B^{-1}(Q'') \supseteq p_B^{-1}(Q') + \mathrm{Ker}(p_A) \supsetneq H'$ and $H', H''$ are adjacent prime ideals, we have $p_B^{-1}(Q'') = H''$. □

## 3. Integral closure of the ring $A \bowtie^f J$

Given a ring extension $R \subseteq S$, the integral closure of $R$ in $S$ will be denoted by $\overline{R}^S$; the integral closure of $R$ in its total ring of fractions $\mathrm{Tot}(R)$ will be simply denoted by $\overline{R}$.

Now, we want to determine the integral closure of the ring $A \bowtie^f J$ in its total ring of fractions. It is easy to compute $\mathrm{Tot}(A \bowtie^f J)$ in some cases.

**Proposition 3.1.** *Let $f : A \to B$ be a ring homomorphism, $J$ an ideal of $B$, and let $A \bowtie^f J$ be as in Proposition 2.1. Assume that $J$ and $f^{-1}(J)$ are regular ideals of $B$ and $A$, respectively. Then $\mathrm{Tot}(A \bowtie^f J)$ is canonically isomorphic to $\mathrm{Tot}(A) \times \mathrm{Tot}(B)$.*

**Proof.** Note that $J_1 := f^{-1}(J) \times J$ is the conductor of $A \bowtie^f J$ in $A \times B$ (i.e., the largest ideal of $A \bowtie^f J$ that is also an ideal of $A \times B$). Since both $f^{-1}(J)$ and $J$ are regular ideals, then $J_1$ is a regular ideal of $A \times B$. Now, the conclusion follows immediately by applying [16, pag. 326]. □

**Remark 3.2.** Note that, in Proposition 3.1, the assumption that $J$ and $f^{-1}(J)$ are regular ideals is essential. For example, let $A$ be an integral domain with quotient field $K$, $B$ an overring of $A$, and let $J = \{0\}$. Then, in this situation, $A \bowtie^f J \cong A$ (Proposition 2.3), and thus $\mathrm{Tot}(A \bowtie^f J)$ is isomorphic to $K$, but $\mathrm{Tot}(A) \times \mathrm{Tot}(B) = K \times K$.

In the previous example, $J$ and $f^{-1}(J)$ are both the zero ideal. Another example, for which $J$ is a nonzero regular ideal, is given next. Let $A$ be an integral domain with quotient field $K$, set $B := A[X]$ and $J := (X)$, and let $f : A \hookrightarrow A[X]$ be the natural inclusion. In this case, from Proposition 2.3 we deduce that $A \bowtie^f J \cong A + XA[X] = A[X]$, and hence $\mathrm{Tot}(A \bowtie^f J) = K(X)$. However, $\mathrm{Tot}(A) \times \mathrm{Tot}(B) = K \times K(X)$. (Note that in this example $f^{-1}(J) = A \cap J = \{0\}$.)

Another example, for which both $J$ and $f^{-1}(J)$ are nonzero and not regular ideals, is the following. Let $K$ be a field and set $A := K^{(3)}$, $B := K^{(2)}$, and $J := \{0\} \times K$, where $K^{(n)}$ is the direct product ring $K \times K \times \cdots \times K$ ($n$-times). If $f$ is the projection defined by $(a, b, c) \mapsto (a, b)$, it is immediately seen that $A \bowtie^f J \cong K^{(4)}$. Then $\mathrm{Tot}(A \bowtie^f J) \cong K^{(4)}$, but $\mathrm{Tot}(A) \times \mathrm{Tot}(B) \cong K^{(5)}$.

We have already observed in [7, Section 5] that the ring $B_\diamond := f(A) + J$ (subring of $B$) plays a relevant role in the construction $A \bowtie^f J$. The next result provides further evidence to this fact.

**Lemma 3.3.** *Let $f : A \to B$ be a ring homomorphism, $J$ an ideal of $B$, and let $A \bowtie^f J$ be as in Proposition 2.1. The ring $A \times (f(A)+J)$, subring of $A \times B$, which contains $A \bowtie^f J$ is integral over $A \bowtie^f J$. More precisely, every element of $A \times (f(A)+J)$ has degree at most two over $A \bowtie^f J$.*

**Proof.** Let $(\alpha, f(a) + j) \in A \times (f(A) + J)$ with $\alpha, a \in A$ and $j \in J$. Assume that $(\alpha, f(a) + j) \notin A \bowtie^f J$, thus, in particular, $\alpha \neq a$. Then, the element $(\alpha, f(a)+j)$ is a root of the monic polynomial $(X - (\alpha, f(\alpha)))(X - (a, f(a)+j)) \in (A \bowtie^f J)[X]$. $\square$

**Proposition 3.4.** *With the notation of Lemma 3.3, assume that $J$ and $f^{-1}(J)$ are regular ideals of $B$ and $A$, respectively. Then $\overline{A \bowtie^f J}$ (i.e., the integral closure of $A \bowtie^f J$ in its total ring of fractions) coincides with $\overline{A} \times \overline{f(A)+J}$. In particular, if $f$ is an integral homomorphism, then $\overline{A \bowtie^f J} = \overline{A} \times \overline{B}$.*

**Proof.** Recall that, under the present hypothesis on $J$ and $f^{-1}(J)$, we have $\mathrm{Tot}(A \bowtie^f J) = \mathrm{Tot}(A \times B)$, which is canonically isomorphic to $\mathrm{Tot}(A) \times \mathrm{Tot}(B)$ (Proposition 3.1). Therefore, it is easy to see that $\overline{A \bowtie^f J} \subseteq \overline{A} \times \overline{f(A)+J}$. On the other hand, the ring $\overline{A} \times \overline{f(A)+J}$ is obviously integral over $A \times (f(A)+J)$ and $A \times (f(A)+J)$ is integral over $A \bowtie^f J$ (Lemma 3.3). Thus $\overline{A} \times \overline{f(A)+J}$ is integral over $A \bowtie^f J$. The conclusion is now straightforward. $\square$

**Remark 3.5.** If we do not assume that $J$ and $f^{-1}(J)$ are regular ideals of $B$ and $A$, respectively, then the argument used in the proof of Proposition 3.4 shows that the integral closure of $A \bowtie^f J$ in $\mathrm{Tot}(A) \times \mathrm{Tot}(B)$ coincides with $\overline{A} \times \overline{f(A)+J}$.

Now, we want to investigate when the ring $A \bowtie^f J$ is integral over $\Gamma(f)(:= \{(a, f(a)) \mid a \in A\})$.

**Lemma 3.6.** *Let $f : A \longrightarrow B$, $J \subseteq B$, and $A \bowtie^f J$ be as in Proposition 2.1. Then, the following conditions are equivalent.*

(i) *$f(A) + J$ is integral over $f(A)$.*
(ii) *$A \bowtie^f J$ is integral over $\Gamma(f)$.*

*In particular, if $f$ is an integral homomorphism, then $A \bowtie^f J$ is integral over $\Gamma(f)$ ($\cong A$).*

**Proof.** (i) implies (ii). Let $(a, f(a)+j)$ be a nonzero element of $A \bowtie^f J$. Thus, by condition (i), there exist a positive integer $n$ and $a_0, a_1, \ldots, a_{n-1} \in A$ such that $(f(a)+j)^n + \sum_{i=0}^{n-1} f(a_i)(f(a)+j)^i = 0$. Therefore, it is easy to verify that $(a, f(a)+j)$ is a root of the monic polynomial $[X - (a, f(a))][X^n + \sum_{i=0}^{n-1}(a_i, f(a_i))X^i] \in \Gamma(f)[X]$. Conversely, consider an element $f(a)+j \in f(A)+J$. By condition (ii), $(a, f(a)+j)$ is integral over $\Gamma(f)$, and hence the equation of integral dependence of $(a, f(a)+j)$ over $\Gamma(f)$ gives us the equation of integral dependence of $f(a)+j$ over $f(A)$. The last statement is straightforward. $\square$

## 4. Krull dimension of $A \bowtie^f J$

Now, we want to study the Krull dimension of the ring $A \bowtie^f J$. We start with an easy observation.

**Proposition 4.1.** *Let $f : A \to B$, $J$, and $A \bowtie^f J$ be as in Proposition 2.1. Then $\dim(A \bowtie^f J) = \max\{\dim(A), \dim(f(A) + J)\}$. In particular, if $f$ is surjective, then $\dim(A \bowtie^f J) = \max\{\dim(A), \dim(B)\} = \dim(A)$.*

**Proof.** By Lemma 3.3 and [18, Theorem 48], it follows immediately that $\dim(A \bowtie^f J) = \dim(A \times (f(A) + J))$. Thus, the conclusion is an easy consequence of the fact that $\mathrm{Spec}(A \times (f(A) + J))$ is canonically homeomorphic to the disjoint union of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(f(A) + J)$. The last statement is straightforward. $\square$

We already observed in [7, Section 5] that the kind of results as in the previous proposition has a moderate interest, because the Krull dimension of $A \bowtie^f J$ is compared to the Krull dimension of $f(A)+J$, which is not easy to evaluate (moreover, if $f^{-1}(J) = \{0\}$, we have $A \bowtie^f J \cong f(A) + J$ (Proposition 2.1(3))).

An easy case for evaluating $\dim(A \bowtie^f J)$ is the following.

**Proposition 4.2.** *Let $f : A \to B$, $J$, and $A \bowtie^f J$ be as in Proposition 2.1. Let $f_\diamond : A \to B_\diamond := f(A) + J$ be the ring homomorphism induced from $f$. If we assume that $f_\diamond$ is integral (e.g., $f$ is integral), then $\dim(A \bowtie^f J) = \dim(A)$.*

**Proof.** By Lemma 3.6 and [18, Theorem 48], it follows immediately that $\dim(A \bowtie^f J) = \dim(\Gamma(f)) = \dim(A)$. $\square$

We proceed our investigation looking for upper and lower bounds of the Krull dimension of $A \bowtie^f J$. By Proposition 2.6, we know that $\mathrm{Spec}(A \bowtie^f J) = X \cup U$, where $X := \mathrm{Spec}(A)$ and $U := \mathrm{Spec}(B) \setminus V(J)$ (for the sake of simplicity, we identify

$X$ and $U$ with their homeomorphic images in $\mathrm{Spec}(A \bowtie^f J)$). Furthermore, again from Proposition 2.6, we deduce that ideals of the form $\overline{Q}^f$ can be contained in ideals of the form $P'^f$, but not vice versa. Therefore, chains in $\mathrm{Spec}(A \bowtie^f J)$ are obtained by juxtaposition of two types of chains, one from $U$ "on the bottom" and the other one from $X$ "on the top" (where either one or the other may be empty or a single element). It follows immediately that both $\dim(X) = \dim(A)$ and $\dim(U)$ are lower bounds for $\dim(A \bowtie^f J)$ and $\dim(A) + \dim(U) + 1$ is an upper bound for $\dim(A \bowtie^f J)$ (where, conventionally, we set $\dim(\emptyset) = -1$).

**Remark 4.3.** Assume that $J \subseteq \mathrm{Jac}(B)$. By Proposition 2.6(5), we get that $U$ does not contain maximal elements of $\mathrm{Spec}(A \bowtie^f J)$. Hence, in this case, $1 + \dim(U) \leq \dim(A \bowtie^f J)$.

Let us define the following subset of $U$:

$$\mathcal{Y}_{(f,J)} := \left\{ Q \in U \mid f^{-1}(Q + J) = \{0\} \right\};$$

it is obvious that $\mathcal{Y}_{(f,J)}$ is *stable under generizations*, i.e., $Q \in \mathcal{Y}_{(f,J)}$, $Q' \in \mathrm{Spec}(B)$ and $Q' \subseteq Q$ imply $Q' \in \mathcal{Y}_{(f,J)}$. Hence $\dim(\mathcal{Y}_{(f,J)}) = \sup\{\mathrm{ht}_B(Q) \mid Q \in \mathcal{Y}_{(f,J)}\}$ and we will denote this integer by $\delta_{(f,J)}$.

**Proposition 4.4.** *Let $f : A \to B$, $J$, and $A \bowtie^f J$ be as in Proposition 2.1; let $U = \mathrm{Spec}(B) \setminus V(J)$ and $\delta_{(f,J)} = \dim(\mathcal{Y}_{(f,J)})$.*

(1) *Let $Q \in \mathrm{Spec}(B)$, then $f^{-1}(Q + J) = \{0\}$ if and only if $\overline{Q}^f$ $(= (A \times Q) \cap A \bowtie^f J)$ is contained in $J_0$ $(= \{0\} \times J)$.*
(2) *for every $Q \in \mathcal{Y}_{(f,J)}$, the corresponding prime $\overline{Q}^f$ of $A \bowtie^f J$ is contained in every prime of the form $P'^f$.*
(3) $\max\{\dim(U), \dim(A) + 1 + \delta_{(f,J)}\} \leq \dim(A \bowtie^f J)$.

**Proof.** (1) Assume that $f^{-1}(Q + J) = \{0\}$. If $(a, f(a) + j) \in \overline{Q}^f$, with $a \in A$ and $j \in J$, then $f(a) + j \in Q$, and so $a \in f^{-1}(Q + J) = \{0\}$, i.e., $a = 0$. Therefore, $(a, f(a) + j) = (0, j) \in J_0$. Conversely, if $a \in f^{-1}(Q + J)$, i.e., $f(a) = q + j$ for some $q \in Q$ and $j \in J$, then $f(a) - j \in Q$, and so $(a, f(a) - j) \in \overline{Q}^f \subseteq J_0$, thus $a = 0$.

(2) By Proposition 2.6(1), we have that every ideal of the form $P'^f$ contains $J_0$. The conclusion follows immediately.

(3) By the observation preceding Remark 4.3, it is enough to show that $\dim(A) + 1 + \delta_{(f,J)} \leq \dim(A \bowtie^f J)$. If $\mathcal{Y}_{(f,J)} = \emptyset$ the statement is obvious. Otherwise, let $Q_0 \subset Q_1 \subset \cdots \subset Q_r$ be a maximal chain in $\mathcal{Y}_{(f,J)}$, thus $r = \delta_{(f,J)}$. Let $P_0 \subset P_1 \subset \ldots \subset P_m$ be a chain realizing $\dim(A)$. By (2) we obtain that

$$\overline{Q}_0^f \subset \ldots \subset \overline{Q}_r^f \subset P_0'^f \subset \ldots \subset P_m'^f,$$

is a chain in $\mathrm{Spec}(A \bowtie^f J)$. $\square$

**Remark 4.5.** (a) In the situation of Proposition 4.4, note that, if $J$ is contained in the nilradical of $B$, i.e., if $V(J) = \mathrm{Spec}(B)$, then $\delta_{(f,J)} = \dim(U) = -1$. Therefore, Proposition 4.4(3) gives $\dim(A) \leq \dim(A \bowtie^f J)$. But, in this (trivial) case, we can say more, precisely that $\mathrm{Spec}(A)$ is homeomorphic to $\mathrm{Spec}(A \bowtie^f J)$ (Proposition 2.6) and so $\dim(A) = \dim(A \bowtie^f J)$. As a matter of fact, with the notation of Propositions 2.3 and 2.1, $\pi_A^* : \mathrm{Spec}(A) \to \mathrm{Spec}(A \bowtie^f J)$ is a homeomorphism.

(b) Note that, if $J \not\subseteq \mathrm{Jac}(B)$, the inequality $1 + \dim(U) \leq \dim(A \bowtie^f J)$ from Remark 4.3 can be false, as the following Example 4.6 will show.

(c) Let $f : A \to B$, $J$, and $A \bowtie^f J$ be as in Proposition 2.1. If we assume that $J \neq \{0\}$ and that $A \bowtie^f J$ and $B$ are integral domains, then, by [7, Proposition 5.2], $f^{-1}(J) = \{0\}$ and the subset $\mathcal{Y}_{(f,J)}$ of $\mathrm{Spec}(B)$, defined in the previous proposition, is nonempty, since $(0) \in \mathcal{Y}_{(f,J)}$, and so $\delta_{(f,J)} \geq 0$. The following Example 4.10 will show that $\delta_{(f,J)}$ may be arbitrarily large. Note that $\delta_{(f,J)}$ may be equal to $-1$ even if $J \neq \{0\}$, $f^{-1}(J) = \{0\}$, but $B$ is not an integral domain. It is sufficient to take $B$ equal to a local zero-dimensional ring not a field, $J$ equal to its maximal ideal, $A$ any subring of $B$ such that $J \cap A = (0)$, and $f$ be the natural embedding of $A$ in $B$ (e.g., $B := K[X]/(X^2)$, where $K$ is a field and $X$ an indeterminate over $K$, and $A$ any domain contained in $K$). In this case, $\mathrm{Spec}(B) = V(J)$ and so $\delta_{(f,J)} = -1$.

(d) Note that, in the situation of Proposition 4.4(1), we can have $\overline{Q}^f \subseteq J_0$ $(= \{0\} \times J)$ with $Q \supsetneq J$. For instance let $A := K$, $B := K[X, Y]$, $Q := (X, Y)B$, $J := XB$, and let $f : A = K \hookrightarrow K[X, Y] = B$ be the natural embedding, where $K$ is a field and $X$ and $Y$ two indeterminates over $K$. In this case, $A \bowtie^f J \cong A + J = K + XK[X, Y]$ (Proposition 2.1(3)). Clearly, $f^{-1}(Q) = f^{-1}(Q + J) = f^{-1}(J) = \{0\}$ and $\overline{Q}^f = J_0 \cong XK[X, Y]$.

**Example 4.6.** Let $K$ be a field and $X$ and $Y$ two indeterminates over $K$. Set $B := K(X)[Y]_{(Y)} \cap K(Y)[X]_{(X)}$. It is well known that $B$ is a one-dimensional semilocal domain, having two maximal ideals $M := YK(X)[Y]_{(Y)} \cap B$ and $N := XK(Y)[X]_{(X)} \cap B$. Let $J := M$, $A := K$ and let $f$ be the natural embedding of $A$ in $B$. Clearly, $f^{-1}(J) = M \cap K = \{0\}$. In this situation, $N \in \mathrm{Spec}(B) \setminus V(J)$ and so $\dim(U) = 1$. It is easy to see that $A \bowtie^f J \cong K + M$ (Proposition 2.1(3)) is a one-dimensional local domain. Therefore, in this case, we have $2 = 1 + \dim(U) > 1 = \dim(A \bowtie^f J)$.

As an immediate consequence of Remark 4.3 and Proposition 4.4, we have:

**Corollary 4.7.** *With the notation of Proposition 4.4, Let $f : A \to B$, $J$, and $A \bowtie^f J$ be as in Section 2. If we assume that $J \subseteq \mathrm{Jac}(B)$ and that $\delta_{(f,J)} \geq 0$ (e.g., $A \bowtie^f J$ and $B$ are integral domains), then*

$$1 + \max\{\dim(A) + \delta_{(f,J)}, \dim(U)\} \leq \dim(A \bowtie^f J).$$

The following observations will be useful for Remark 4.13

**Remark 4.8.** Let $f : A \to B, J$, and $A \bowtie^f J$ as in Proposition 2.1, and let $Q$ be a prime ideal of $B$.

(i) By Proposition 4.4(1), it follows immediately that $\overline{Q}^f := (A \times Q) \cap A \bowtie^f J \subsetneq J_0 := \{0\} \times J$ if and only if $Q \in \mathcal{Y}_{(f,J)}$ (as defined in Proposition 4.4), i.e. $f^{-1}(Q + J) = \{0\}$, and $Q \not\supseteq J$. Therefore, $\mathcal{Y}_{(f,J)}$ is homeomorphic to $\{H \in \mathrm{Spec}(A \bowtie^f J) \mid H \subsetneq J_0\}$.

(ii) If $A \bowtie^f J$ and $B$ are integral domains and $J \neq \{0\}$ then, in this situation, $J_0 = (0)'^f \in \mathrm{Spec}(A \bowtie^f J)$ and $f^{-1}(J) = \{0\}$ by [7, Proposition 5.2]. Therefore, $Q = (0) \in \mathcal{Y}_{(f,J)} (\neq \emptyset)$ and $\overline{Q}^f = f^{-1}(J) \times \{0\} = (0) \subsetneq J_0$; thus, if $\mathrm{ht}_{A \bowtie^f J}(J_0) < \infty$, $\delta_{(f,J)} (= \dim \mathcal{Y}_{(f,J)}) = \mathrm{ht}_{A \bowtie^f J}(J_0) - 1$.

The next goal is to determine upper bounds to $\dim(A \bowtie^f J)$, possibly sharper than $\dim(A) + \dim(U) + 1$.

**Theorem 4.9.** *Let $f : A \to B, J$, and $A \bowtie^f J$ be as in Proposition 2.1. With the notation of Proposition 4.4, assume that $A \bowtie^f J$ has finite Krull dimension. Then*

$$\begin{aligned} \dim(A \bowtie^f J) &\leq \max\{\dim(A), \dim(A/f^{-1}(J)) + \min\{\dim(B), 1 + \dim(U)\}\} \\ &\leq \min\{\dim(A) + \dim(U) + 1, \max\{\dim(A), \dim(A/f^{-1}(J)) + \dim(B)\}\}. \end{aligned}$$

**Proof.** We can assume that $\mathrm{Spec}(B) \neq V(J)$, because otherwise we already know that $\dim(A \bowtie^f J) = \dim(A)$ (Remark 4.5(a)) and so the inequalities hold.

Let $H_0 \subset H_1 \subset \ldots \subset H_n$ be a chain of prime ideals of $A \bowtie^f J$ realizing $\dim(A \bowtie^f J)$. Two extreme cases are possible.

(1) If $H_0 \supseteq \{0\} \times J$ then, by Proposition 2.6(1), the chain $H_0 \subset H_1 \subset \ldots \subset H_n$ induces a chain of prime ideals of $A$ of length $n$. From Proposition 4.4(2), we conclude that $\dim(A \bowtie^f J) = \dim(A)$.

(2) If $H_n \not\supseteq \{0\} \times J$. From Proposition 2.6(2), the chain $H_0 \subset H_1 \subset \ldots \subset H_n$ induces a chain of prime ideals of $U$ of length $n$. From Proposition 4.4(2), we conclude that $\dim(A \bowtie^f J) = \sup\{\mathrm{ht}(Q) \mid Q \in U\} = \dim(U)$.

We now consider the general case.

(3) Let $t$ be the maximum index such that $H_t \not\supseteq \{0\} \times J$, with $0 \leq t \lneq n$. According to the notations of Proposition 2.6, rewrite the given chain as follows:

$$\overline{Q}_0^f \subset \overline{Q}_1^f \subset \ldots \subset \overline{Q}_t^f \subset P_{t+1}^{\prime f} \subset P_{t+2}^{\prime f} \subset \ldots \subset P_n^{\prime f},$$

where $Q_0 \subset Q_1 \subset \ldots \subset Q_t$ is an increasing chain of prime ideals of $B$, with $Q_t \not\supseteq J$ (Proposition 2.6(2)), and $P_{t+1} \subset P_{t+2} \subset \ldots \subset P_n$ is an increasing chain of prime ideals of $A$ (Proposition 2.6(1)). Furthermore, by Proposition 2.7, we can find a prime ideal $Q$ in $V(J)$ $(\subseteq \mathrm{Spec}(B))$ such that the prime ideal $H_{t+1} = P_{t+1}^{\prime f}$ coincides also with the restriction to $A \bowtie^f J$ of the prime ideal $A \times Q$ of $A \times B$, i.e., $H_{t+1} = P_{t+1}^{\prime f} = \overline{Q}^f$. It follows immediately that $P_k \in V(f^{-1}(J))$, for $t+1 \leq k \leq n$. Therefore, $\dim(A \bowtie^f J) = (1 + t) + (n - t - 1)$ with $1 + t \leq \min\{1 + \dim(U), \dim(B)\}$ and $n - t - 1 \leq \dim\left(A/f^{-1}(J)\right)$.

Finally, it is obvious that $\min\{\dim(B), 1 + \dim(U)\} \leq \dim(B)$ and that $\dim\left(A/f^{-1}(J)\right) + \min\{\dim(B), 1 + \dim(U)\} \leq \dim(A) + \dim(U) + 1$. $\square$

**Example 4.10.** Let $V$ be a valuation domain with maximal ideal $\mathfrak{M}$ such that $V = K + \mathfrak{M}$, where $K$ is a field isomorphic to the residue field $V/\mathfrak{M}$. Let $D$ be an integral domain with quotient field $K$, and set $B := D + \mathfrak{M}$. Assume that $\dim(V) = n \geq 1$ and that $\mathfrak{Q}$ is a prime ideal of $V$ with $\mathrm{ht}_V(\mathfrak{Q}) = t + 1$, $n \geq t + 1 \geq 0$. Set $J := \mathfrak{Q} \cap B$. By the well known properties of the "$D + \mathfrak{M}$ constructions," $B_{\mathfrak{M}} = V$ [16, Exercise 13(1), page 203], so $J$ is a prime ideal of $B$ and $\mathrm{ht}_B(J) = t + 1$. More precisely, if $(0) \subset \mathfrak{Q}_1 \subset \mathfrak{Q}_2 \subset \ldots \subset \mathfrak{Q}_t \subset \mathfrak{Q}_{t+1} = \mathfrak{Q}$ is the chain of prime ideals of $V$ realizing the height of $\mathfrak{Q}$, then $Q_0 := (0) \subset Q_1 := \mathfrak{Q}_1 \cap B \subset Q_2 := \mathfrak{Q}_2 \cap B \subset \ldots \subset Q_t := \mathfrak{Q}_t \cap B \subset Q_{t+1} := \mathfrak{Q}_{t+1} \cap B = J$. Set $A := D$ and let $f : A = D \hookrightarrow D + \mathfrak{M} = B$ be the canonical embedding. Clearly, $f^{-1}(J) = \{0\}$ and so it is easy to verify that, in the present situation,

$$\begin{aligned} \mathcal{Y}_{(f,J)} &:= \{Q \in \mathrm{Spec}(B) \mid Q \notin V(J), f^{-1}(Q + J) = \{0\}\} \\ &= \{Q_k \mid 0 \leq k \leq t\} = \mathrm{Spec}(B) \backslash V(J) = U \end{aligned}$$

(see also [16, Exercise 12(1), page 202]). Therefore, $\delta_{(f,J)} = t = \dim(U)$. Moreover, if $m := \dim(D)$ $(= \dim(A))$ then, again by the well known properties of the "$D + \mathfrak{M}$ constructions", $\dim(B) = m + n$ [16, Exercise 12(4), page 203]. Henceforth, in the present example, we have $\max\{\dim(A) + 1 + \delta_{(f,J)}, 1 + \dim(U)\} = \dim(A) + 1 + \delta_{(f,J)} = m + 1 + t$.

On the other hand, since $f^{-1}(J) = \{0\}$, clearly $A/f^{-1}(J) = A$ and so $\max\{\dim(A), \dim\left(A/f^{-1}(J)\right) + \min\{\dim(B), 1 + \dim(U)\}\} = \dim(A) + \min\{\dim(B), 1 + \dim(U)\} = m + \min\{m + n, 1 + t\}$. Since $n \geq t + 1$, then $\min\{m + n, 1 + t\} = 1 + t$. Furthermore, by the fact that $f^{-1}(J) = \{0\}$, we have $A \bowtie^f J \cong A + J = D + J$ (Proposition 2.1(3)). Therefore, from Corollary 4.3 and Theorem 4.9, we deduce that $\dim(D + J) = m + 1 + t$.

Let $A \subset B$ be an arbitrary ring extension. We will apply the previous results to the polynomial rings of the form $A + XB[X]$ and we will show that the bounds given by Fontana, Izelgue and Kabbaj [14, Theorem 2.1] in the very special case where $B$ and $A$ are integral domains coincide to the bounds obtained specializing the general setting of amalgamated algebras.

**Remark 4.11.** Recall that, by [7, Example 2.5], the ring $A+XB[X]$ (respectively, $A+XB[\![X]\!]$) is naturally isomorphic to $A\bowtie^{\sigma'}XB[X]$ (respectively, $A\bowtie^{\sigma''}XB[\![X]\!]$), where $\sigma'$ (respectively, $\sigma''$) is the inclusion of $A$ into $B' := B[X]$ (respectively, into $B'' := B[\![X]\!]$).

**Corollary 4.12.** *Let $A \subseteq B$ be a ring extension and $X$ an indeterminate over $B$. Set*

$$\delta'_{(A,B)} := \sup\{\mathrm{ht}_{B[X]}(Q) \mid Q \in \mathrm{Spec}(B[X]), X \notin Q, (Q+XB[X]) \cap A = \{0\}\}.$$

*Then*

$$\max\{\dim(A)+1+\delta'_{(A,B)}, \dim(B[X,X^{-1}])\} \le \dim(A+XB[X]) \le \dim(A)+\dim(B[X]).$$

**Proof.** Let $B' := B[X]$ and $J' := XB[X]$. As observed above (Remark 4.11), we know that $A\bowtie^{\sigma'}J' = A+XB[X]$. From the definitions, it is easy to see that $\delta_{(\sigma',J')} = \delta'_{(A,B)}$. Moreover, since $\dim(B[X,X^{-1}]) = \sup\{\mathrm{ht}_{B[X]}(Q) \mid Q \in \mathrm{Spec}(B[X]), X \notin Q\} = \dim(U)$ (where $U$, in this case, is homeomorphic to $\mathrm{Spec}(B[X]) \setminus V(J')$) and $\sigma'^{-1}(J') = A \cap XB[X] = \{0\}$, the conclusion follows from Proposition 4.4(3) and Theorem 4.9. □

**Remark 4.13.** Let $A \subseteq B$ integral domains and let $N := A\setminus\{0\}$. In [14, Theorem 2.1], Fontana, Izelgue and Kabbaj proved that

$$\max\{\dim(A)+\dim(N^{-1}B[X]), \dim(B[X])\} \le \dim(A+XB[X]) \le \dim(A)+\dim(B[X]).$$

By [14, Theorem 1.2(a) and Lemma 1.3], we know that

$$\dim(N^{-1}B[X]) = \mathrm{ht}_{A+XB[X]}(XB[X]) = 1+\lambda'_{(A,B)},$$

where

$$\lambda'_{(A,B)} := \sup\{\dim\left(B[X]_{\boldsymbol{q}[X]}\right) \mid \boldsymbol{q} \in \mathrm{Spec}(B), \boldsymbol{q} \cap A = (0)\}.$$

From Remark 4.8(iii) and the proof of Corollary 4.12, we deduce the equality $\mathrm{ht}_{A+XB[X]}(XB[X]) = 1+\delta'_{(A,B)} = 1+\lambda'_{(A,B)}$, hence $\delta'_{(A,B)} = \lambda'_{(A,B)}$; moreover, we have $\dim B[X] = \dim B[X,X^{-1}]$, by [2, Proposition 1.14]. Therefore, in particular, we reobtain Fontana, Izelgue and Kabbaj's result on the dimension of the integral domain $A+XB[X]$. This fact provides further evidence on the sharpness of the bounds obtained in Proposition 4.4(3) and Theorem 4.9, in the general setting of amalgamated algebras.

We consider now the case of power series rings of the type $A+XB[\![X]\!]$ for arbitrary ring extensions $A \subset B$.

**Corollary 4.14.** *Let $A \subset B$ be a ring extension and $X$ an indeterminate over $B$. Set*

$$\delta''_{(A,B)} := \sup\{\mathrm{ht}_{B[\![X]\!]}(Q) \mid Q \in \mathrm{Spec}(B[\![X]\!])\setminus V(X), (Q+XB[\![X]\!]) \cap A = \{0\}\}.$$

*Then*

$$\max\{\dim(A)+1+\delta''_{(A,B)}, 1+\dim(B[\![X]\!][X^{-1}])\} \le \dim(A+XB[\![X]\!]) \le 1+\dim(A)+\dim(B[\![X]\!][X^{-1}]).$$

**Proof.** Keeping in mind the statements and the notation of Remark 4.11, it follows immediately that $\delta_{(\sigma'',XB[\![X]\!])} = \delta''_{(A,B)}$. Moreover, recalling that $U$, in this case, is homeomorphic to $\mathrm{Spec}(B[\![X]\!]) \setminus V(X)$, it is easy to see that $\dim(U) = \dim(B[\![X]\!][X^{-1}])$. Finally, note that $\min\{\dim(B[\![X]\!]), 1+\dim(U)\} = 1+\dim(U)$, since every maximal ideal of $B[\![X]\!]$ contains $X$ [3, Chapter 1, Exercise 5(iv)]. The conclusion is now a straightforward consequence of Corollary 4.3 and Theorem 4.9. □

**Remark 4.15.** By applying Corollary 4.14 and Remark 4.5, it follows that, if $B$ is an integral domain, then

$$1+\max\{\dim(A)+\delta''_{(A,B)}, \dim(B[\![X]\!][X^{-1}])\} \le \dim(A+XB[\![X]\!]) \le 1+\dim(A)+\dim(B[\![X]\!][X^{-1}]).$$

Now, we can compare our lower bound with that given by Dobbs and Khalis's Theorem [11, Theorem 11]. Setting

$$\lambda''_{(A,B)} := \sup\{\dim\left(B[\![X]\!]_{\boldsymbol{q}[\![X]\!]}\right) \mid \boldsymbol{q} \in \mathrm{Spec}(B), \boldsymbol{q} \cap A = (0)\},$$

they prove that

$$1+\max\{\dim(A)+\lambda''_{(A,B)}, \dim(B[\![X]\!][X^{-1}])\} \le \dim(A+XB[\![X]\!]) \le 1+\dim(A)+\dim(B[\![X]\!][X^{-1}]).$$

It is clear that $\dim\left(B[\![X]\!]_{\boldsymbol{q}[\![X]\!]}\right) = \mathrm{ht}_{B[\![X]\!]}(\boldsymbol{q}[\![X]\!])$. Moreover, it is immediately seen that, if $\boldsymbol{q} \in \mathrm{Spec}(B)$ and $\boldsymbol{q} \cap A = (0)$, then $(\boldsymbol{q}[\![X]\!]+XB[\![X]\!]) \cap A = (0)$. Since the set $\{\boldsymbol{q}[\![X]\!] \in \mathrm{Spec}(B[\![X]\!]) \mid \boldsymbol{q} \in \mathrm{Spec}(B)$ and $\boldsymbol{q} \cap A = \{0\}\}$ is a subset of $\{Q \in \mathrm{Spec}(B[\![X]\!]) \mid X \notin Q$ and $(Q+XB[\![X]\!]) \cap A = \{0\}\}$, we have $\lambda''_{(A,B)} \le \delta''_{(A,B)}$. It is natural to ask, as in the polynomial case: does $\lambda''_{(A,B)} = \delta''_{(A,B)}$ hold? At the moment, the answer to this question is open. However, by [11, Theorem 7], we observe that the answer could be negative if

$$\mathrm{ht}_{A+XB[\![X]\!]}(XB[\![X]\!]) = 1+\delta''_{(A,B)}$$

and $\lambda''_{(A,B)} \lneq \sup\{\mathrm{ht}_{B[\![X]\!]}(Q) \mid Q \in \Lambda_{(A,B)}\}$, where $\Lambda_{(A,B)}$, as in [11, Theorem 7], is defined to be $\Lambda_{(A,B)} = \{Q \in \mathrm{Spec}(B[\![X]\!]) \mid X \notin Q,\ Q \subset (\boldsymbol{q}, X),$ for some $\boldsymbol{q} \in \mathrm{Spec}(B)$ with $\boldsymbol{q} \cap A = (0)\}$.

**Example 4.16.** It is possible to construct an infinite-dimensional ring of the type $A \bowtie^f J$, where $A$ is a finite-dimensional ring. In this situation, $B$ must be a infinite-dimensional ring by Theorem 4.9. For instance, let $A := \mathbb{C}$ be the field of complex numbers, let $Y$ be an indeterminate over $\mathbb{C}$, and let $R := \mathbb{C}[\{Y^{1/n} \mid n \in \mathbb{N}\backslash\{0\}\}]$. Consider the maximal ideal $\mathfrak{M}$ of $R$ generated by the set $\{Y^{1/n} \mid n \in \mathbb{N}\backslash\{0\}\}$. Set $B := R_{\mathfrak{M}}$, and consider the ring $A + XB[\![X]\!]$ ($\cong A \bowtie^{\sigma''} XB[\![X]\!]$, according to notation of Remark 4.11). Then, by [11, Example 3], $B$ is a one-dimensional non-discrete valuation domain and $\mathrm{ht}_{A+XB[\![X]\!]}(XB[\![X]\!]) = \infty$, and thus $\dim(A + XB[\![X]\!]) = \infty$.

The next two examples show that the upper bound and lower bound of Theorem 4.9 and Proposition 4.4(3) are "sharp", in the sense that $\dim(A \bowtie^f J)$ may be equal to each of the two numerical terms appearing in the first inequality (respectively, in the inequality) of Theorem 4.9 (respectively, Proposition 4.4(3)).

**Example 4.17.** Let $A$ be a valuation domain such that $\dim(A) = n \geq 3$, let $\{0\} \subset P_1 \subset P_2 \subset \ldots \subset P_n$ be a chain of prime ideals of $A$ realizing $\dim(A)$, and let $x_h \in P_{h+1}\backslash P_h$, with $1 \leq h \leq n-2$ and $(x_h) \neq P_{h+1}$. Since $A$ is a valuation domain, it is easily seen that $V(x_h) = V(P_{h+1})$, and thus $\dim(A/(x_h)) = \dim(A/P_{h+1}) = n - (h+1)$. Set $B := A/(x_h)$, $f : A \twoheadrightarrow B$ the canonical projection, $Q_k := P_k/(x_h)$ for $h+1 \leq k \leq n$, and $J := Q_{h+j}$ for some $1 \leq j \leq n-h$. In this case, by Proposition 4.1, $\dim(A \bowtie^f J) = \dim(A \times B) = \max\{\dim(A), \dim(B)\} = \dim(A) = n$. Note also that $\dim\left(A/f^{-1}(J)\right) = n - (h+j) \leq n - (h+1) = \dim(B)$, and

$$\dim(U) = \begin{cases} -1, & \text{if } j = 1, \\ j-2, & \text{if } 1 < j \leq n-h. \end{cases}$$

It is also easy to see that $f^{-1}(Q+J) \neq \{0\}$ for all $Q \in \mathrm{Spec}(B)$ and so in this case $\delta_{(f,J)} = -1$, for all $1 \leq j \leq n-h$. Moreover, in the present situation, $A \bowtie^f J$ is a local ring, but it is not an integral domain since $f^{-1}(J) \neq \{0\}$ (see [7, Proposition 5.2]).

Consider now a chain $H_0 \subset H_1 \subset \ldots \subset H_n$ of prime ideals of $A \bowtie^f J$ realizing $\dim(A \bowtie^f J)$. Two cases are possible.

- If $1 \leq j \leq n-h$, then the previous chain (realizing $\dim(A \bowtie^f J)$) is of the type:

$$\begin{aligned} ((0) \neq) P_0'^f &\subset P_1'^f \subset \ldots \subset P_h'^f \\ &\subset P_{h+1}'^f = \overline{Q}_{h+1}^f \subset \ldots \subset P_{h+j-1}'^f = \overline{Q}_{h+j-1}^f \\ &\subset P_{h+j}'^f \subset \ldots \subset P_n'^f \end{aligned}$$

(where $P_k'^f = \overline{Q}_k^f$ also for $h+j \leq k \leq n$, but in this case $Q_k \supseteq J$);

- If $j = 1$, then the previous chain realizing $\dim(A \bowtie^f J)$ is of the type:

$$((0) \neq) P_0'^f \subset P_1'^f \subset \ldots \subset P_h'^f \subset \ldots \subset P_n'^f$$

and none of the $P_k'^f$ is equal to a $\overline{Q}_k^f$ for $Q_k \not\supseteq J$.

In the present example, the inequality of Corollary 4.3 gives back the inequality $\max\{\dim(A) + 1 + \delta_{(f,J)},\ 1 + \dim(U)\} = \max\{n + 1 - 1,\ 1 + (j-2)\} \leq n = \dim(A \bowtie^f J)$. The first inequality of Theorem 4.9 gives $\dim(A \bowtie^f J) = n \leq \max\{n,\ n - (h+j) + \min\{n - (h+1),\ 1 + (j-2)\}\} = \max\{\dim(A),\ \dim\left(A/f^{-1}(J)\right) + \min\{\dim(B),\ 1 + \dim(U)\}\}$.

**Example 4.18.** Let $K$ be a field and let $V$ and $W$ be two incomparable finite-dimensional valuation domains having same field of quotients $F$. Assume that $V$ and $W$ are $K$-algebras, that $V = K + \mathfrak{M}$ and $W = K + \mathfrak{N}$ where $\mathfrak{M}$ (respectively, $\mathfrak{N}$) is the maximal ideal of $V$ (respectively, $W$), and that $\dim(V) = m \geq 1$ and $\dim(W) = n \geq 1$. Set $T := V \cap W$. It is well known that $T$ is a finite-dimensional Bézout domain with quotient field $F$ and with two maximal ideals $M := \mathfrak{M} \cap T$ and $N := \mathfrak{N} \cap T$ such that $T_M = V$ and $T_N = W$, and so $\dim(T) = \max\{m, n\}$ [18, Theorem 101]. Let $D$ be an integral domain of Krull dimension $d$ with quotient field $K$. Since $D$ is embedded naturally in $V$ ($= K + \mathfrak{M}$) and $W$ ($= K + \mathfrak{N}$), we have also a natural embedding $\iota : D \hookrightarrow T$.

In this situation, using the standard notation of the $A \bowtie^f J$ construction, when $A := D, B := T, J := M$, and $f := \iota$, we have that the ring $D + M$ (subring of $T$) is canonically isomorphic to $D \bowtie^{\iota} M$, by [7, Example 2.6]. Moreover, $f^{-1}(J) = M \cap D = \{0\}$ and so $\dim(A/f^{-1}(J)) = \dim(D) = d$, and $\dim(U) = \max\{m-1, n\}$ .

It is easy to verify that if $(0) = \mathfrak{Q}_0 \subset \mathfrak{Q}_1 \subset \cdots \subset \mathfrak{Q}_m = \mathfrak{M}$ are the prime ideals of $V$, then $\{Q_k := \mathfrak{Q}_k \cap B \mid 0 \leq k \leq m-1\}$ coincides with the set $\{Q \in \mathrm{Spec}(B)\backslash V(J) \mid f^{-1}(Q+J) = (Q+J) \cap D = \{0\}\}$. Therefore, $\delta_{(f,J)} = m-1$. On the other hand, it is easy to verify that $\dim(D+M) = \max\{m+d, n\}$.

In the present example, the inequality of Proposition 4.4(3) gives back the inequality $\max\{\dim(A) + 1 + \delta_{(f,J)},\ \dim(U)\} = \max\{d + 1 + m - 1,\ \max\{m-1, n\}\} \leq \max\{m+d, n\} = \dim(A \bowtie^f J) = \dim(D+M)$. Therefore, if $n > m+d$, then $\dim(A \bowtie^f J) = \dim(U)$. By the first inequality of Theorem 4.9, it follows that $\dim(A \bowtie^f J) = \max\{m+d, n\} \leq \max\{d,\ d + \min\{\max\{m, n\},\ 1 + \max\{m-1, n\}\}\} = \max\{\dim(A), \dim\left(A/f^{-1}(J)\right) + \min\{\dim(B),\ 1 + \dim(U)\}\}$. Therefore, if $m + d \leq n$, then $n = \dim(D+M) = \dim(D \bowtie^{\iota} M) = d + \min\{\max\{m, n\},\ 1 + \max\{m-1, n\}\}$.

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