Soluzioni 4-AM4

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16 ottobre 2003

1.

1) Integrando per parti si ottiene

$$\int x^3 \arcsin \frac{1}{x}\, dx = \frac{x^4}{4} \arcsin \frac{1}{x} + \frac{1}{4} \int \frac{x^3}{\sqrt{x^2-1}}\, dx.$$

Con la sostituzione $x = \cosh t$ si ha

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2-1}}\, dx &= \int \cosh^3 t\, dt \\ &= \int \cosh t (1 + \sinh^2 t)\, dt \\ &= \int d(\sinh t) \int \sinh^2 t\, d(\sinh t) \\ &= \sinh t + \frac{\sinh^3 t}{3} \end{aligned}$$

quindi il risultato sarà:

$$\int x^3 \arcsin \frac{1}{x}\, dx = \frac{x^4}{4} \arcsin \frac{1}{x} + \frac{\sqrt{x^2-1}}{4} + \frac{(x^2-1)^{\frac{3}{2}}}{12}.$$

2) Mediante la sostituzione $x = e^t$ si ottiene

$$\begin{aligned} \int \cos(\ln x) dx &= \int e^t \cos t\, dt \\ &= e^t \cos t + \int e^t \sin t\, dt \end{aligned}$$

$$= \quad e^t \cos t + e^t \sin t - \int e^t \cos t \, dt$$

da cui segue che

$$\int e^t \cos t \, dt = \frac{e^t}{2}(\cos t + \sin t)$$

e in termini della variabile di partenza x

$$\int \cos(\ln x) \, dx = \frac{1}{2} x \big(\cos(\ln x) + \sin(\ln x) \big)$$

3) Si osservi che l'integrale esiste per $x < 1$.

$$\begin{aligned}
\int x^2 \ln \sqrt{1-x} \, dx &= \frac{x^2}{2} \ln|1-x| \, dx \\
&= \frac{x^3}{6} \ln|1-x| + \int \frac{x^3}{6} \frac{1}{1-x} \\
&= \frac{x^3}{6} \ln|1-x| - \frac{1}{6} \int \left(x^2 + x + 1 - \frac{1}{1-x} \right) dx \\
&= \frac{x^3}{6} \ln|1-x| - \frac{1}{6} \left[\frac{x^3}{3} + \frac{x^2}{2} + x - \ln|1-x| \right]
\end{aligned}$$

4) Scrivendo

$$\frac{x^2-1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{2Bx+C}{x^2+1} = x^2 - 1$$

e risolvendo il sistema nelle incognite A, B, C si ottiene che

$$\int \frac{x^2-1}{(x-2)(x^2+1)} \, dx = A \ln|x-2| + B \ln|x^2+1| + C \arctan x.$$

5) Scrivendo

$$\frac{2x-3}{(x-2)(x+1)^2}\,dx = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

e risolvendo il sistema nelle incognite A, B, C si ottiene che

$$\int \frac{2x-3}{(x-2)(x+1)^2}\,dx = A\ln|x-2| + B\ln|x+1| - \frac{C}{x+1}.$$

2. In coordinate polari si ottiene

$$\begin{aligned}\int_S f\,d\sigma &= \int_0^{2\pi}\int_0^1 \rho(\rho^4\cos^4\theta + \rho^2\sin^2\theta)\sqrt{1+\rho^2}\,d\rho\,d\theta \\ &= \int_0^{2\pi}\cos^4\theta\int_0^1 \rho^5\sqrt{1+\rho^2}\,d\rho\,d\theta + \int_0^{2\pi}\sin^2\theta\int_0^1 \rho^3\sqrt{1+\rho^2}\,d\rho\,d\theta\end{aligned}$$

Usando la relazione $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ si ottiene che

$$\int_0^{2\pi}\cos^4\theta\,d\theta = \frac{3}{4}\pi$$

$$\int_0^{2\pi}\sin^2\theta\,d\theta = \pi.$$

Inoltre

$$\begin{aligned}\int_0^1 \rho^5\sqrt{1+\rho^2}\,d\rho &= \int_0^1 \rho^4(\rho\sqrt{1+\rho^2})\,d\rho \\ &= \left[\frac{1}{3}(1+\rho^2)^{\frac{3}{2}}\rho^4\right]_0^1 - \frac{4}{3}\int_0^1 \rho^3(1+\rho^2)^{\frac{3}{2}}\,d\rho \\ &= \left[\frac{1}{3}(1+\rho^2)^{\frac{3}{2}}\rho^4 - \frac{4}{3\cdot 5}(1+\rho^2)^{\frac{5}{2}}\rho^2\right]_0^1 + \frac{4\cdot 2}{3\cdot 5}\int_0^1 \rho(1+\rho^2)^{\frac{5}{2}}\,d\rho \\ &= \left[\frac{1}{3}(1+\rho^2)^{\frac{3}{2}}\rho^4 - \frac{4}{3\cdot 5}(1+\rho^2)^{\frac{5}{2}}\rho^2 + \frac{4\cdot 2}{3\cdot 5\cdot 7}(1+\rho^2)^{\frac{7}{2}}\right]_0^1\end{aligned}$$

Analogamente si procede per il calcolo di

$$\int_0^1 \rho^3\sqrt{1+\rho^2}\,d\rho = \left[\frac{\rho^2}{3}(\sqrt{1+\rho^2})^{\frac{3}{2}} - \frac{2}{3\cdot 5}(1+\rho^2)^{\frac{5}{2}}\right]_0^1$$

3. Risulta $I_n = -x^n e^{-x} + nI_{n-1}$.