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The Weak Stability Boundary

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Historical background of the problem

Human mind has always been attracted by the fascinating motion of planets and stars in the sky.
From the Babylonians up to our days there are so many models describing the motion of the celestial elements each representing an important step to knowledge, and many other still have to come.
Mach has been improved from Ptolemy's theory with planets moving over circles about circles around the Earth (epicycles) to today's questions about the form of the Universe; but, in particular the basic steps have been done by three important mathematicians.
The first to give a huge contribution was Kepler in 1600 with is three laws of motion; later in time it was Newton to discover the Law of Universal Gravitation which totally explained Kepler's discoveries; he also completely solved the problem of the motion of two point masses under an inverse square law of attraction, while he declared that the problem of finding the same for three point masses *"exceeds, if I am not mistaken, the force of any human mind".*
. The last deep contribution was by Henry Pioncaré, who discovered that *"...it may happen that small differences in the initial conditions produce very great ones in the final phenomena. [...] Prediction becomes impossible, and we have the fortuitous phenomenon."* And this is the main feature of chaos.

Probably the Golden age of solar system exploration is that since 1959 up to our days. Advancements in rocketry after World War II enabled our machines to break the grip of Earth's gravity and sent automated spacecraft and human-crewed expeditions to explore (lending or only orbiting) the whole system. Through the electronic sight and other "senses" of our automated spacecraft, color and complexion have been given to worlds before seen as indistinct points of light, and dozens of previously unknown objects have been discovered.Future historians will likely view these pioneering flights through the solar system as some of the most remarkable achievements of the 20th century.

While manned travel in Earth-Moon space is today quite common, we are also concentrating on sophisticated applications of Earth and interplanetary satellites in the fields of communications, navigation and other research. The perturbing theory has been more and more studied by the arising concept of the sphere of influence that brakes an n-body motion into more two-body (perturbed) motions.
In fact, generally speaking, interplanetary spacecraft spends most of its flight time moving under the gravitational influence of a single body: the Sun. Only for brief periods, compared with the total mission duration, is its path shaped by the gravitational field of the departure or arrival planet also the

perturbations caused by any other planet are negligible.
The computation of a mission orbit is a trial-and-error procedure involving numerical integration of the complete equations of motion where all perturbation effects are considered. The best method available for such an analysis is called the patched-conic approximation; it permits us to ignore the gravitational influence of the Sun until the spacecraft is at great distance from the Earth, so the first step of mission design is to find sphere of influence of the departing planet; for the most planets the departure and arriving orbits can be considered as circular and coplanar, and among the possible transfers with this initial and final conditions Hohmann transfer have been considered for very many years the most economical, although wasting in time. It is is a fuel efficient way to transfer from one circular orbit to another both lying in same plane (same inclination), but at different altitude. The first Lunar missions, as well as the future mission to Mars are mainly planned with this kind of transfer, unfortunately it also is too wasting in time so that, for example, up to now there is no way to send men to Mars.

Waiting the necessary technology progress that will bring to new possibilities in space traveling, the new frontier of research is the application of the dynamics to mission design; this is the way the best mathematicians efforts about Celestial Mechanics are directed to, and actually focusing on. This kind of research has a relatively brief story.

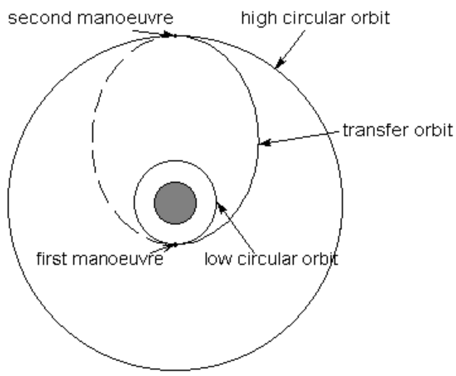

Figure 1: Hohmann transfer, discoveder by Walter Hohmann in 1925

Tomorrow's concept of traveling in the space will probably be to reach a certain point of the space points under determinate conditions and then automatically be transported by gravitational forces, without the necessity of fuel, but this necessarily means a deeper use (and comprehension) of the dynamics beyond the three or more body problem.

In this direction, perhaps, one of the main concept is that of Ballistic Capture mainly developed by Belbruno. It is the most natural way to use the dynamics till to now.

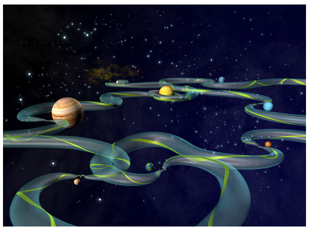

Figure 2: Interplanetary Highway

The Low Energy Transfer, as are called the transfer which use this kind of Capture, has unique property that upon arrival at the Moon, a spacecraft is automatically captured in an elliptical orbit without needing to reduce it's velocity with the use of rockets. Transfers using this process were first precisely numerically demonstrated in 1987 using a Weak Stability Boundary (WSB) theory. These were initially designed for shuttles requiring electric propulsion. These transfers are of relevant interest for applications because of their fuel-saving properties.

This transfer (also called WSB transfer), when compared to Hohmann transfer, has many advantages. If one considers, for example, a transfer from a Low Earth Orbit (LEO) at 200 kilometers altitude to a Low Lunar Orbit (LLO) at altitude 100 kilometers, the WSB transfer saves about 25% of the fuel required for lunar capture, since it eliminates the necessity of using rockets to reduce the velocity at lunar periapsis upon arrival (the so called hyperbolic excess velocity). This 25% improvement can, in certain circumstances, double the payload that can be placed into LLO. Furthermore they use the dynamics of the problem in a more natural way than the Hohmann transfers do that is, the capture process is more gradual, not requiring big Newton thrusters, and it directly implies that less fuel is also required for orbit maintenance. By the way, WSB transfer has the increased flight time between 60 and 100 days and an Earth apocenter of approximately 1.5 million kilometers which should be considered in mission design.
This transfer is being considered in a number of mission because of the possible doubling payload aspect and it may be very advantageous for any future lunar base development.

The problem of a transfer requiring less fuel than an Hohmann transfer needs vas firstly proposed by the Japanese Isas Institute that wanted to get MUSES-A and into lunar orbit as a replacements for MUSES-B, a smaller shuttle that, when detached from the bigger one get lost in the space instead of going around the Moon as expected. MUSES-A had a very small fuel capability of approximately, far less than what is necessary to be placed

into lunar orbit using an Hohmann transfer. This is because it was never designed to go to the Moon.
A solution was found by Belbruno and Miller in June 1990 to enable MUSES-A, renamed Hiten, to reach the Moon on a ballistic capture transfer to the region $\mathcal{W}$ (the Weak Stability Boundary). This transfer rescued the Japanese lunar mission. Without it, there was not enough propellent to get Hiten to the Moon by any other means.

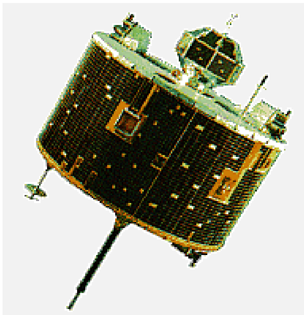

Figure 3: Muses-A renamed Hiten

Arguments of this thesis

◇ **What is the planar circular restricted three body problem?**

The circular, planar, restricted, three-body problem represents one of the long standing problems in orbital dynamics it represents a particular case of the more general three-body problem, that is the problem of three point-masses that move under the action of their mutual gravitational action only. The restricted circular three body problem concerns the motion of a small object of negligible mass (restricted case)in the vicinity of two larger primary masses. The two Primaries move about each other on circular orbits (circular case), unaffected by the presence of the small one, since that is assumed not to perturb the motion of the two primaries. Moreover the three objects move on the same plane (planar case). In spite of the simplifying assumptions, the resulting motion of the small mass under the gravitational action of the primary bodies can even be chaotic.

◇ **What is a Weak Stability Boundary?**

Roughly speaking WSB region (also named Fuzzy Boundary Region) is described as “a transition region between gravitational capture and escape from the Moon in the phase space.” To give another definition we could define it as the zone of the phase space that allow us to transfer (for example)

from the Earth to the Moon maximally using the attraction forces of the Primaries, or paraphrasing it, as a complicate zone around the Moon that once we are on a point of, we are naturally captured by the Moon without spending any propellent.
We now introduce a more mathematically definition of WSB that was given by Belbruno when he first approached to this concept and that is substantially algorithmical; but firstly we have to introduce the concept of “Ballistic Capture”. This concept is defined for the circular restricted problem and is based on monitoring the sign of Kepler’s energy function (i.e. the function that describes the energy of the Shuttle/Moon system) with respect to the smaller primary.
Consider a radial line l coming out from the center of the Moon, inclined of an angle θ from the Earth-Moon line; now take that line as the major axes of the Shuttle’s orbit; we want the Shuttle to start it’s motion on a determined point of that line that we impose to be the periapsis of it’s osculating orbit (this implies that the initial velocity of the Shuttle is perpendicular to l). We also impose that the initial two-body energy of the system is negative. Than we can say that:

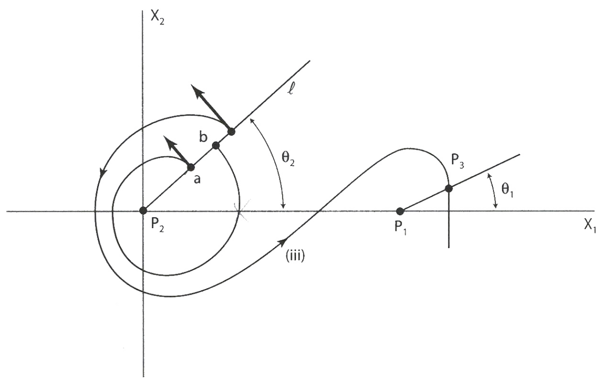

Figure 4: Belbruno’s scheme for defining the WSB.

Definition 1 *The motion of the Shuttle about the Moon is* ***stable*** *if, after leaving* l*, it makes a full cycle about the Moon without going around the Earth (i.e., calling* θ_1 *the polar angle of the Shuttle with respect to the Earth, as in Figure 4, we want that* $\theta_1 \neq 0$*) and returns to a point on* l *still with negative energy.*

Instead

Definition 2 *The motion of the Shuttle about the Moon is* ***unstable*** *if:*
- it performs a full cycle about the Moon without going around the bigger

Primary but returns to a point on l with positive energy
- it moves away from the Moon towards the Earth and makes a full cycle about the second or collides with it. P_1.

We can finally introduce the definition of the WSB that is:

Definition 3 *The Weak Stability Boundary is the two dimensional stability transition region of position and velocity space described by*

$$\mathcal{W} := \{r^*(\theta, e) \in \mathbb{R}^1 | \theta \in [0, 2\pi], e \in [0, 1]\}. \tag{1}$$

With r^* being distance between the Shuttle and the Moon at the periselenium (a function of the eccentricity and the inclination of the Shuttle's orbit), such that

- ⋄ *If $r < r^*$, the motion is stable.*
- ⋄ *If $r > r^*$, the motion is unstable.*

As pointed out by Garcia and Gomez in [1] the above definition is not accurate in the following point: it is not clear that, for fixed values of θ and e, there is a finite distance $r^*(\theta, e)$ defining the boundary of the stable and unstable orbits. Indeed they gave *numerical evidence* that "along $l(\theta)$ there are several transitions from stability to instability and, in particular, for a fixed value of θ, the set of stable points recalls a Cantor set". Garcia and Gomez also discussed an alternative definition of $\mathcal{W}$ that, however, is also not satisfactory. As a matter of fact it seems that the weak stability boundary has actually not a so simple structure as Belbruno says.

Anyway as an attempt to clarify the question, at least from an analytical point of view, let us give the following alternative definition of r^*:

$$r^* := \sup \{ r > 0 \text{ s.t. if } \ r_2 < r \text{ the motion is stable} \} . \tag{2}$$

where, of course, "stable" is as Definition 1.
We will prove that now r^* is well defined; in particular we will show that the set defined on the right hand side of (2) is not empty. This fact is quite intuitive: if the Shuttle starts very close to the Moon the gravitational influence of the Earth will be negligible and the problem substantially reduces to a two body system with the Shuttle performing an almost elliptic orbit around the Moon (at least for short times), which is stable according to the above definition. Then it is immediate to see that $0 < r^* < 1$ (in the scaled unity of measure).
With the definition of r^* given in (2) the resulting $\mathcal{W}$ will actually be a subset of the weak stability boundary numerically considered by Belbruno himself and Garcia and Gomez.

The aim of this thesis is to prove that the definition of r^* in (2) is well posed and, mainly, *to give an analytical estimate from below on* r^*.
To achieve this result we first put the system in Delaunay variables that are a set of canonical action-angle variables, which are commonly used in general perturbation theories.
The element set consists of two conjugate action-angle pairs, namely L, ℓ, G, g.

- $L$is related to the Energy of the orbit of the Shuttle around the Moon; more specifically it's square is related tho the major axes of the osculating ellipse that, as we will see, could be seen as an expression of the energy of the orbit;
- G is the magnitude of the orbital angular momentum;
- ℓ is the Mean Anomaly. It is equal to the area of the ellipse swept by the angle between the position of the Shuttle and the line joining the Moon and the Earth traced at the initial instant of the time t over the Area of the ellipse, so it is deeply related to the angle ν in the figure called Eccentric Anomaly.
- g is the angle between the line that contains the major axes of the osculating ellipse and the Earth/Moon ray. It is oriented anticlockwise as in the figure below, and summed with the angle following the Earth gives the so called "Argument of the Periselenium" φ_0 that is the inclination of the major axes of the Shuttle's orbit with respect to the initial Earth/Moon joining line.

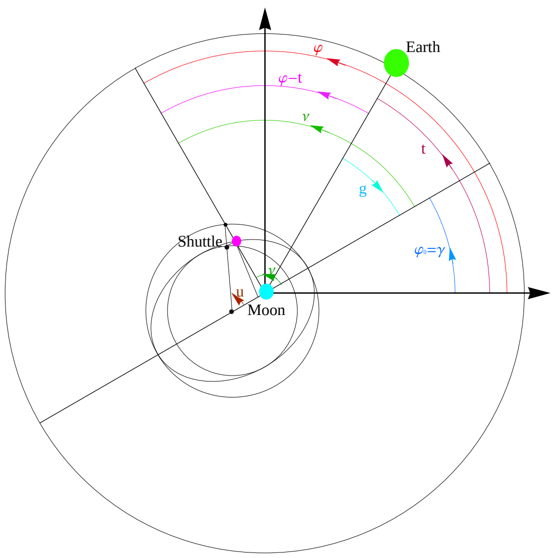

Figure 5: **Delaunay angle variables.**

Then we will proceed in two alternative ways:

1. by Normal Forms,

2. by direct evaluations.

Normal Forms are typical tools in Perturbation Theory. If the perturbative parameter (in this case the distance between the Shuttle and the Moon) is sufficiently small one can construct a canonical change of coordinates transforming the Hamiltonian in an integrable system plus (exponentially) small terms. The Normal Form has therefore the great advantage to give a very precise description of the dynamics on very long interval of times. The drawback is that usually the perturbative parameter must be very small. Also in the present case the estimates are not realistic: the resulting bound on r^* is actually smaller than the radius of the Moon!

To obtain realistic estimates we faced the problem from a more direct point of view. Looking at the equation of motion one sees that, if the Shuttle starts quite close to the Moon the mean anomaly is a "fast angle". Then by direct evaluations we will be able to estimate form below the derivative with respect to time of the mean anomaly, proving that the Shuttle actually comes back to the starting line l in a time $\mathcal{T}$ which is quite short. On this short interval the variation of the other involved quantities will be small and the orbit will be stable according to Definition 1. This gives a *realistic analytic estimate* from below on r^*: it results to be of the same order of magnitude as the escaping ray numerically found by Belbruno and actually used in the recovering of the Hiten mission.

Thesis description

- Chapter 1: Basic Laws. In the first chapter we have presented the basic laws of dynamics and astrodynamics that are Kepler's three laws of gravitation, Newton's laws of motion and the law of universal gravitation.

- Chapter 2: The Three Body Orbital Mechanics. The second Chapter is an introduction to the problem; setting the fundamental equations of motions we have deduced the expressions for the potential and kinetic energy in order to find the total energy of the system.
Than we have restricted to the case of circular planar restricted three body problem introducing Jacobi coordinates.
Finally we have passed to a Moon centered system of reference and again we have found the total energy in the restricted case.

- Chapter 3: The Delaunay Action-Angle variables. In order to introduce the Delaunay's action angle variables we have first analyzed the

two body problem with the introduction of polar coordinates, effective potential and eccentric anomaly.
We have deduced from it the conservation of the angular momentum and also Kepler's laws Than we have introduced the Delaunay coordinates for two-body system also explicitly finding the period of revolution and the frequency for the circular case. Moreover we have then passed to the three body again introducing Polar coordinates then Action-angle variables and finally finding explicit expressions for Delaunay coordinates.
The last section is dedicated to finding the solution of the Kepler's equation thet naturally comes out from the analysis above. Firstly we have performed the Classic Lagrange's expansion, than extending the solution to higher eccentricities in terms of derivatives of Bessel's functions.

- Chapter 4: Jacobi's Integral. Here comes some important knowledge to understand Belbruno's explanation of WSB. We have firstly defined the Jacobi's integral and domonstrated it to be an integral of motion for the three body problem. We have then explicitly found its relation with the two-body energy and the angular momentum and also its expressions in variou systems of reference.
Than we have briefly treated Hill's regions and their practical meaning obviously including Lagrangian points.

- Chapter 5: Weak Stability Boundary. In chapter five we have briefly explained the born of the WSB concept and compared this new arising way of transfer with the classical and repeatedly collaudated Hohmann transfer.
Than, giving the definition of Ballistic Capture and Stability we have naturally arrived to Belbruno's definition of Weak Stability Boundary.
In the final part of the chapter we have reported and than analyzed Garcia and Gomez "A note on Weak Stability Boundary" also expressing some doubts about the article.

- Chapter 6: Searching for the upper bound of stable radii. Since the WSB through Garcia and Gomez's article we have understood that much still have to be understood about the Low Energy way of transfer we have given our own definition of WSB, then demonstrating it to be consistent. by to different approaches. First approach by Normal Form Theorem: demonstration, theoretical and practical application. It led to a deep knowledge comprehension of the dynamics of the system for long period but to a not very realistic estimate, as usual in KAM theory. Second approach by direct estimation - existence of a "fast angle", estimation of its period , evaluation of the variation of the other variables, conclusion finding a realistic analytical estimation of stable zone (with the stble radius considered equal to 1/6 of Belbruno's ray of escaping used for Hiten's mission).

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