AM120-2014 Settimana 7

SERIE DI POTENZE NEL CAMPO COMPLESSO

1. Definizione $z_n \in \mathbf{C}$ converge a z (e scriveremo $z_n \to_n z$) se e solo se $|z_n - z| \to_n 0$, ovvero, se $\forall \epsilon > 0 \quad \exists n_\epsilon : \quad |z_n - z| \le \epsilon \quad \forall n \ge n_\epsilon$.

Siccome $|z_n - z|^2 = |Rez_n - Rez|^2 + |Imz_n - Imz|^2$, si ha che:

$$z_n \to_n z \qquad \Leftrightarrow \qquad Rez_n \to_n Rez \quad \text{e} \quad Imz_n \to_n Imz$$

La condizione necessaria e sufficente di Cauchy

$$z_n \to_n z \quad \Leftrightarrow \quad \forall \epsilon > 0, \exists n_\epsilon : \quad n, m \ge n_\epsilon \quad \Rightarrow \quad |z_n - z_m| \le \epsilon$$

2. Definizione $\sum_{n=1}^{\infty} z_n$ converge sse $S_N := \sum_{n=1}^{N} z_n$ converge.
$\sum_n z_n$ si dice assolutamente convergente se $\sum_n |z_n| < +\infty$.

(**Cauchy**) $\sum_n z_n$ converge $\Leftrightarrow$ $\forall \epsilon > 0, \exists N_\epsilon : \ |\sum_{n=N}^{N+p} z_n| \le \epsilon \quad \forall N \ge N_\epsilon, \ \forall p$.

In particolare, $\sum_n |z_n| < +\infty \quad \Rightarrow \quad \sum_n z_n$ converge e in particolare,

$\limsup_n |z_n|^{\frac{1}{n}} < 1 \ \Rightarrow \ \sum_n |z_n| < +\infty \quad \Rightarrow \quad \sum z_n$ converge. Si ha cosí

3. Cauchy-Hadamard Sia $a_n \in \mathbf{C}$, $r := \limsup_n |a_n|^{-\frac{1}{n}}$. Allora

$$z \in \mathbf{C}, \quad |z| < r \quad \Rightarrow \quad \sum_{n=0}^{\infty} |a_n z^n| < +\infty \ , \qquad |z| > r \quad \Rightarrow \quad \sum_{n=0}^{\infty} |a_n z^n| = +\infty$$

r:= raggio di convergenza, $D_r := \{z : \ |z| < r\}$:= disco di convergenza .

ESEMPIO. $\sum_{n=0}^{\infty} z^n$ converge in $|z| < 1$ e $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

4. La funzione esponenziale nel campo complesso

$$\exp\ z \ := \ \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \text{converge} \qquad \forall z \in \mathbf{C}$$

5. Proposizione $\exp(z + w) = \exp z + \exp w \qquad \forall\, z, \ w \in \mathbf{C}$.

Segue da

6. Lemma (Prodotto secondo Cauchy). $\sum\limits_{n=0}^{\infty}|z_n| + \sum\limits_{n=0}^{\infty}|w_n| < +\infty \Rightarrow$

$$\sum_{n=0}^{\infty}|\sum_{j+k=n} z_j\, w_k| < +\infty \qquad e \qquad \sum_{n=0}^{\infty}(\sum_{j+k=n} z_j\, w_k) = (\sum_{n=0}^{\infty} z_n)\ (\sum_{n=0}^{\infty} w_n)$$

Da 6. segue 5.: $\quad \exp(z+w) = \sum\limits_{n=0}^{\infty}\frac{(z+w)^n}{n!} = \sum\limits_{n=0}^{\infty}(\frac{1}{n!}\sum\limits_{j+k=n}\frac{n!}{j!\,k!}\, z^j\, w^k) =$

$$\sum_{n=0}^{\infty}(\sum_{j+k=n}\frac{z^j}{j!}\,\frac{w^k}{k!}) = (\sum_{n=0}^{\infty}\frac{z^j}{j!})\,(\sum_{n=0}^{\infty}\frac{w^k}{k!}) = \exp z\,\exp w$$

In particolare, $(\exp z)^p = \exp(pz)\ \ \forall p \in \mathbf{N},\ z \in \mathbf{C}$,exp $p = (\exp\ 1)^p =:$ e^p, $(\exp(\frac{1}{p}))^p = e$. Dunque $\exp(\frac{1}{p}) = e^{\frac{1}{p}}$ e quindi $\exp(\frac{p}{q}) = (\exp\frac{1}{q})^p = (e^{\frac{1}{q}})^p = e^{\frac{p}{q}}$: $x \to \exp x,\ \ x \in \mathbf{R}$ é prolungamento continuo di $r \to e^r, r \in \mathbf{Q}$.

7. $(i)\ \ \exp(-z) = (\exp z)^{-1}\ \ (ii)\ \ \overline{\exp z} = \exp\overline{z}\ \ (iii)\ |\exp(it)| = 1\ \ \forall t \in \mathbf{R}$

$(i)\ \exp z\ \exp(-z) = 1\ \ (ii)\ \exp\overline{z} = \lim\limits_{N\to+\infty}\sum\limits_{n=0}^{N}\frac{\overline{z}^n}{n!} = \lim\limits_{N\to+\infty}\overline{\sum\limits_{n=0}^{N}\frac{z^n}{n!}} = \overline{\exp z}$

$(iii)\ |\exp(it)|^2 = \exp(it)\ \overline{\exp(it)} = \exp(it)\ \exp(-it) = 1.$

8. $Re\,(\exp\ it) = \sum\limits_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!} = \cos t \qquad Im\,(\exp\ it) = \sum\limits_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sin t$

Formule di Eulero

$$\exp(\pm\, it) = \cos t \pm i\sin t$$

$$\sin t = \frac{\exp(it) - \exp(-it)}{2i}, \qquad \cos t = \frac{\exp(it) + \exp(-it)}{2} \qquad \forall\, t \in \mathbf{R}$$

Da ció segue in particolare che

- $\exp(2k\pi i) = 1\ \ \forall k \in \mathbf{Z}$, cioé $t \to \exp(it)\ \ t \in \mathbf{R}$ é 2π-periodica,
- $\exp(x+iy) = \exp x\,\exp(iy) = e^x(\cos y + i\sin y)$ e $\exp(z+2\pi i) = \exp z,\ \ \forall z$
- $z = x+iy,\ \ x^2+y^2 = 1\ \ y \ge 0 \quad \Rightarrow\ \exists\ !\ t \in [0,\pi]:\ \ x = \cos t$
- $|z| = 1 \quad \Rightarrow \quad \exists\ !\ t \in (-\pi,\pi]:\ \ z = \exp(it)$

9. Funzioni circolari ed iperboliche sui complessi

$$\cos z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!} \qquad \forall z \in \mathbf{C}$$

$$\sinh z := \frac{1}{2}(\exp z - \exp(-z)) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad \forall z \in \mathbf{C}$$

$$\cosh z := \frac{1}{2}(\exp z + \exp(-z)) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad \forall z \in \mathbf{C}$$

10. (i) $\exp(iz) \equiv \cos z + i \sin z, \qquad \exp(-iz) \equiv \cos z - i \sin z$

(ii) $\cos z \equiv \dfrac{\exp(iz) + \exp(-iz)}{2} \equiv \cosh iz \qquad \sin z \equiv \dfrac{\exp(iz) - \exp(-iz)}{2i} \equiv \dfrac{\sinh(iz)}{i}$

Da (ii) segue: $\sin z, \cos z$ sono funzioni 2π-periodiche mentre $\sinh z, \cosh z$ sono $2\pi i$-periodiche. Inoltre, $\sin^2 z + \cos^2 z \equiv 1$, $\cosh^2 z - \sinh^2 z \equiv 1$

11. Definizione di $\arg z$, $\log z$, $z \in \mathbf{C}$

Dato $z \in \mathbf{C}$, $\arg z$ (**argomento di** z) é l'unico reale in $(-\pi, \pi]$ tale che

$$z = |z| \exp(i \arg z)$$

Notiamo che, per periodicitá, $z = |z| \exp(i(\arg z + 2k\pi)) \;\; \forall\, k \in \mathbf{Z}$. Scriveremo

$$Arg\, z := \{\arg z + 2k\pi, \quad k \in \mathbf{Z}\}$$

Ora, dato $w \in \mathbf{C}$, $w \neq 0$

$$\exp z = w \quad \Leftrightarrow \quad \exp(Re\, z) \exp(i\, Im\, z) = |w| \exp(i \arg w) \quad \Leftrightarrow$$

$$\exp\, Re\, z = |w| \quad \text{e} \quad Im\, z - \arg w \in 2\pi\, \mathbf{Z} \qquad \text{cioé}$$

$$\exp z = w \quad \Leftrightarrow \quad z \in \{\log |w| + i\, Arg\, w\}$$

Porremo $\qquad Log\, w := \{\log |w| + i\, Arg\, w\} \qquad \forall w \in \mathbf{C}, \quad w \neq 0$

La funzione $log\, w := \log |w| + i \arg w$ si chiama valore principale del logaritmo.

Esempi. $Log\, x = \log x + 2k\pi i$, $\forall x > 0$, $Log\, x = \log |x| + (2k+1)\pi i$, $\forall x < 0$.
$log\,(-1) = \pi i$, $log\, i = \frac{\pi}{2} i$, $Log\,(1-i) = \log \sqrt{2} + (2k - \frac{1}{4})\pi i$.

12. Potenze in C Se $w, z \in C, w \neq 0$

$$w^z := \exp(z\, Log\, w) = \exp\{z\, [\log |w| + i(arg\, w + 2k\pi)]\} \qquad k \in \mathbf{Z}$$

Esempi. Sia $z = n \in \mathbf{N}$; $\quad w^n = \exp\{n\, [\log |w| + i(arg\, w + 2k\pi)]\} = \exp\{n \log |w|\} \exp\{n\, i(arg\, w + 2k\pi)\} = |w|^n [\exp\{i(arg\, w + 2k\pi)\}]^n = w \times \ldots \times w$ (n volte).

Se $z = \frac{1}{n}$, $n \in \mathbf{N}$, $a^{\frac{1}{n}} = \{|a|^{\frac{1}{n}} \exp\, i\, \frac{arg\, a + 2k\pi}{n},\ k = 0, \ldots, n-1\}$ (le n radici complesse di a). Se $z \notin \mathbf{Q}$, a^z é un insieme infinito. In particolare, $e^z = \exp z$ se e solo se $z \in \mathbf{Z}$.

APPENDICE

A1. Funzioni complesse di variabile complessa.

Sia $O \subset \mathbf{C}$ aperto, ovvero $\forall z_0 \in O,\ \exists D_r(z_0) := \{z \in \mathbf{C}:\ |z - z_0| < r\} \subset O$ e quindi, $z_0 \in O, z_n \to_n z_0 \ \Rightarrow z_n \in O$ definitivamente.
Sia $f : O \to \mathbf{C}$.

f é **continua** in $z_0 \in O \ \Leftrightarrow\ \forall \epsilon > 0\, \exists \delta > 0:\ |z - z_0| \le \delta \Rightarrow |f(z) - f(z_0)| \le \epsilon$

f é **derivabile** in $z_0 \in O$ con derivata $f'(z_0)$ $\Leftrightarrow$

$$\forall\ \epsilon > 0\, \exists\ \delta > 0:\ |z - z_0| \le \delta \quad \Rightarrow \quad |\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)| \le \epsilon$$

Anche qui, come nel caso reale: f é derivabile in $z_0 \Rightarrow f$ é continua in z_0.
Esercizio Sia $f(z) := \sum_{n=0}^{\infty} a_n z^n$ somma di una serie di potenze avente raggio di convergenza $r > 0$. Allora $f \in C^\infty(D_r)$.
Proviamo che

$$\frac{d^k f}{dz^k}(z) = \sum_{n=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k} z^n \qquad \forall z \in D_r$$

Basta provare la formula per $k = 1$. Sia $z \in D_\rho(z_0) \subset D_{r+\epsilon}$. É

$$|\frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} n a_n z_0^{n-1}| \le \sum_{n=2}^{\infty} |a_n|\, |\frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1}|$$

Ora, $|z^k - z_0^k| = |(z - z_0)(z^{k-1} + z_0 z^{k-2} + \ldots + z_0^{k-2} z + z_0^{k-1})| \le |z - z_0| k (r+\epsilon)^{k-1} \Rightarrow$

$$|\frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1}| = |z^{n-1} - z_0^{n-1} + z_0(z^{n-2} - z_0^{n-2}) + \ldots + z_0^{n-2}(z - z_0)| \le$$

$$\le |z^{n-1} - z_0^{n-1}| + |z_0|\, |z^{n-2} - z_0^{n-2}| + \ldots + |z_0|^{n-2}\, |z - z_0| \le$$

$$\le |z - z_0| \left[(n-1)(r+\epsilon)^{n-2} + (n-2)|z_0|(r+\epsilon)^{n-3} + \ldots + |z_0|^{n-2} \right] \le$$

$$\le \frac{n(n-1)}{2} |z - z_0|\, (r+\epsilon)^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} |a_n|\, |\frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1}| \le |z - z_0| \sum_{n=2}^{\infty} |a_n|\, \frac{n(n-1)}{2}\, (r+\epsilon)^{n-2} \to_{z \to z_0} 0$$

perché $\rho < r \Rightarrow \sum_{n=2}^{\infty} |a_n| \frac{n(n-1)}{2} \rho^{n-2} < +\infty.$

A1: Prova di 6. Siano $s_N := \sum_{n=0}^{N} z_n, \quad \sigma_N := \sum_{n=0}^{N} w_n$

$$p_N := \sum_{n=0}^{N} (\sum_{j+k=n} z_j w_k) = z_0w_0+(z_0w_1+z_1w_0)+\ldots+(z_0w_N+z_1w_{N-1}+\ldots+z_{N-1}w_1+z_Nw_0)$$

$= z_0(w_0+w_1+\ldots+w_N)+z_1(w_0+\ldots+w_{N-1})+\ldots+z_Nw_0.$ Dunque

$$|s_N\,\sigma_N - p_N| =$$

$$|z_0(w_0+\ldots+w_N)+z_1(w_0+\ldots+w_N)+\ldots+z_{N-1}(w_0+\ldots+w_N)+z_N(w_0+\ldots+w_N)-$$

$$[z_0\,(w_0+w_1+\ldots+w_N)+z_1\,(w_0+\ldots+w_{N-1})+\ldots+z_{N-1}(w_0+w_1)+z_N\,w_0]| =$$

$$|z_1\,w_N + z_2(w_{N-1}+w_N)+\ldots+z_{N-1}(w_2+\ldots+w_N)+z_N\,(w_1+\ldots+w_N)| \le$$

$$\le \sum_{j=1}^{n} \left[|z_j|\,|\sum_{i=1}^{j} w_{N-j+i}| \right] + \sum_{j=n+1}^{N} \left[|z_j|\,|\sum_{i=1}^{j} w_{N-j+i}| \right] \le$$

$$\le \left[\sum_{j=1}^{n} |z_j| \right] \left[\sum_{k=N-n+1}^{\infty} |w_k| \right] + \left[\sum_{j\ge n+1} |z_j| \right] \left[\sum_{k=1}^{\infty} |w_k| \right] \qquad n := [\frac{N}{2}].$$ Da

$\sum_{k=N-[\frac{N}{2}]+1}^{\infty} |w_k| \to_{N\to+\infty} 0, \quad \sum_{j\ge[\frac{N}{2}]+1} |z_j| \to_{N\to+\infty} 0, \quad \sum_j |z_j| < +\infty, \quad \sum_{k=1}^{\infty} |w_k| < \infty$
segue $|s_N\,\sigma_N - p_N| \to_{N\to+\infty} 0$ e quindi $\lim_N p_N = \lim_N s_N\,\sigma_N.$

ESERCIZIO. $e := \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to+\infty} (1+\frac{1}{n})^n$ É

$(1+\frac{1}{n})^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \frac{1}{n^k} < \sum_{k=0}^{n} \frac{1}{k!}$ perché $\frac{n!}{n^k\,(n-k)!} = \frac{(n-k)!(n-k-1)\ldots n}{(n-k)!n\,n\ldots n} < 1$

e quindi $$\limsup_n (1+\frac{1}{n})^n \le \sum_{n=0}^{\infty} \frac{1}{n!}$$

Viceversa, $n > n_0 \Rightarrow (1+\frac{1}{n})^n > \sum_{k=0}^{n_0} \frac{n!}{k!\,(n-k)!} \frac{1}{n^k} \Rightarrow \liminf_n (1+\frac{1}{n})^n \ge \sum_{k=0}^{n_0} \frac{1}{k!}, \ \forall n_0$

perché $\frac{n!}{n^k\,(n-k)!} = (1-\frac{1}{n})\,(1-\frac{2}{n})\,\ldots(1-\frac{k-1}{n}) \to_n 1.$ Quindi

$$\liminf_n (1+\frac{1}{n})^n \ge \sum_{k=0}^{\infty} \frac{1}{k!}$$