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Teorema di Fubini

(1) Calcolare $\int_Q f(x,y)\mathrm{d}\,x\,\mathrm{d}\,y$ dove $Q=[0,1]\times[0,1]$ e $f(x,y)=1-x-y$.

(2) Calcolare $\int_R f(x,y)\mathrm{d}\,x\,\mathrm{d}\,y$ dove $R=[3,4]\times[1,2]$ e $f(x,y)=\frac{1}{x^2+y^2}$.

(3) Calcolare $\int_Q f(x,y)\mathrm{d}\,x\,\mathrm{d}\,y$ dove $Q=[0,1]\times[0,1]$ e $f(x,y)=\frac{x^2}{1+y^2}$.

(4) Calcolare $\int_R f(x,y)\mathrm{d}\,x\,\mathrm{d}\,y$ dove $R=[0,1]\times[1,2]$ e $f(x,y)=\frac{e^{\frac{x}{y}}}{y^3}$.

(5) Sia

$$E=\{\,(x,y)\in\mathbb{R}^2\;:\;-1\le x\le 1,\;\;0\le y\le\sqrt{1-x^2}\,\}.$$

Calcolare $\int_E x^2\mathrm{d}\,x\,\mathrm{d}\,y$.

(6) ESERCIZI:
- Sia E il quarto di cerchio di centro $(0,0)$ e raggio 1, nel primo quadrante. Calcolare $\int_E \arcsin y\mathrm{d}\,x\,\mathrm{d}\,y$.
- Sia $R=[1,2]\times[1,2]$. Calcolare $\int_R x\,y\,\log(x\,y)\mathrm{d}\,x\,\mathrm{d}\,y$.
- Sia E il triangolo di vertici $(1,0)$, $(0,1)$, $(1,1)$. Calcolare $\int_E \frac{y}{\sqrt{1+x^2}}\mathrm{d}\,x\,\mathrm{d}\,y$ e $\int_E (1+y)\,e^x\mathrm{d}\,x\mathrm{d}\,y$.

Soluzioni:

(1)

$$\begin{aligned}\int_Q f(x,y)\mathrm{d}\,x\,\mathrm{d}\,y &= \int_0^1 \mathrm{d}\,x\Big(\int_0^1 1-x-y\mathrm{d}\,y\Big)=\int_0^1\Big[y-xy-\frac{y^2}{2}\Big]_{y=0}^{y^1}\mathrm{d}\,x\\ &= \int_0^1\Big(1-x-\frac{1}{2}\Big)\mathrm{d}\,x=\int_0^1\frac{1}{2}-x\mathrm{d}\,x=\Big[\frac{x}{2}-\frac{x^2}{2}\Big]_0^1=0.\end{aligned}$$

(2)

$$\begin{aligned}\int_R f(x,y)\mathrm{d}\,x\,\mathrm{d}\,y &= \int_3^4 \mathrm{d}\,x\int_1^2 \frac{\mathrm{d}\,y}{(x+y)^2} = \int_3^4 \mathrm{d}\,x\Big[-\frac{1}{x+y}\Big]_{y=1}^{y=2} \\ &= \int_3^4 \frac{1}{x+1}-\frac{1}{x+2}\mathrm{d}\,x = \Big[\log(x+1)-\log(x+2)\Big]_3^4 \\ &= \Big[\log\frac{x+1}{x+2}\Big]_3^4 = \frac{5}{6}-\log 45 = \log\frac{25}{24}.\end{aligned}$$

(3)

$$\begin{aligned}\int_Q f(x,y)\mathrm{d}\,x\,\mathrm{d}\,y &= \int_0^1 \mathrm{d}\,x\Big(\int_0^1 \frac{x^2}{1+y^2}\mathrm{d}\,y\Big) = \int_0^1 x^2\mathrm{d}\,x\int_0^1 \frac{1}{1+y^2}\mathrm{d}\,y \\ &= \int_0^1 x^2\mathrm{d}\,x[\arctan y]_0^1 = \frac{\pi}{4}\Big[\frac{x^3}{3}\Big]_0^1 = \frac{\pi}{12}\end{aligned}$$

(4)

$$\begin{aligned}\int_R f(x,y)\mathrm{d}\,x\,\mathrm{d}\,y &= \int_1^2 \mathrm{d}\,x\int_0^1 \frac{e^{\frac{x}{y}}}{y^3}\mathrm{d}\,x = \int_1^2 \frac{1}{y^2}\mathrm{d}\,y\int_0^1 \frac{e^{\frac{x}{y}}}{y}\mathrm{d}\,x \\ &= \int_1^2 \frac{1}{y^2}\mathrm{d}\,y[e^{\frac{x}{y}}]_{x=0}^{x=1} = \int_1^2 \frac{e^{\frac{1}{y}}}{y^2}-\frac{1}{y^2}\mathrm{d}\,y = \Big[-e^{-\frac{1}{y}}+\frac{1}{y}\Big]_1^2 = -\sqrt{e}-\frac{1}{2}+e.\end{aligned}$$

(5)

$$\begin{aligned}\int_E x^2\mathrm{d}\,x &= \int_{-1}^1 x^2\mathrm{d}\,x\int_0^{\sqrt{1-x^2}}\mathrm{d}\,y = \int_{-1}^1 x^2\sqrt{1-x^2}\mathrm{d}\,x = \int_0^\pi \cos^2 t\sqrt{1-cos^2 t}(-\sin t)\mathrm{d}\,t \\ &= -\int_0^\pi \cos^2\sin^2 t\mathrm{d}\,t = \int_0^\pi (\cos^2 t-1)\cos^2 t\mathrm{d}\,t = \int_0^\pi \cos^4 t - cos^2 t\mathrm{d}\,t \\ &= \int_0^\pi \Big(\frac{\cos(2t)+1}{2}\Big)^2-\Big(\frac{\cos(2t)+1}{2}\Big)\mathrm{d}t \\ &= \int_0^\pi \Big(\frac{\cos^2(2t)}{4}+\frac{\cos(2t)}{2}+\frac{1}{4}-\frac{\cos(2t)}{2}-\frac{1}{2}\Big)\mathrm{d}\,t = \frac{1}{4}\int_0^\pi \Big(\frac{\cos(4t)+1}{2}-1\Big)\mathrm{d}\,t \\ &= \frac{1}{4}\int_0^\pi \Big(\frac{\cos(4t)}{2}-\frac{1}{2}\Big)\mathrm{d}\,t = \Big[\frac{\sin(4t)}{32}-\frac{t}{8}\Big]_0^\pi = -\frac{\pi}{8}.\end{aligned}$$