Am1c – Soluzioni Tutorato VIII

Integrali III

Martedì 29 Aprile 2008

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Esercizio 1 Dato che $1+x^4=\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)$ si ha che

$$\frac{1}{1+x^4}=\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)}=\frac{Ax+B}{\left(x^2+\sqrt{2}x+1\right)}+\frac{Cx+D}{\left(x^2-\sqrt{2}x+1\right)}$$

da cui si ottiene che $A=-\frac{1}{2\sqrt{2}}$ $\quad B=\frac{1}{2}$ $\quad C=\frac{1}{2\sqrt{2}}$ $\quad D=\frac{1}{2}$. Otteniamo quindi:

$$\int\frac{1}{1+x^4}dx=\int\frac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{\left(x^2+\sqrt{2}x+1\right)}dx+\int\frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{\left(x^2-\sqrt{2}x+1\right)}dx=$$

$$=-\frac{1}{4\sqrt{2}}\int\frac{2x+\sqrt{2}}{\left(x^2+\sqrt{2}x+1\right)}dx+\frac{3}{4}\int\frac{dx}{\left(x^2+\sqrt{2}x+1\right)}+\frac{1}{4\sqrt{2}}\int\frac{2x-\sqrt{2}}{\left(x^2-\sqrt{2}x+1\right)}dx+\frac{3}{4}\int\frac{dx}{\left(x^2-\sqrt{2}x+1\right)}=$$

$$=-\frac{1}{4\sqrt{2}}\ln\left(x^2+\sqrt{2}x+1\right)+\frac{1}{4\sqrt{2}}\ln\left(x^2-\sqrt{2}x+1\right)+\frac{3}{4}\int\frac{dx}{\left(x^2+\sqrt{2}x+1\right)}+\frac{3}{4}\int\frac{dx}{\left(x^2-\sqrt{2}x+1\right)}=$$

$$=-\frac{1}{4\sqrt{2}}\ln\left(x^2+\sqrt{2}x+1\right)+\frac{1}{4\sqrt{2}}\ln\left(x^2-\sqrt{2}x+1\right)+\frac{3}{4}\int\frac{dx}{\left(x^2+\sqrt{2}x+\frac{1}{2}+\frac{1}{2}\right)}+\frac{3}{4}\int\frac{dx}{\left(x^2-\sqrt{2}x+\frac{1}{2}+\frac{1}{2}\right)}=$$

$$=-\frac{1}{4\sqrt{2}}\ln\left(x^2+\sqrt{2}x+1\right)+\frac{1}{4\sqrt{2}}\ln\left(x^2-\sqrt{2}x+1\right)+\frac{3}{4}\int\frac{dx}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}+\frac{3}{4}\int\frac{dx}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}=$$

$$=-\frac{1}{4\sqrt{2}}\ln\left(x^2+\sqrt{2}x+1\right)+\frac{1}{4\sqrt{2}}\ln\left(x^2-\sqrt{2}x+1\right)+\frac{3}{2}\int\frac{dx}{\left(\sqrt{2}x+1\right)^2+1}+\frac{3}{2}\int\frac{dx}{\left(\sqrt{2}x-1\right)^2+1}=$$

$$=-\frac{1}{4\sqrt{2}}\ln\left(x^2+\sqrt{2}x+1\right)+\frac{1}{4\sqrt{2}}\ln\left(x^2-\sqrt{2}x+1\right)+\frac{3}{2\sqrt{2}}\arctan\left(\sqrt{2}x+1\right)+\frac{3}{2\sqrt{2}}\arctan\left(\sqrt{2}x-1\right)+k$$

Esercizio 2 Aggiungendo e sottraendo x^2 al numeratore si ottiene

$$\int\frac{1}{\left(1+x^2\right)^2}dx=\int\frac{1+x^2-x^2}{\left(1+x^2\right)^2}dx=\int\frac{1}{1+x^2}dx+\frac{1}{2}\int\frac{-2x^2}{\left(1+x^2\right)^2}dx$$

dai cui, integrando per parti con $f' = \frac{-2x}{\left(1+x^2\right)^2} \quad g = x$ si ha

$$\int \frac{1}{1+x^2}\,dx + \frac{1}{2}\int \frac{-2x^2}{\left(1+x^2\right)^2}\,dx = \arctan x + \frac{1}{2}\left(\frac{x}{1+x^2} - \int \frac{1}{1+x^2}\,dx\right) =$$

$$= \frac{1}{2}\arctan x + \frac{1}{2}\frac{x}{1+x^2} + k$$

Esercizio 3 Consideriamo I_n : integrando per parti con $f = \frac{1}{\left(1+x^2\right)^{n-1}} \quad g' = 1$, si ha

$$I_{n-1} = \int \frac{1}{\left(1+x^2\right)^{n-1}}\,dx = \frac{x}{\left(1+x^2\right)^{n-1}} + 2(n-1)\int \frac{x^2}{\left(1+x^2\right)^n}\,dx =$$

$$= \frac{x}{\left(1+x^2\right)^{n-1}} + 2(n-1)\int \frac{1+x^2-1}{\left(1+x^2\right)^n}\,dx =$$

$$= \frac{x}{\left(1+x^2\right)^{n-1}} + 2(n-1)\int \frac{1}{\left(1+x^2\right)^{n-1}}\,dx - 2(n-1)\int \frac{1}{\left(1+x^2\right)^n}\,dx =$$

$$= \frac{x}{\left(1+x^2\right)^{n-1}} + 2(n-1)I_{n-1} - 2(n-1)I_n$$

da cui la tesi.

Esercizio 4 (1) Utilizzando il metodo del completamento del quadrato otteniamo

$$\int \frac{1}{\left(7x^2+4x+3\right)^3}\,dx = \int \frac{1}{\left[\left(\sqrt{7}x + \frac{2}{\sqrt{7}}\right)^2 + \frac{17}{7}\right]^3}\,dx = \left(\frac{17}{7}\right)^3 \int \frac{1}{\left[\left(\frac{7}{\sqrt{17}}x + \frac{2}{\sqrt{17}}\right)^2 + 1\right]^3}\,dx$$

ora posto $t = \frac{7}{\sqrt{17}}x + \frac{2}{\sqrt{17}}$ e quindi $dt = \frac{7}{\sqrt{17}}dx$ si ha

$$\left(\frac{17}{7}\right)^3 \int \frac{1}{\left[\left(\frac{7}{\sqrt{17}}x + \frac{2}{\sqrt{17}}\right)^2 + 1\right]^3}\,dx = \left(\frac{17}{7}\right)^3 \frac{\sqrt{17}}{7}\int \frac{1}{\left(t^2+1\right)^3}dt$$

utilizzando quindi la formula dell'esercizio precedente si ottiene

$$\left(\frac{17}{7}\right)^3\frac{\sqrt{17}}{7}\int\frac{1}{\left(t^2+1\right)^3}dt=\left(\frac{17}{7}\right)^3\frac{\sqrt{17}}{7}\left(\frac{3}{8}\arctan t+\frac{3}{8}\frac{t}{t^2+1}+\frac{1}{4}\frac{t}{\left(t^2+1\right)^2}+k\right)=$$

$$=\left(\frac{17}{7}\right)^3\frac{\sqrt{17}}{7}\left(\frac{3}{8}\arctan\left(\frac{7}{\sqrt{17}}x+\frac{2}{\sqrt{17}}\right)+\frac{3}{8}\frac{\frac{7}{\sqrt{17}}x+\frac{2}{\sqrt{17}}}{\left(\frac{7}{\sqrt{17}}x+\frac{2}{\sqrt{17}}\right)^2+1}+\frac{1}{4}\frac{\frac{7}{\sqrt{17}}x+\frac{2}{\sqrt{17}}}{\left[\left(\frac{7}{\sqrt{17}}x+\frac{2}{\sqrt{17}}\right)^2+1\right]^2}\right)+k$$

(2) Analogamente a prima si ha

$$\int\frac{x+1}{\left(x^2+2\right)^4}dx=\frac{1}{2}\int\frac{2x}{\left(x^2+2\right)^4}dx+\int\frac{1}{\left(x^2+2\right)^4}dx=$$

$$=-\frac{1}{6\left(x^2+2\right)^3}+\frac{\sqrt{2}}{16}\int\frac{1}{\left[\left(\frac{x}{\sqrt{2}}\right)^2+1\right]^4}\frac{dx}{\sqrt{2}}=$$

$$=-\frac{1}{6\left(x^2+2\right)^3}+\frac{\sqrt{2}}{16}\int\frac{1}{\left(t^2+1\right)^4}dt$$

dalla formula dell'esercizio precedente si ottiene

$$\int\frac{1}{\left(t^2+1\right)^4}dt=\frac{5}{16}\arctan t+\frac{5}{16}\frac{t}{t^2+1}+\frac{5}{24}\frac{t}{\left(t^2+1\right)^2}+\frac{1}{6}\frac{t}{\left(t^2+1\right)^3}+k$$

e quindi

$$\int\frac{x+1}{\left(x^2+2\right)^4}dx=\frac{5\sqrt{2}}{256}\arctan\frac{x}{\sqrt{2}}+\frac{5}{128}\frac{x}{x^2+2}+\frac{5}{96}\frac{x}{\left(x^2+2\right)^2}+\frac{1}{12}\frac{x-2}{\left(x^2+2\right)^3}+k$$